*Research Article*

# Hermite-Hadamard Inequality on Time Scales

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We discuss some variants of the Hermite-Hadamard inequality for convex functions on time scales. Some improvements and applications are also included.

## 1. Introduction

Recently, new developments of the theory and applications of dynamic derivatives on time scales were made. The study provides an unification and an extension of traditional differential and difference equations and, in the same time, it is a unification of the discrete theory with the continuous theory, from the scientific point of view. Moreover, it is a crucial tool in many computational and numerical applications. Based on the well-known $\Delta$ (delta) and $\nabla$ (nabla) dynamic derivatives, a combined dynamic derivative, so-called $\diamond_\alpha$ (diamond-$\alpha$) dynamic derivative, was introduced as a linear combination of $\Delta$ and $\nabla$ dynamic derivatives on time scales. The diamond-$\alpha$ dynamic derivative reduces to the $\Delta$ derivative for $\alpha = 1$ and to the $\nabla$ derivative for $\alpha = 0$. On the other hand, it represents a "weighted dynamic derivative" on any uniformly discrete time scale when $\alpha = 1/2$. See [1–5] for the basic rules of calculus associated with the diamond-$\alpha$ dynamic derivatives.

The classical Hermite-Hadamard inequality gives us an estimate, from below and from above, of the mean value of a convex function. The aim of this paper is to establish a full analogue of this inequality if we compute the mean value with the help of the delta, nabla, and diamond-$\alpha$ integral.

The left-hand side of the Hermite-Hadamard inequality is a special case of the Jensen inequality.

Recently, it has been proven a variant of diamond-$\alpha$ Jensen's inequality (see [6]).

**Theorem 1.1.** *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $g \in C([a,b]_{\mathbb{T}}, (c,d))$, and $f \in C((c,d), \mathbb{R})$ is convex, then*

$$f\left(\frac{\int_a^b g(s) \diamond_\alpha s}{b-a}\right) \leq \frac{\int_a^b f(g(s)) \diamond_\alpha s}{b-a}. \tag{1.1}$$

In the same paper appears the following generalized version of the diamond-$\alpha$ Jensen's inequality.

**Theorem 1.2.** *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $g \in C([a,b]_\mathbb{T}, (c,d))$, $h \in C([a,b]_\mathbb{T}, \mathbb{R})$ with $\int_a^b |h(s)| \diamond_\alpha s > 0$, and $f \in C((c,d), \mathbb{R})$ is convex, then*

$$f\left(\frac{\int_a^b |h(s)| g(s) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}\right) \leq \frac{\int_a^b |h(s)| f(g(s)) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}. \tag{1.2}$$

In Section 2, we review some necessary definitions and the calculus on time scales. In Section 3, we give our main results concerning the Hermite-Hadamard inequality. Some improvements and applications are presented in Section 4, together with an extension of Hermite-Hadamard inequality for some symmetric functions. A special case is that of diamond-1/2 integral, which enables us to gain a number of consequences of our Hermite-Hadamard type inequality; we present them in Section 5 together with a discussion concerning the case of convex-concave symmetric functions.

## 2. Preliminaries

A *time scale* (or measure chain) is any nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$ (endowed with the topology of subspace of $\mathbb{R}$).

Throughout this paper, $\mathbb{T}$ will denote a time scale and $[a,b]_\mathbb{T} = [a,b] \cap \mathbb{T}$ a time-scaled interval.

For all $t, r \in \mathbb{T}$, we define the *forward jump operator* $\sigma$ and the *backward jump operator* $\rho$ by the formulas

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \qquad \rho(r) = \sup\{\tau \in \mathbb{T} : \tau < r\} \in \mathbb{T}. \tag{2.1}$$

We make the convention:

$$\inf \varnothing := \sup \mathbb{T}, \qquad \sup \varnothing := \inf \mathbb{T}. \tag{2.2}$$

If $\sigma(t) > t$, then $t$ is said to be *right-scattered*, and if $\rho(r) < r$, then $r$ is said to be *left-scattered*. The points that are simultaneously right-scattered and left-scattered are called *isolated*. If $\sigma(t) = t$, then $t$ is said to be *right dense*, and if $\rho(r) = r$, then $r$ is said to be *left dense*. The points that are simultaneously right-dense and left-dense are called *dense*.

The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by

$$\mu(t) := \sigma(t) - t,$$
$$\nu(t) := t - \rho(t) \tag{2.3}$$

are called, respectively, the *forward* and *backward graininess functions*.

If $\mathbb{T}$ has a right-scattered minimum $m$, then define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum $M$, then define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Finally, put $\mathbb{T}_\kappa^\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

*Definition 2.1.* For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, one defines the delta derivative of $f$ in $t$, to be the number denoted by $f^\Delta(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ such that

$$\left| \left[ f(\sigma(t)) - f(s) \right] - f^\Delta(t) \left[ \sigma(t) - s \right] \right| < \varepsilon \left| \sigma(t) - s \right|, \tag{2.4}$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, one defines the nabla derivative of $f$ in $t$, to be the number denoted by $f^\nabla(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood $V$ of $t$ such that

$$\left| \left[ f(\rho(t)) - f(s) \right] - f^\nabla(t) \left[ \rho(t) - s \right] \right| < \varepsilon \left| \rho(t) - s \right|, \tag{2.5}$$

for all $s \in V$.

We say that $f$ is *delta differentiable* on $\mathbb{T}^\kappa$, provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and that $f$ is *nabla differentiable* on $\mathbb{T}_\kappa$, provided that $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

If $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f^\nabla(t) = f'(t). \tag{2.6}$$

If $\mathbb{T} = \mathbb{Z}$, then

$$f^\Delta(t) = f(t+1) - f(t) \tag{2.7}$$

is the forward difference operator, while

$$f^\nabla(t) = f(t) - f(t-1) \tag{2.8}$$

is the backward difference operator.

For a function $f : \mathbb{T} \to \mathbb{R}$, we define $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$, for all $t \in \mathbb{T}$, (i.e., $f^\sigma = f \circ \sigma$). We also define $f^\rho : \mathbb{T} \to \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$, for all $t \in \mathbb{T}$, (i.e., $f^\rho = f \circ \rho$).

For all $t \in \mathbb{T}^\kappa$, we have the following properties.

(i) If $f$ is delta differentiable at $t$, then $f$ is continuous at $t$.

(ii) If $f$ is left continuous at $t$ and $t$ is right-scattered, then $f$ is delta differentiable at $t$ with $f^\Delta(t) = (f^\sigma(t) - f(t))/\mu(t)$.

(iii) If $t$ is right-dense, then $f$ is delta differentiable at $t$, if and only if, the limit $\lim_{s \to t}((f(t) - f(s))/(t - s))$ exists as a finite number. In this case, $f^\Delta(t) = \lim_{s \to t}((f(t) - f(s))/(t - s))$.

(iv) If $f$ is delta differentiable at $t$, then $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$.

In the same manner, for all $t \in \mathbb{T}_\kappa$ we have the following properties.

(i) If $f$ is nabla differentiable at $t$, then $f$ is continuous at $t$.

(ii) If $f$ is right continuous at $t$ and $t$ is left-scattered, then $f$ is nabla differentiable at $t$ with $f^\nabla(t) = (f(t) - f^\rho(t))/\nu(t)$.

(iii) If $t$ is left-dense, then $f$ is nabla differentiable at $t$, if and only if, the limit $\lim_{s \to t}((f(t) - f(s))/(t - s))$ exists as a finite number. In this case, $f^\nabla(t) = \lim_{s \to t}((f(t) - f(s))/(t - s))$.

(iv) If $f$ is nabla differentiable at $t$, then $f^\rho(t) = f(t) - \nu(t)f^\nabla(t)$.

*Definition 2.2.* A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. One denotes by $C_{\mathrm{rd}}$ the set of all rd-continuous functions.

   A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous, if it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits are finite at all right-dense points in $\mathbb{T}$. One denotes by $C_{\mathrm{ld}}$ the set of all ld-continuous functions.

   It is easy to remark that the set of continuous functions on $\mathbb{T}$ contains both $C_{\mathrm{rd}}$ and $C_{\mathrm{ld}}$.

*Definition 2.3.* A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}^{\kappa}$. Then, one defines the delta integral by $\int_a^t f(s) \Delta s = F(t) - F(a)$.

   A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $G^{\nabla}(t) = f(t)$, for all $t \in \mathbb{T}_{\kappa}$. Then, one defines the nabla integral by $\int_a^t f(s) \Delta s = G(t) - G(a)$.

   According to [2, Theorem 1.74], every rd-continuous function has a delta antiderivative, and every ld-continuous function has a nabla antiderivative.

**Theorem 2.4** (see [2, Theorem 1.75]). *(i) If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(s) \Delta s = \mu(t) f(t). \tag{2.9}$$

*(ii) If $f \in C_{ld}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_{\rho(t)}^t f(s) \nabla s = \nu(t) f(t). \tag{2.10}$$

**Theorem 2.5** (see [2, Theorem 1.77]). *If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

   (i) $\int_a^b (f(t) + g(t)) \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t$;

   (ii) $\int_a^b \beta f(t) \Delta t = \beta \int_a^b f(t) \Delta t$;

   (iii) $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t$;

   (iv) $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t$;

   (v) $\int_a^b f(\sigma(t)) g^{\Delta}(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(t) \Delta t$;

   (vi) $\int_a^b f(t) g^{\Delta}(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(\sigma(t)) \Delta t$;

   (vii) $\int_a^a f(t) \Delta t = 0$;

   (viii) *if $f(t) \geq 0$ for all $t$, then $\int_a^b f(t) \Delta t \geq 0$;*

   (ix) *if $|f(t)| \leq g(t)$ on $[a, b)$, then*

$$\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b g(t) \Delta t. \tag{2.11}$$

Using Theorem 2.5, (viii) we get

(i) if $f(t) \leq g(t)$ for all $t$, then $\int_a^b f(t)\Delta t \leq \int_a^b g(t)\Delta t$;

(ii) if $f(t) \geq 0$ for all $t$, then $f \equiv 0$ if and only if $\int_a^b f(t)\Delta t = 0$;

and if in (ix), we choose $g(t) = |f(t)|$ on $[a,b]$, we obtain

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b |f(t)|\Delta t. \tag{2.12}$$

A similar theorem works for the nabla antiderivative (for $f, g \in C_{\mathrm{ld}}$).

Now, we give a brief introduction of the diamond-$\alpha$ dynamic derivative and of the diamond-$\alpha$ integral.

*Definition 2.6.* Let $\mathbb{T}$ be a time scale and for $s, t \in \mathbb{T}_\kappa^\kappa$ put $\mu_{\mathrm{ts}} = \sigma(t) - s$, and $\nu_{\mathrm{ts}} = \rho(t) - s$. One defines the diamond-$\alpha$ dynamic derivative of a function $f : \mathbb{T} \to \mathbb{R}$ in $t$ to be the number denoted by $f^{\diamond_\alpha}(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ such that for all $s \in U$

$$\left| \alpha \left[ f(\sigma(t)) - f(s) \right] \nu_{\mathrm{ts}} + (1 - \alpha) \left[ f(\rho(t)) - f(s) \right] \mu_{\mathrm{ts}} - f^{\diamond_\alpha}(t)\mu_{\mathrm{ts}}\nu_{\mathrm{ts}} \right| < \varepsilon \left| \mu_{\mathrm{ts}}\nu_{\mathrm{ts}} \right|. \tag{2.13}$$

A function is called *diamond-$\alpha$ differentiable* on $\mathbb{T}_\kappa^\kappa$ if $f^{\diamond_\alpha}(t)$ exists for all $t \in \mathbb{T}_\kappa^\kappa$. If $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}$ in the sense of $\Delta$ and $\nabla$, then $f$ is diamond-$\alpha$ differentiable at $t \in \mathbb{T}_\kappa^\kappa$, and the diamond-$\alpha$ derivative $f^{\diamond_\alpha}(t)$ is given by

$$f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t), \quad 0 \leq \alpha \leq 1. \tag{2.14}$$

As it was proved in [5, Theorem 3.9], if $f$ is diamond-$\alpha$ differentiable for $0 < \alpha < 1$ then $f$ is both $\Delta$ and $\nabla$ differentiable. It is obvious that for $\alpha = 1$ the diamond-$\alpha$ derivative reduces to the standard $\Delta$ derivative and for $\alpha = 0$ the diamond-$\alpha$ derivative reduces to the standard $\nabla$ derivative. For $\alpha \in (0, 1)$, it represents a "weighted dynamic derivative."

We present here some operations with the diamond-$\alpha$ derivative. For that, let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-$\alpha$ differentiable at $t \in \mathbb{T}$. Then,

(i) $f + g : \mathbb{T} \to \mathbb{R}$ is diamond-$\alpha$ differentiable at $t \in \mathbb{T}$ and

$$(f + g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) + g^{\diamond_\alpha}(t); \tag{2.15}$$

(ii) if $c \in \mathbb{R}$ and $cf : \mathbb{T} \to \mathbb{R}$ is diamond-$\alpha$ differentiable at $t \in \mathbb{T}$ and

$$(cf)^{\diamond_\alpha}(t) = cf^{\diamond_\alpha}(t); \tag{2.16}$$

(iii) $fg : \mathbb{T} \to \mathbb{R}$ is diamond-$\alpha$ differentiable at $t \in \mathbb{T}$ and

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)g^\Delta(t) + (1 - \alpha)f^\rho(t)g^\nabla(t). \tag{2.17}$$

Let $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. The diamond-$\alpha$ integral of $f$ from $a$ to $b$ is defined by

$$\int_a^b f(t) \diamond_\alpha t = \alpha \int_a^b f(t) \Delta t + (1 - \alpha) \int_a^b f(t) \nabla t, \quad 0 \leq \alpha \leq 1, \tag{2.18}$$

provided that $f$ has a delta and a nabla integral on $[a, b]_\mathbb{T}$. Obviously, each continuous function has a diamond-$\alpha$ integral. The combined derivative $\diamond_\alpha$ is not a dynamic derivative, since we do not have a $\diamond_\alpha$ antiderivative. See [6, Example 2.1]. In general,

$$\left( \int_a^t f(s) \diamond_\alpha s \right)^{\diamond_\alpha} \neq f(t), \quad t \in \mathbb{T}, \tag{2.19}$$

but we still have some of the "classical" properties, as one can easily be deduced from Theorem 2.5 and its analogue for the nabla integral.

**Theorem 2.7.** *If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$, and $f, g$ are continuous functions, then*

(i) $\int_a^b (f(t) + g(t)) \diamond_\alpha t = \int_a^b f(t) \diamond_\alpha t + \int_a^b g(t) \diamond_\alpha t$;

(ii) $\int_a^b \beta f(t) \diamond_\alpha t = \beta \int_a^b f(t) \diamond_\alpha t$;

(iii) $\int_a^b f(t) \diamond_\alpha t = - \int_b^a f(t) \diamond_\alpha t$;

(iv) $\int_a^b f(t) \diamond_\alpha t = \int_a^c f(t) \diamond_\alpha t + \int_c^b f(t) \diamond_\alpha t$;

(v) $\int_a^a f(t) \diamond_\alpha t = 0$;

(vi) *if $f(t) \geq 0$ for all $t$, then $\int_a^b f(t) \diamond_\alpha t \geq 0$*;

(vii) *if $f(t) \leq g(t)$ for all $t$, then $\int_a^b f(t) \diamond_\alpha t \leq \int_a^b g(t) \diamond_\alpha t$*;

(viii) *if $f(t) \geq 0$ for all $t$, then $f \equiv 0$ if and only if $\int_a^b f(t) \diamond_\alpha t = 0$*;

(ix) *if $|f(t)| \leq g(t)$ on $[a, b)$, then*

$$\left| \int_a^b f(t) \diamond_\alpha t \right| \leq \int_a^b g(t) \diamond_\alpha t. \tag{2.20}$$

In Theorem 2.7, (ix), if we choose $g(t) = |f(t)|$ on $[a, b]$, we have

$$\left| \int_a^b f(t) \diamond_\alpha t \right| \leq \int_a^b |f(t)| \diamond_\alpha t. \tag{2.21}$$

## 3. The Hermite-Hadamard inequality

In this section, we present an extension of the Hermite-Hadamard inequality, for time scales. For that, we need to find the conditions fulfilled by the functions defined on a time scale. We want to evaluate $\int_a^b t \Delta t$ and $\int_a^b t \nabla t$ on such sets, because they provide us with a useful tool for the proof of Hermite-Hadamard inequality. We start with a few technical lemmas.

**Lemma 3.1.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function and $a, b \in \mathbb{T}$.*

(i) *If $f$ is nondecreasing on $\mathbb{T}$, then*

$$(b - a)f(a) \leq \int_a^b f(t)\Delta t \leq \int_a^b \tilde{f}(t)\, dt \leq \int_a^b f(t)\nabla t \leq (b - a)f(b), \tag{3.1}$$

*where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $f(t) = \tilde{f}(t)$, for all $t \in \mathbb{T}$.*

(ii) *If $f$ is nonincreasing on $\mathbb{T}$, then*

$$(b - a)f(a) \geq \int_a^b f(t)\Delta t \geq \int_a^b \tilde{f}(t)\, dt \geq \int_a^b f(t)\nabla t \geq (b - a)f(b), \tag{3.2}$$

*where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a continuous nonincreasing function such that $f(t) = \tilde{f}(t)$, for all $t \in \mathbb{T}$.*

*In both cases, there exists an $\alpha_T \in [0, 1]$ such that*

$$\int_a^b f(t)\diamond_{\alpha_T} t = \int_a^b \tilde{f}(t)\, dt. \tag{3.3}$$

*Proof.* (i) We start by noticing that if $\mathbb{T} = \{a, b\}$ then by Theorem 2.4, we have

$$\int_a^b f(t)\Delta t = \int_a^{\sigma(a)} f(t)\Delta t = f(a)(b - a), \tag{3.4}$$

while if $\mathbb{T} = [a, b]$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\, dt. \tag{3.5}$$

It suffices to prove that, for monotone functions, the value of $\int_a^b f(t)\Delta t$, for a general time scale $\mathbb{T}$, remains between the values of $\int_a^b f(t)\Delta t$ for $\mathbb{T} = \{a, b\}$ and for $\mathbb{T} = [a, b]$.

Now, let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $f(t) = \tilde{f}(t)$, for all $t \in \mathbb{T}$. First, we will show that by adding a point or an interval, the corresponding integral increases.

Let us suppose that we add a point $c$ to $\mathbb{T}$, where $a < c < b$. If $\mathbb{T}_1 = \mathbb{T} \cup \{c\}$, and $c \notin \mathbb{T}$ is an isolated point of $\mathbb{T}_1$ (with $\int_a^b f(t)\Delta_1 t$ the corresponding integral), then

$$
\begin{aligned}
\int_a^b f(t)\Delta_1 t &= \int_a^c f(t)\Delta_1 t + \int_c^b f(t)\Delta_1 t \\
&= \int_a^{\rho_1(c)} f(t)\Delta_1 t + \int_{\rho_1(c)}^c f(t)\Delta_1 t + \int_c^{\sigma_1(c)} f(t)\Delta_1 t + \int_{\sigma_1(c)}^b f(t)\Delta_1 t \\
&= \int_a^{\rho_1(c)} f(t)\Delta t + \int_{\rho_1(c)}^c f(t)\Delta_1 t + \int_c^{\sigma_1(c)} f(t)\Delta_1 t + \int_{\sigma_1(c)}^b f(t)\Delta t \\
&= \int_a^b f(t)\Delta t - \int_{\rho_1(c)}^{\sigma_1(c)} f(t)\Delta t + \int_{\rho_1(c)}^c f(t)\Delta_1 t + \int_c^{\sigma_1(c)} f(t)\Delta_1 t \\
&= \int_a^b f(t)\Delta t - f(\rho_1(c))(\sigma_1(c) - \rho_1(c)) + f(\rho_1(c))(c - \rho_1(c)) + f(c)(\sigma_1(c) - c) \\
&= \int_a^b f(t)\Delta t + (f(c) - f(\rho_1(c)))(\sigma_1(c) - c) \\
&\geq \int_a^b f(t)\Delta t.
\end{aligned}
\tag{3.6}
$$

In the same manner, we prove that if we add an interval, the corresponding integral remains in the same interval. So, let us denote $\mathbb{T}_1 = \mathbb{T} \cup [c,d]$, with $a < c < d < b$ and $\mathbb{T} \cap [c,d] = \varnothing$, then

$$
\begin{aligned}
\int_a^b f(t)\Delta_1 t &= \int_a^{\rho_1(c)} f(t)\Delta_1 t + \int_{\rho_1(c)}^c f(t)\Delta_1 t + \int_c^d f(t)\Delta_1 t + \int_d^{\sigma_1(d)} f(t)\Delta_1 t + \int_{\sigma_1(d)}^b f(t)\Delta_1 t \\
&= \int_a^{\rho_1(c)} f(t)\Delta t + \int_{\rho_1(c)}^c f(t)\Delta_1 t + \int_c^d f(t)\Delta_1 t + \int_d^{\sigma_1(d)} f(t)\Delta_1 t + \int_{\sigma_1(d)}^b f(t)\Delta t \\
&= \int_a^b f(t)\Delta t - \int_{\rho_1(c)}^{\sigma_1(d)} f(t)\Delta t + \int_{\rho_1(c)}^c f(t)\Delta_1 t + \int_c^d f(t)\Delta_1 t + \int_d^{\sigma_1(d)} f(t)\Delta_1 t \\
&= \int_a^b f(t)\Delta t - f(\rho_1(c))(\sigma_1(d) - \rho_1(c)) + f(\rho_1(c))(c - \rho_1(c)) \\
&\quad + \int_c^d \tilde{f}(t)\,dt + f(d)(\sigma_1(d) - d) \\
&\geq \int_a^b f(t)\Delta t - f(\rho_1(c))(d - c) + (d - c)\tilde{f}(s) \\
&\geq \int_a^b f(t)\Delta t,
\end{aligned}
\tag{3.7}
$$

where $s \in (c,d)$ is the point from mean value theorem.

Using the same methods, we show that if we "extract" an isolated point or an interval from an initial times scale, the corresponding integral decreases. And so, the value of $\int_a^b f(t)\Delta t$ is between its minimum value (corresponding to $\mathbb{T} = \{a,b\}$) and its maximum value (corresponding to $\mathbb{T} = [a,b]$), that is

$$(b-a)f(a) \le \int_a^b f(t)\Delta t \le \int_a^b \tilde{f}(t)\,dt. \tag{3.8}$$

The proof is similar in the case of nonincreasing functions and also, for the nabla integral. The final conclusion of the Lemma 3.1 is obvious for any $\alpha \in [0,1]$ if $\int_a^b f(t)\Delta t$ is equal to $\int_a^b f(t)\nabla t$, while if the two integrals differ, it is all clear taking

$$\alpha_T = \frac{\int_a^b \tilde{f}(t)\,dt - \int_a^b f(t)\nabla t}{\int_a^b f(t)\Delta t - \int_a^b f(t)\nabla t}. \tag{3.9}$$

Then,

$$\int_a^b \tilde{f}(t)\,dt = \alpha_T \int_a^b f(t)\Delta t + (1-\alpha_T)\int_a^b f(t)\nabla t, \tag{3.10}$$

that is

$$\int_a^b f(t)\diamond_{\alpha_T} t = \int_a^b \tilde{f}(t)\,dt. \tag{3.11}$$

$\square$

*Remark 3.2.* The above proof covers the case of adding or extracting a set of the form $\{l_1, l_1, \ldots, l_n, \ldots, l\}$, where $n \in \mathbb{N}$ and $(l_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that $\lim_{n\to\infty} = l$. For that, suppose that $(l_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence (the proof works in the same way for nonincreasing sequences, while the case of nonmonotone sequences can be split in two subcases with monotone sequences). Let $\varepsilon > 0$. Since $(l_n)_{n\in\mathbb{N}}$ is convergent, we have $N_1 \in \mathbb{N}$ such that $|l - l_n| < \varepsilon$, for all $n \ge N_1$. Since $f$ is rd-continuous and $l$ is left dense, the limit $\lim_{n\to\infty} f(l_n)$ exists and it is finite. Denoting by $b$ this limit, we have $N_2 \in \mathbb{N}$ such that $|b - f(l_n)| < \varepsilon$, for all $n \ge N_2$ and so $f(l_n) \in (b-\varepsilon, b+\varepsilon)$, for all $n \ge N_2$. Using Theorem 2.5(iv), we have, for $N = \max\{N_1, N_2\}$,

$$\begin{aligned}\int_{l_1}^l f(t)\Delta t &= \sum_{i=1}^{N-1}\int_{l_i}^{l_{i+1}} f(t)\Delta t + \int_{l_N}^l f(t)\Delta t \\ &= \sum_{i=1}^{N-1}\int_{l_i}^{\sigma(l_i)} f(t)\Delta t + \int_{l_N}^l f(t)\Delta t \\ &= \sum_{i=1}^{N-1}\mu(l_i)f(l_i) + \int_{l_N}^l f(t)\Delta t.\end{aligned} \tag{3.12}$$

Taking the delta integral in the following inequality $b - \varepsilon < f(l_n) < b + \varepsilon$ and using Theorem 2.5(viii), we have

$$(b-\varepsilon)(l - l_N) < \int_{l_N}^l f(t)\Delta t < (b+\varepsilon)(l - l_N). \tag{3.13}$$

Taking the modulus in the last inequality and using $|l - l_N| < \varepsilon$, we get

$$0 \leq \left| \int_{l_N}^{l} f(t) \Delta t \right| \leq (b + \varepsilon) \varepsilon. \tag{3.14}$$

If $\varepsilon$ goes to 0 and $N$ goes to $\infty$, then $\lim_{N \to \infty} \int_{a_N}^{b} f(t) \Delta t = 0$. Passing to the limit as $N \to \infty$, in (3.12), we get

$$\int_{l_1}^{l} f(t) \Delta_1 t = \lim_{n \to \infty} \sum_{i=1}^{n} f(l_i)(l_{i+1} - l_i) \tag{3.15}$$

and so

$$\int_{l_1}^{l} f(t) \Delta_1 t \geq \lim_{n \to \infty} \sum_{i=1}^{n} f(l_1)(l_{i+1} - l_i) = f(l_1)(l - l_1), \tag{3.16}$$

while

$$\int_{l_1}^{l} f(t) \Delta_1 t \leq \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_i)(\xi_{i+1} - \xi_i) = \int_{l_1}^{l} f(t) \, dt, \tag{3.17}$$

which are, respectively, the case of adding two points $l_1$, $l$ and the case of adding an interval $[l_1, l]$.

*Remark 3.3.* (i) If $f$ is nondecreasing on $\mathbb{T}$, then for $\alpha \leq \alpha_T$, we have

$$\int_{a}^{b} f(t) \diamond_\alpha t \geq \int_{a}^{b} \widetilde{f}(t) \, dt, \tag{3.18}$$

while if $\alpha \geq \alpha_T$, we have

$$\int_{a}^{b} f(t) \diamond_\alpha t \leq \int_{a}^{b} \widetilde{f}(t) \, dt. \tag{3.19}$$

(ii) If $f$ is nonincreasing on $\mathbb{T}$, then for $\alpha \leq \alpha_T$, we have

$$\int_{a}^{b} f(t) \diamond_\alpha t \leq \int_{a}^{b} \widetilde{f}(t) \, dt, \tag{3.20}$$

while if $\alpha \geq \alpha_T$, we have

$$\int_{a}^{b} f(t) \diamond_\alpha t \geq \int_{a}^{b} \widetilde{f}(t) \, dt. \tag{3.21}$$

(iii) If $\mathbb{T} = [a, b]$ or if $f$ is constant, then $\alpha_T$ can be any real number from $[0, 1]$. Otherwise, $\alpha_T \in (0, 1)$

Now we will prove that if $f : \mathbb{T} \to \mathbb{R}$ is a linear function, (i.e., $f(t) = ut + v$) then $\int_a^b f(t)\Delta t$ and $\int_a^b f(t)\nabla t$ are symmetric with respect to $\int_a^b \tilde{f}(t)\,dt$, where $\tilde{f} : [a,b] \to \mathbb{R}$, $\tilde{f}(t) = ut + v$ is the corresponding linear function, defined on the interval $[a,b]$.

**Lemma 3.4.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a linear function and let $\tilde{f} : [a,b] \to \mathbb{R}$ be the corresponding linear function. If $\int_a^b f(t)\Delta t = \int_a^b \tilde{f}(t)\,dt - C$, with $C \in \mathbb{R}$, then $\int_a^b f(t)\nabla t = \int_a^b \tilde{f}(t)\,dt + C$.*

*Proof.* We will start by considering the case of $f : \mathbb{T} \to \mathbb{R}$, $f(t) = t$. If $\mathbb{T} = [a,b]$, then $C = 0$ and the conclusion is clear. If $\mathbb{T} = [a,b] \setminus (c,d)$, then

$$\int_a^b t\Delta t = \int_a^c t\Delta t + \int_c^d t\Delta t + \int_d^b t\Delta t$$

$$= \int_a^c t\,dt + \int_c^{\sigma(c)} t\Delta t + \int_d^b t\,dt$$

$$= \int_a^b t\,dt - \int_c^d t\,dt + c(d-c) \qquad (3.22)$$

$$= \int_a^b t\,dt - (d-c)\frac{d+c}{2} + c(d-c)$$

$$= \int_a^b t\,dt - \frac{(d-c)^2}{2},$$

while

$$\int_a^b t\nabla t = \int_a^c t\nabla t + \int_c^d t\nabla t + \int_d^b t\nabla t$$

$$= \int_a^c t\,dt + \int_{\rho(d)}^d t\nabla t + \int_d^b t\,dt$$

$$= \int_a^b t\,dt - \int_c^d t\,dt + d(d-c) \qquad (3.23)$$

$$= \int_a^b t\,dt - (d-c)\frac{d+c}{2} + d(d-c)$$

$$= \int_a^b t\,dt + \frac{(d-c)^2}{2}$$

and, obvious, if we choose $C = (d-c)^2/2$ the conclusion is clear.

By repeating the same arguments several times, we can "extract" any number of intervals from $[a,b]$ and get the same conclusion.

If we "extract" an interval, but we "add" an isolated point (i.e., $\mathbb{T} = [a,b] \setminus ((c,e) \cup (e,d)) = [a,c] \cup \{e\} \cup [d,b])$, then

$$
\begin{aligned}
\int_a^b t\,\Delta t &= \int_a^c t\,\Delta t + \int_c^e t\,\Delta t + \int_e^d t\,\Delta t + \int_d^b t\,\Delta t \\
&= \int_a^c t\,dt + \int_c^{\sigma(c)} t\,\Delta t + \int_e^{\sigma(e)} t\,\Delta t + \int_d^b t\,dt \\
&= \int_a^b t\,dt - \int_c^d t\,dt + c(e-c) + e(d-e) \\
&= \int_a^b t\,dt - (d-c)\frac{d+c}{2} + e(c+d) - c^2 - e^2 \\
&= \int_a^b t\,dt - \frac{d^2}{2} - \frac{c^2}{2} + e(c+d) - e^2,
\end{aligned}
\tag{3.24}
$$

while

$$
\begin{aligned}
\int_a^b t\,\nabla t &= \int_a^c t\,\nabla t + \int_c^e t\,\nabla t + \int_e^d t\,\nabla t + \int_d^b t\,\nabla t \\
&= \int_a^c t\,dt + \int_{\rho(e)}^e t\,\nabla t + \int_{\rho(d)}^d t\,\nabla t + \int_d^b t\,dt \\
&= \int_a^b t\,dt - \int_c^d t\,dt + e(e-c) + d(d-e) \\
&= \int_a^b t\,dt - (d-c)\frac{d+c}{2} - e(c+d) + d^2 + e^2 \\
&= \int_a^b t\,dt + \frac{d^2}{2} + \frac{c^2}{2} - e(c+d) + e^2
\end{aligned}
\tag{3.25}
$$

and thus, for $C = (e-c)^2/2 + (d-e)^2/2$, we get the conclusion.

For a general linear function, $f(t) = ut + v$, we have

$$
\begin{aligned}
\int_a^b f(t)\,\Delta t &= \int_a^b (ut+v)\,\Delta t = u\left(\int_a^b t\,dt - C\right) + v(b-a) = u\int_a^b t\,dt - uC + v(b-a), \\
\int_a^b f(t)\,\nabla t &= \int_a^b (ut+v)\,\nabla t = u\left(\int_a^b t\,dt + C\right) + v(b-a) = u\int_a^b t\,dt + uC + v(b-a),
\end{aligned}
\tag{3.26}
$$

so that $\int_a^b f(t)\,\Delta t = \int_a^b \tilde{f}(t)\,dt - uC$ and $\int_a^b f(t)\,\nabla t = \int_a^b \tilde{f}(t)\,dt + uC$.   □

*Definition 3.5.* Let $\mathbb{T}$ be a bounded time scale and $a, b \in \mathbb{T}$. One defines the *measure of graininess* between $a$ and $b$ to be the function $G : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_+$ by

$$
G(a,b) = \sum_{a \le t < b} \frac{\mu(t)^2}{2} = \sum_{a < t \le b} \frac{\nu(t)^2}{2}.
\tag{3.27}
$$

It is clear that the two sums are equal, noticing that

$$G(a,b) = \sum_{\substack{a \le t < b \\ t \text{ left-scattered}}} \frac{\mu(t)^2}{2} = \sum_{\substack{a < t \le b \\ t \text{ right-scattered}}} \frac{\nu(t)^2}{2}, \tag{3.28}$$

and using the fact that $\mu(t) = \nu(\sigma(t))$ for all $t$ right-scattered and that $[a,b]_{\mathbb{T}}$ is a bounded set. We have

$$G(a,b) = \sum_{a \le t < b} \frac{\mu(t)^2}{2} \le \frac{\left(\sum_{a \le t < b} \mu(t)\right)^2}{2} = \frac{(b-a)^2}{2}, \tag{3.29}$$

and so $G(a,b)$ is finite.

In other words, the function $G$ measures the square of distances between all scattered points between $a$ and $b$ and it depends on the "geometry" of the time scale $\mathbb{T}$.

*Remark 3.6.* The difference between $\int_a^b t \Delta t$ and $\int_a^b t\, dt$ depends on the measure of graininess function. In fact, we have

$$\int_a^b t \Delta t = \int_a^b t\, dt - G(a,b). \tag{3.30}$$

The proof uses the same methods as the proof of **Lemma 3.4**, so we will omit the details.

Notice that

$$\begin{aligned} \int_a^b t \nabla t &= \int_a^b t\, dt + G(a,b), \\ \int_a^b t \diamond_{1/2} t &= \frac{b^2 - a^2}{2}. \end{aligned} \tag{3.31}$$

*Remark 3.7.* For all time scales $\mathbb{T}$ and all $\alpha \in [0,1]$, we have

$$\frac{1}{b-a} \int_a^b t \diamond_\alpha t \in [a,b]. \tag{3.32}$$

Indeed, using **Lemma 3.1** for the nondecreasing function $f(t) = t$, we have

$$a \le \frac{1}{b-a} \int_a^b t \Delta t \le \frac{a+b}{2} \le \frac{1}{b-a} \int_a^b t \nabla t \le b, \tag{3.33}$$

and the conclusion is clear.

We denote by $x_\alpha = (1/(b-a)) \int_a^b t \diamond_\alpha t$ and call it the $\alpha$-center of the time-scaled interval $[a,b]_{\mathbb{T}}$.

Based on the previous remarks, we can compute $\int_a^b |t-s| \diamond_\alpha s$.

**Corollary 3.8.** *Let $\mathbb{T}$ be a time scale. Then,*

$$\int_a^b |t-s| \diamond_\alpha s = \frac{(t-a)^2 + (b-t)^2}{2} + (1-2\alpha)\big(G(t,b) - G(a,t)\big), \tag{3.34}$$

*where $G$ is the function introduced in Definition 3.5.*

*Proof.* Using Remark 3.6, we have

$$\int_a^b |t - s| \diamond_\alpha s = \int_a^t (t - s) \diamond_\alpha s + \int_t^b (s - t) \diamond_\alpha s$$

$$= t(t - a) - \int_a^t s \diamond_\alpha s - t(b - t) + \int_t^b s \diamond_\alpha s \qquad (3.35)$$

$$= \frac{(t - a)^2 + (b - t)^2}{2} + (1 - 2\alpha)(G(t, b) - G(a, t)).$$

□

Now, we are able to give the Hermite-Hadamard inequality for the time scales.

**Theorem 3.9** (Hermite-Hadamard inequality). *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f : [a, b] \to \mathbb{R}$ be a continuous convex function. Then,*

$$f(x_\alpha) \leq \frac{1}{b - a} \int_a^b f(t) \diamond_\alpha t \leq \frac{b - x_\alpha}{b - a} f(a) + \frac{x_\alpha - a}{b - a} f(b). \qquad (3.36)$$

*Proof.* For every convex function, we have

$$f(t) \leq f(a) + \frac{f(b) - f(a)}{b - a}(t - a). \qquad (3.37)$$

By taking the diamond-$\alpha$ integral side by side, we get

$$\int_a^b f(t) \diamond_\alpha t \leq f(a)(b - a) + \frac{f(b) - f(a)}{b - a}\left(\int_a^b t \diamond_\alpha t - a(b - a)\right), \qquad (3.38)$$

that is,

$$\frac{1}{b - a} \int_a^b f(t) \diamond_\alpha t \leq \frac{b - x_\alpha}{b - a} f(a) + \frac{x_\alpha - a}{b - a} f(b), \qquad (3.39)$$

and so we have proved the right-hand side.

For the left-hand side, we use Theorem 1.1, by taking $g : \mathbb{T} \to \mathbb{T}$, $g(s) = s$ for all $s \in \mathbb{T}$. We have

$$f\left(\frac{\int_a^b s \diamond_\alpha s}{b - a}\right) \leq \frac{\int_a^b f(s) \diamond_\alpha s}{b - a}, \qquad (3.40)$$

and, hence, we get

$$f(x_\alpha) \leq \frac{1}{b - a} \int_a^b f(t) \diamond_\alpha t. \qquad (3.41)$$

□

*Remark 3.10.* The right-hand side of Hermite-Hadamard inequality (3.36) remains true for all $0 \leq \alpha \leq \lambda$, including for the nabla integral, if $f(b) \leq f(a)$ and for all $\lambda \leq \alpha \leq 1$, including for the delta integral, if $f(b) \geq f(a)$, where $x_\lambda$ is the $\lambda$-center of the time-scaled interval $[a, b]_\mathbb{T}$.

Indeed, let us suppose that $f(b) \geq f(a)$. Then, by taking the diamond-$\alpha$ integral side by side to the inequality $f(t) \leq f(a) + ((f(b) - f(a))/(b-a))(t-a)$, we get

$$
\begin{aligned}
\int_a^b f(t) \diamond_\alpha t &\leq f(a)(b-a) + \frac{f(b) - f(a)}{b - a} \left( \int_a^b t \diamond_\alpha t - a(b-a) \right) \\
&\leq f(a)(b-a) + (f(b) - f(a))(x_\lambda - a) \\
&= (b - x_\lambda) f(a) + (x_\lambda - a) f(b).
\end{aligned}
\tag{3.42}
$$

According to Lemma 3.1, the last inequality is true for $\int_a^b t \diamond_\alpha t \leq \int_a^b t \diamond_\lambda t$, that is, for $\alpha \geq \lambda$. The same arguments work for $\lambda \geq \alpha$.

*Remark 3.11.* The left-hand side of Hermite-Hadamard inequality (3.36) remains true for all $0 \leq \alpha \leq \lambda$, including the nabla integral, if $f$ is nonincreasing and for all $\lambda \leq \alpha \leq 1$, including the delta integral, if $f$ is nondecreasing.

Indeed, let us suppose that $f$ is nonincreasing. Then, using Theorem 1.1, let $g : \mathbb{T} \to \mathbb{T}$, $g(s) = s$ for all $s \in \mathbb{T}$. We have

$$
f \left( \frac{\int_a^b s \diamond_\alpha s}{b - a} \right) \leq \frac{\int_a^b f(s) \diamond_\alpha s}{b - a}.
\tag{3.43}
$$

For $\alpha \geq \lambda$, we have $\int_a^b s \diamond_\alpha s \leq \int_a^b s \diamond_\lambda s$ and so

$$
f \left( \frac{\int_a^b s \diamond_\lambda s}{b - a} \right) \leq f \left( \frac{\int_a^b s \diamond_\alpha s}{b - a} \right) \leq \frac{\int_a^b f(s) \diamond_\alpha s}{b - a},
\tag{3.44}
$$

that is,

$$
f(x_\lambda) \leq \frac{1}{b - a} \int_a^b f(t) \diamond_\alpha t.
\tag{3.45}
$$

The same arguments are used to prove the case of $f$ nondecreasing function.

Using the last remarks, we can give a more general Hermite-Hadamard inequality for time scales.

**Theorem 3.12** (a general version of Hermite-Hadamard inequality). *Let $\mathbb{T}$ be a time scale, $\alpha, \lambda \in [0,1]$ and $a, b \in \mathbb{T}$. Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function. Then,*

(i) *if $f$ is nondecreasing on $[a,b]_\mathbb{T}$, then, for all $\alpha \in [0, \lambda]$ one has*

$$
f(x_\lambda) \leq \frac{1}{b - a} \int_a^b f(t) \diamond_\alpha t,
\tag{3.46}
$$

*and for all $\alpha \in [\lambda, 1]$, one has*

$$
\frac{1}{b - a} \int_a^b f(t) \diamond_\alpha t \leq \frac{b - x_\lambda}{b - a} f(a) + \frac{x_\lambda - a}{b - a} f(b).
\tag{3.47}
$$

(ii) *If $f$ is nonincreasing on $[a,b]_\mathbb{T}$, then, for all $\alpha \in [0, \lambda]$ one has the above inequality (3.47) and for all $\alpha \in [0, \lambda]$ one has the above inequality (3.46).*

*Remark 3.13.* In the above inequalities (3.46) and (3.47), we have equalities if $f$ is a constant function and $\alpha, \lambda \in [0, 1]$ or if $f$ is a linear function and $\alpha = \lambda$.

**Theorem 3.14** (a weighted version of Hermite-Hadamard inequality). *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f : [a, b] \to \mathbb{R}$ be a continuous convex function and let $w : \mathbb{T} \to \mathbb{R}$ be a continuous function such that $w(t) \geq 0$ for all $t \in \mathbb{T}$ and $\int_a^b w(t) \diamond_\alpha t > 0$. Then,*

$$f(x_{w,\alpha}) \leq \frac{1}{\int_a^b w(t) \diamond_\alpha t} \int_a^b f(t) w(t) \diamond_\alpha t \leq \frac{b - x_{w,\alpha}}{b - a} f(a) + \frac{x_{w,\alpha} - a}{b - a} f(b), \tag{3.48}$$

*where $x_{w,\alpha} = \int_a^b t w(t) \diamond_\alpha t / \int_a^b w(t) \diamond_\alpha t$.*

*Proof.* For every convex function, we have

$$f(t) \leq f(a) + \frac{f(b) - f(a)}{b - a}(t - a). \tag{3.49}$$

Multiplying this inequality with $w(t)$ which is nonnegative, we get

$$f(t) w(t) \leq f(a) w(t) + \frac{f(b) - f(a)}{b - a}(t - a) w(t). \tag{3.50}$$

By taking the diamond-$\alpha$ integral side by side, we get

$$\int_a^b f(t) w(t) \diamond_\alpha t \leq f(a) \int_a^b w(t) \diamond_\alpha t + \frac{f(b) - f(a)}{b - a} \left( \int_a^b t w(t) \diamond_\alpha t - a \int_a^b w(t) \diamond_\alpha t \right), \tag{3.51}$$

that is,

$$\frac{1}{\int_a^b w(t) \diamond_\alpha t} \int_a^b f(t) w(t) \diamond_\alpha t \leq \frac{b - x_{w,\alpha}}{b - a} f(a) + \frac{x_{w,\alpha} - a}{b - a} f(b), \tag{3.52}$$

and so we have proved the right-hand side.

For the left-hand side, we use Theorem 1.2, by taking $g : \mathbb{T} \to \mathbb{T}$, $g(s) = s$ for all $s \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$, $h(t) = w(t)$. We have

$$f\left( \frac{\int_a^b s w(s) \diamond_\alpha s}{\int_a^b w(s) \diamond_\alpha s} \right) \leq \frac{\int_a^b w(s) f(s) \diamond_\alpha s}{\int_a^b w(s) \diamond_\alpha s}, \tag{3.53}$$

and, hence, we get

$$f(x_{w,\alpha}) \leq \frac{1}{b - a} \int_a^b w(t) f(t) \diamond_\alpha t. \tag{3.54}$$

$\square$

*Remark 3.15.* If we consider concave functions instead of convex functions, the above Hermite-Hadamard inequalities (3.36), (3.46), (3.47), and (3.48) are reversed.

## 4. The Hermite-Hadamard inequality for $(w, \alpha)$-symmetric functions

In [7], Florea and Niculescu proved the following theorem.

**Theorem 4.1** (see [7, Theorem 3]). *Suppose that $f : I \to \mathbb{R}$ verifies a symmetry condition (i.e., $f(x) + f(2m - x) = 2f(m)$ for all $x \in I \cap (-\infty, m]$) and is convex over the interval $I \cap (-\infty, m]$ and concave over the interval $I \cap [m, \infty)$.*

*If $(a + b)/2 \geq m$ and $\mu$ is a Hermite-Hadamard measure on each of the intervals $[a, 2m - a]$ and $[2m - a, b]$, and is invariant with respect to the map $T(x) = 2m - x$ on $[a, 2m - a]$, then*

$$f(x_\mu) \geq \frac{1}{\mu([a,b])} \int_a^b f(x) d\mu \geq \frac{b - x_\mu}{b - a} f(a) + \frac{x_\mu - a}{b - a} f(b). \qquad \text{(GHH)}$$

*If $(a + b)/2 \leq m$, then the inequalities* (GHH) *work in a reverse way, provided $\mu$ is a Hermite-Hadamard measure on each of the intervals $[a, 2m - b]$ and $[2m - b, b]$, and is invariant with respect to the map $T(x) = 2m - x$ on $[2m - b, b]$.*

We will give an extension of this theorem, for time scales, using functions not necessarily symmetric in the usual sense. For that, we need the following definition.

*Definition 4.2.* Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, $w : \mathbb{T} \to \mathbb{R}_+$ be a positive weight and $\alpha \in [0, 1]$. One says that a function $f : [a, b] \to \mathbb{R}$ is $(w, \alpha)$-symmetric on $[a, b]_\mathbb{T}$ if the following conditions are satisfied:

(i)

$$\frac{b - x_{w,\alpha}}{b - a} f(a) + \frac{x_{w,\alpha} - a}{b - a} f(b) = f(x_{w,\alpha}), \qquad (4.1)$$

(ii)

$$\int_a^b f(t) w(t) \diamond_\alpha t = f(x_{w,\alpha}) \int_a^b w(t) \diamond_\alpha t. \qquad (4.2)$$

Here, $x_{w,\alpha} = (\int_a^b t w(t) \diamond_\alpha t) / (\int_a^b w(t) \diamond_\alpha t)$.

Notice that the function $f$ should be continuous only on $[a, b]_\mathbb{T}$ not on $[a, b]$. An example of such a function is the following.

*Example 4.3.* Let $\mathbb{T} = \{1\} \cup [3, 4]$, $w : \{1\} \cup [3, 4] \to \mathbb{R}_+$, $w(1) = 1$, $w(t) = 2$ for all $t \in [3, 4]$ and $\alpha = 1/2$. Then, $f : [1, 4] \to \mathbb{R}$,

$$f(t) = \begin{cases} 0, & \text{if } t \in \left[1, \dfrac{14}{5}\right), \\ 1, & \text{if } t \in \left[\dfrac{14}{5}, 3\right), \\ \dfrac{5}{3}, & \text{if } t \in [3, 4], \end{cases} \qquad (4.3)$$

is a $(w, 1/2)$-symmetric function on $[1, 4]_\mathbb{T}$.

We can provide also a continuous function on $[1, 4]$, such as

$$f(t) = \begin{cases} \dfrac{5}{9}t - \dfrac{5}{9}, & \text{if } t \in \left[1, \dfrac{14}{5}\right), \\[2ex] \dfrac{10}{3}t - \dfrac{25}{3}, & \text{if } t \in \left[\dfrac{14}{5}, 3\right), \\[2ex] \dfrac{5}{3}, & \text{if } t \in [3, 4], \end{cases} \tag{4.4}$$

which is $(w, 1/2)$-symmetric on $[1, 4]_{\mathbb{T}}$.

Indeed, since $\int_1^4 w(t) \diamond_{1/2} t = 5$ and $\int_1^4 tw(t) \diamond_{1/2} t = 14$, we have $x_{w,1/2} = 14/5$.

Condition (i) can be restated as

$$\frac{2}{5}f(1) + \frac{3}{5}f(4) = f\left(\frac{14}{5}\right), \tag{4.5}$$

while condition (ii) can be restated as

$$\int_1^4 f(t)w(t) \diamond_{1/2} t = 5f\left(\frac{14}{5}\right), \tag{4.6}$$

and it is easy to check that both are fulfilled.

Now, we can state our theorem, that is a generalization of Theorem 4.1.

**Theorem 4.4.** *Let $\mathbb{T}$ be a time scale, $a \leq c \leq b \in \mathbb{T}$, $w : \mathbb{T} \to \mathbb{R}_+$ be a positive weight and $\alpha \in [0, 1]$. Let $p = \int_a^c tw(t) \diamond_\alpha t / \int_a^c w(t) \diamond_\alpha t$ and $q = \int_c^b tw(t) \diamond_\alpha t / \int_c^b w(t) \diamond_\alpha t$.*

(i) *If the function $f : [a, b] \to \mathbb{R}$ is $(w, \alpha)$-symmetric on $[a, c]_{\mathbb{T}}$ and convex on $[p, b]$ then*

$$f(x_{w,\alpha}) \leq \frac{1}{\int_a^b w(t) \diamond_\alpha t} \int_a^b f(t)w(t) \diamond_\alpha t \leq \frac{b - x_{w,\alpha}}{b - a}f(a) + \frac{x_{w,\alpha} - a}{b - a}f(b). \tag{4.7}$$

*If $f$ is concave on $[p, b]$ then the inequalities in (4.7) are reversed.*

(ii) *If the function $f : [a, b] \to \mathbb{R}$ is $(w, \alpha)$-symmetric on $[c, b]_{\mathbb{T}}$ and concave on $[a, q]$ then one has (4.7).*

*If $f$ is convex on $[a, q]$, then the inequalities in (4.7) are reversed.*

*Proof.* Suppose first that $f$ is $(w, \alpha)$-symmetric on $[a, c]_{\mathbb{T}}$ and convex on $[p, b]$. We will prove the left-hand side inequality in (4.7). For that, we notice that

$$\int_a^b f(t)w(t) \diamond_\alpha t = \int_a^c f(t)w(t) \diamond_\alpha t + \int_c^b f(t)w(t) \diamond_\alpha t$$
$$= f(p)\int_a^c w(t) \diamond_\alpha t + \int_c^b f(t)w(t) \diamond_\alpha t, \tag{4.8}$$

using the $(w, \alpha)$-symmetry property of the function $f$. Since $f$ is convex on $[p, b]$ and $c \geq p$, then, using Theorem 3.14 the last integral is more or equal to $f(q) \int_c^b w(t) \diamond_\alpha t$ and so

$$
\frac{1}{\int_a^b w(t) \diamond_\alpha t} \int_a^b f(t) w(t) \diamond_\alpha t \geq \frac{\int_a^c w(t) \diamond_\alpha t}{\int_a^b w(t) \diamond_\alpha t} f(p) + \frac{\int_c^b w(t) \diamond_\alpha t}{\int_a^b w(t) \diamond_\alpha t} f(q)
$$

$$
\geq f\left( \frac{\int_a^c w(t) \diamond_\alpha t}{\int_a^b w(t) \diamond_\alpha t} p + \frac{\int_c^b w(t) \diamond_\alpha t}{\int_a^b w(t) \diamond_\alpha t} q \right) \tag{4.9}
$$

$$
= f\left( \frac{\int_a^b t w(t) \diamond_\alpha t}{\int_a^b w(t) \diamond_\alpha t} \right)
$$

$$
= f(x_{w,\alpha}),
$$

using the definitions of $p$ and $q$, combined with the convexity of $f$ on $[p, b]$.

Now, we prove the right-hand side inequality in (4.7). Since $f$ is $(w, \alpha)$-symmetric on $[a, c]_{\mathbb{T}}$ and convex on $[p, b]$, using Theorem 3.14 we have

$$
\int_a^b f(t) w(t) \diamond_\alpha t = \int_a^c f(t) w(t) \diamond_\alpha t + \int_c^b f(t) w(t) \diamond_\alpha t
$$

$$
\leq f(p) \int_a^c w(t) \diamond_\alpha t + \left( \frac{b - q}{b - c} f(c) + \frac{q - c}{b - c} f(b) \right) \int_c^b w(t) \diamond_\alpha t. \tag{4.10}
$$

Using again the definition of $p$ and $q$, we have

$$
q = \frac{\int_c^b t w(t) \diamond_\alpha t}{\int_c^b w(t) \diamond_\alpha t}
$$

$$
= \frac{1}{\int_c^b w(t) \diamond_\alpha t} \left( \int_a^b t w(t) \diamond_\alpha t - \int_a^c t w(t) \diamond_\alpha t \right) \tag{4.11}
$$

$$
= \frac{x_{w,\alpha} \int_a^b w(t) \diamond_\alpha t - p \int_a^c w(t) \diamond_\alpha t}{\int_c^b w(t) \diamond_\alpha t}.
$$

To complete the proof, it suffices to show that

$$
f(p) \int_a^c w(t) \diamond_\alpha t + \left( \frac{b - q}{b - c} f(c) + \frac{q - c}{b - c} f(b) \right) \int_c^b w(t) \diamond_\alpha t
$$

$$
\leq \left( \frac{b - x_{w,\alpha}}{b - a} f(a) + \frac{x_{w,\alpha} - a}{b - a} f(b) \right) \int_a^b w(t) \diamond_\alpha t. \tag{4.12}
$$

We put $\lambda = \int_a^c w(t)\diamond_\alpha t / \int_a^b w(t)\diamond_\alpha t$. Then, $\int_c^b w(t)\diamond_\alpha t / \int_a^b w(t)\diamond_\alpha t = 1 - \lambda$ and the previous inequality becomes

$$
\lambda f(p) + (1-\lambda)\left(\frac{b - (x_{w,\alpha} - \lambda p)/(1-\lambda)}{b-c}f(c) + \frac{(x_{w,\alpha} - \lambda p)/(1-\lambda) - c}{b-c}f(b)\right)
$$
$$
\leq \frac{b - x_{w,\alpha}}{b-a}f(a) + \frac{x_{w,\alpha} - a}{b-a}f(b),
$$
(4.13)

and can be restated as

$$
\lambda f(p) + \frac{(1-\lambda)b - x_{w,\alpha} - \lambda p}{b-c}f(c) + \frac{x_{w,\alpha} - \lambda p - (1-\lambda)c}{b-c}f(b) \leq \frac{b - x_{w,\alpha}}{b-a}f(a) + \frac{x_{w,\alpha} - a}{b-a}f(b).
$$
(4.14)

Since $f$ is $(w,\alpha)$-symmetric on $[a,c]_{\mathbb{T}}$, we have $((c-p)/(c-a))f(a) + ((p-a)/(c-a))f(c) = f(p)$, that means $f(a) = ((c-a)/(c-p))f(p) - ((p-a)/(c-p))f(c)$. And so, the last inequality becomes

$$
\lambda f(p) + \frac{(1-\lambda)b - x_{w,\alpha} - \lambda p}{b-c}f(c) + \frac{x_{w,\alpha} - \lambda p - (1-\lambda)c}{b-c}f(b)
$$
$$
\leq \frac{b - x_{w,\alpha}}{b-a}\left(\frac{c-a}{c-p}f(p) - \frac{p-a}{c-p}f(c)\right) + \frac{x_{w,\alpha} - a}{b-a}f(b).
$$
(4.15)

After making some calculation, including a simplification, we get

$$
f(c) \leq \frac{b-c}{b-p}f(p) + \frac{c-p}{b-p}f(b),
$$
(4.16)

which is true since $f$ is convex on $[p,b]_{\mathbb{T}}$, and $c$ is a convex combination of $p$ and $b$:

$$
c = \frac{b-c}{b-p}p + \frac{c-p}{b-p}b.
$$
(4.17)

The other cases are treated similarly.                                                                    □

*Remark 4.5.* If $c = a$ or $c = b$, we get Theorem 3.14 as a particular case of Theorem 4.4.

## 5. Some extensions of the diamond-$1/2$ integral

Using Remark 3.6, we get the following corollary, which is a "middle point" variant of Theorem 3.9.

**Corollary 5.1** (middle point Hermite-Hadamard inequality). *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function. Then,*

$$
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\diamond_{1/2}t \leq \frac{f(a) + f(b)}{2}.
$$
(5.1)

*Remark 5.2.* (i) If $\mathbb{T} = \{a, (a+b)/2, b\}$ and $\alpha_T = 1/2$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[\frac{f(a)}{2} + \frac{f((a+b)/2)}{2}\right] + \frac{1}{2}\left[\frac{f((a+b)/2)}{2} + \frac{f(b)}{2}\right] \le \frac{f(a)+f(b)}{2}, \qquad (5.2)$$

that is,

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)}{4} + \frac{f((a+b)/2)}{2} + \frac{f(b)}{4} \le \frac{f(a)+f(b)}{2}. \qquad (5.3)$$

(ii) If $\mathbb{T} = \{a, (a+b)/4, (a+b)/2, 3((a+b)/4), b\}$, and $\alpha_T = 1/2$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)}{8} + \frac{f((a+b)/4)}{4} + \frac{f((a+b)/2)}{2} + \frac{f(3((a+b)/4))}{4} + \frac{f(b)}{8} \le \frac{f(a)+f(b)}{2}. \qquad (5.4)$$

(iii) In general, if $\mathbb{T}$ has $2^n + 1$ points at equal distance, then

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)}{2^{n+1}} + \frac{f((a+b)/2^n)}{2^n} + \frac{f((a+b)/2^{n-1})}{2^{n-1}} + \cdots$$

$$+ \frac{f((2^{n-1}-1)((a+b)/2^n))}{2^2} + \frac{f((a+b)/2)}{2} + \frac{f((2^{n-1}+1)((a+b)/2^n))}{2^2} + \cdots$$

$$+ \frac{f((2^n-1)((a+b)/2^n))}{2^n} + \frac{f(b)}{2^{n+1}} \le \frac{f(a)+f(b)}{2}. \qquad (5.5)$$

*Remark 5.3* (an improvement on Hermite-Hadamard inequality). Suppose $\mathbb{T}$ is a symmetric time scale such that if we divide it in $2^n$ all of them are symmetric. An example of such a time scale is the set $\mathbb{T}$ with $2^n + 1$ points at equal distance. Then, by applying Hermite-Hadamard inequality to the time scales $\mathbb{T} \cap [a, (a+b)/2]$ and $\mathbb{T} \cap [(a+b)/2, b]$, we get

$$f\left(\frac{3a+b}{4}\right) \le \frac{2}{b-a}\int_a^{(a+b)/2} f(t)\diamond_{1/2}t \le \frac{1}{2}\left(f(a) + f\left(\frac{a+b}{2}\right)\right),$$

$$f\left(\frac{a+3b}{4}\right) \le \frac{2}{b-a}\int_{(a+b)/2}^b f(t)\diamond_{1/2}t \le \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) + f(b)\right). \qquad (5.6)$$

By summing them, side by side, we obtain the following refinement of the inequality (3.36):

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right)$$

$$\le \frac{1}{b-a}\int_a^b f(t)\diamond_{1/2}t$$

$$\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]$$

$$\le \frac{1}{2}(f(a) + f(b)). \qquad (5.7)$$

By continuing this process, we can obtain approximations of $\int_a^b f(t) \diamond_{1/2} t$ as good as we want, by the value of the function in some of the dyadic points of $\mathbb{T}$.

### 5.1. The Hermite-Hadamard inequality for convex-concave symmetric functions

In [8], Czinder and Páles proved an interesting and useful extension of Hermite-Hadamard inequality for convex-concave symmetric functions.

**Theorem 5.4** (see [8, Theorem 2.2 ]). *Let* $f : I \to \mathbb{R}$ *be symmetric with respect to an element* $m \in I$, *that is,*

$$f(x) + f(2m - x) = 2f(m), \quad \forall x \in I \cap (-\infty, m]. \tag{S}$$

*Furthermore, suppose that* $f$ *is convex over the interval* $I \cap (-\infty, m]$ *and concave over* $I \cap [m, -\infty]$. *Then, for any interval* $[a, b] \subset I$ *with* $(a + b)/2 \geq m$, *the following inequalities hold true:*

$$f\left(\frac{a + b}{2}\right) \geq \frac{1}{b - a} \int_a^b f(x) \, dx \geq \frac{f(a) + f(b)}{2}. \tag{CP}$$

*If* $(a + b)/2 \leq m$, *then the inequalities* (CP) *should be reversed.*

We will try to give a similar version of the previous theorem. For that, we need some definitions.

*Definition 5.5.* A set $M \subseteq \mathbb{T}$ is called *symmetric with respect to an element* $m \in M$ provided that

$$m - t \in M \text{ implies } m + t \in M, \tag{5.8}$$

for all $t \in \mathbb{R}$ such that $m - t \in M$.

*Definition 5.6.* Let $\mathbb{T}$ be a time scale and let $I \subset \mathbb{R}$ be an interval such that $I_\mathbb{T} = I \cap \mathbb{T}$ is symmetric with respect to $m \in \mathbb{T}$. A function $f : I_\mathbb{T} \to \mathbb{R}$ is called *symmetric with respect to* $m$ if the equality

$$f(m - t) + f(m + t) = 2f(m) \tag{5.9}$$

is true for all $t \in \mathbb{R}$ such that $m - t \in I_\mathbb{T}$.

We will need also two technical lemmas. The first one concerns the functions defined on intervals (and its proof is similar to [8, Theorem 2.1]), while the second one concerns the functions defined on a time scale $\mathbb{T}$.

**Lemma 5.7.** *Let* $f : I \to \mathbb{R}$ *be a function which is symmetric with respect to* $m \in I$. *Then,*

$$\int_{m-a}^{m-b} f(t) \, dt + \int_{m+b}^{m+a} f(t) \, dt = 2(a - b) f(m), \tag{5.10}$$

*for any positive* $a, b \in (I - m) \cap (I + m)$, *with* $a > b$.

**Lemma 5.8.** *Let $f : I \to \mathbb{R}$ and $I_{\mathbb{T}}$ be symmetric with respect to $m \in I$. Then,*

$$\int_{m-a}^{m+a} f(t) \diamond_{1/2} t = 2af(m), \tag{5.11}$$

*for any positive $a \in \mathbb{R}$ such that $m - a \in I_{\mathbb{T}}$.*

*Proof.* First, we split the integral with respect to scattered points

$$\int_{m-a}^{m+a} f(t) \diamond_{1/2} t = \sum_{i=0}^{n} \int_{m-a_i}^{m-a_{i+1}} f(t) \diamond_{1/2} t + \sum_{i=0}^{n} \int_{m+a_{i+1}}^{m+a_i} f(t) \diamond_{1/2} t, \tag{5.12}$$

where $a_i \in \mathbb{R}$ are descending numbers such that $m - a_i$, $m + a_i$ are all scattered points, for any $i \in \{0, \dots, n\}$ such that $a_0 = a$ and $a_n = 0$.

If $m - a_i$, $m - a_{i+1}$ are not isolated (that means, $m - a_i$ is right dense, while $m - a_{i+1}$ is left dense) then $[a_i, a_{i+1}]_{\mathbb{T}}$ is an interval and thus, according to Lemma 5.7, we have

$$\int_{m-a_i}^{m-a_{i+1}} f(t) \diamond_{1/2} t + \int_{m+a_{i+1}}^{m+a_i} f(t) \diamond_{1/2} t = \int_{m-a_i}^{m-a_{i+1}} f(t) \, dt + \int_{m+a_{i+1}}^{m+a_i} f(t) \, dt$$

$$= 2(a_i - a_{i+1}) f(m). \tag{5.13}$$

If $m - a_i$, $m - a_{i+1}$ are isolated then, we have

$$\int_{m-a_i}^{m-a_{i+1}} f(t) \Delta t = (a_i - a_{i+1}) f(m - a_i), \tag{5.14}$$

while

$$\int_{m-a_i}^{m-a_{i+1}} f(t) \nabla t = (a_i - a_{i+1}) f(m - a_{i+1}). \tag{5.15}$$

Furthermore,

$$\int_{m+a_{i+1}}^{m+a_i} f(t) \Delta t = (a_i - a_{i+1}) f(m + a_{i+1}), \tag{5.16}$$

while

$$\int_{m+a_{i+1}}^{m+a_i} f(t) \nabla t = (a_i - a_{i+1}) f(m + a_i), \tag{5.17}$$

and so,

$$\int_{m-a_i}^{m-a_{i+1}} f(t) \diamond_{1/2} t + \int_{m+a_{i+1}}^{m+a_i} f(t) \diamond_{1/2} t$$

$$= \frac{1}{2} (a_i - a_{i+1}) [f(m - a_i) + f(m - a_{i+1}) + f(m + a_i) + f(m + a_{i+1})] \tag{5.18}$$

$$= 2(a_i - a_{i+1}) f(m).$$

Since these are the only possibilities, the proof is complete. □

Now, we can give a theorem similar to [8, Theorem 2.2].

**Theorem 5.9.** *Let* $f : I \to \mathbb{R}$ *and* $I_\mathbb{T}$ *be symmetric with respect to* $m \in I$ *and suppose that* $f$ *is concave over the interval* $I \cap (-\infty, m]$ *and convex over* $I \cap [m, -\infty]$*. Then, for any* $a, b \in I_\mathbb{T}$ *with* $(a + b)/2 \geq m$*, and* $(a + b)/2 \in \mathbb{T}$*, the following inequalities hold true:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\diamond_{1/2} \leq \frac{f(a) + f(b)}{2}. \tag{Hs}$$

*If* $(a + b)/2 \leq m$*, then the inequalities* (Hs) *should be reversed.*

  *If* $f$ *is convex over the interval* $I \cap (-\infty, m]$ *and concave over* $I \cap [m, -\infty]$*. Then, for any* $a, b \in I_\mathbb{T}$ *with* $(a+b)/2 \leq m$*, and* $(a+b)/2 \in \mathbb{T}$ *the inequalities* (Hs) *hold true, while if* $(a+b)/2 \geq m$*, the inequalities* (Hs) *are reversed.*

  Using the previous lemmas, we could give a proof in the same manner as in [8]. We will use, instead, Theorem 4.4.

*Proof.* Let $a, b \in I_\mathbb{T}$ with $(a + b)/2 \geq m$, and suppose that $f$ is concave over the interval $I \cap (-\infty, m]$ and convex over $I \cap [m, -\infty]$. Further, we can assume that $a < m < b$ (the other cases are covered by Theorem 3.9). Due to the fact that $a < m < b$ and $(a + b)/2 \geq m$, we have $m < 2m - a < b$.

  According to Lemma 5.8, we have

$$\int_a^{2m-a} f(t)\diamond_{1/2} = 2(m - a)f(m), \tag{5.19}$$

while

$$x_{1/2} = \int_a^{2m-a} t\diamond_{1/2} = m, \tag{5.20}$$

and so $f$ is $(\mathbf{1}, 1/2)$-symmetric (that means, with respect to the weight $w \equiv 1$ and $\alpha = 1/2$). Now, it is obvious that we can apply Theorem 4.4, considering $p = m$, $c = 2m - a$, and $w \equiv 1$.

  If $(a + b)/2 \leq m$, then we will consider $q = m$, $c = 2b - m$, and $w \equiv 1$, and the proof is clear. The other cases can be treated in a similar way.  □

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