Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 195137, 8 pages
doi:10.1155/2008/195137

Research Article

Euler-Lagrange Type Cubic Operators and Their Norms on X_λ Space

Abbas Najati[1] and Asghar Rahimi[2]

[1] *Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabil, P.O. Box 56199-11367 Ardabil, Iran*
[2] *Department of Mathematics, University of Maragheh, P.O. Box 55181-83111, Maragheh, East Azarbayjan, Iran*

Correspondence should be addressed to Abbas Najati, a.najati@uma.ac.ir

Received 17 April 2008; Accepted 1 July 2008

Recommended by Jong Kim

We will introduce linear operators and obtain their exact norms defined on the function spaces X_λ and Z_λ^5. These operators are constructed from the Euler-Lagrange type cubic functional equations and their Pexider versions.

Copyright © 2008 A. Najati and A. Rahimi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let X and Y be complex normed spaces. For a fixed nonnegative real number λ, we denote by X_λ the linear space of all functions $f : X \to Y$ (with pointwise operations) for which there exists a constant $M_f \geq 0$ with

$$\|f(x)\| \leq M_f e^{\lambda\|x\|} \tag{1.1}$$

for all $x \in X$. It is easy to show that the space X_λ with the norm

$$\|f\| := \sup_{x\in X}\{e^{-\lambda\|x\|}\|f(x)\|\} \tag{1.2}$$

is a normed space. Let us denote by X_λ^n the linear space of all functions $\phi : \underbrace{X \times \cdots \times X}_{n \text{ times}} \to Y$ (with pointwise operations) for which there exists a constant $M_\phi \geq 0$ with

$$\|\phi(x_1, \ldots, x_n)\| \leq M_\phi e^{\lambda\sum_{i=1}^{n}\|x_i\|} \tag{1.3}$$

for all $x_1, \ldots, x_n \in X$. It is not difficult to show that the space X_λ^n with the norm

$$\|\phi\| := \sup_{x_1,\ldots,x_n \in X} \left\{ \|\phi(x_1,\ldots,x_n)\| e^{-\lambda \sum_{i=1}^n \|x_i\|} \right\} \tag{1.4}$$

is a normed space.

We denote by Z_λ^m the normed space $\bigoplus_{i=1}^m X_\lambda = \{(f_1,\ldots,f_m) : f_1,\ldots,f_m \in X_\lambda\}$ (with pointwise operations) together with the norm

$$\|(f_1,\ldots,f_m)\| := \max\{\|f_1\|,\ldots,\|f_m\|\}. \tag{1.5}$$

The norms of the Pexiderized Cauchy, quadratic, and Jensen operators on the function space X_λ have been investigated by Czerwik and Dlutek [1, 2]. In [3], Moslehian et al. have extended the results of [2] to the Pexiderized generalized Jensen and Pexiderized generalized quadratic operators on the function space X_λ and provided more general results regarding their norms.

In [4], Jung investigated the norm of the cubic operator on the function space Z_λ^5.

A function $f : X \to Y$ is called a cubic function if and only if f is a solution function of the cubic functional equation

$$f(x+y) + f(x-y) = 2f\left(\frac{1}{2}x + y\right) + 2f\left(\frac{1}{2}x - y\right) + 12f\left(\frac{1}{2}x\right). \tag{1.6}$$

Jun and Kim [5] proved that when both X and Y are real vector spaces, a function $f : X \to Y$ satisfies (1.6) if and only if there exists a function $B : X \times X \times X \to Y$ such that $f(x) = B(x,x,x)$ for all $x \in X$, and B is symmetric for each fixed one variable and is additive for fixed two variables.

In [6], the authors introduced the following Euler-Lagrange-type cubic functional equation, which is equivalent to (1.6),

$$f(x+y) + f(x-y) = af\left(\frac{1}{a}x + y\right) + af\left(\frac{1}{a}x - y\right) + 2a(a^2-1)f\left(\frac{1}{a}x\right) \tag{1.7}$$

for fixed integers a with $a \neq 0, \pm 1$. Moreover, Jun and Kim [7] introduced the following Euler-Lagrange-type cubic functional equation

$$f\left(\frac{1}{a}x+\frac{1}{b}y\right)+f\left(\frac{1}{b}x+\frac{1}{a}y\right)=(a+b)(a-b)^2\left[f\left(\frac{1}{ab}x\right)+f\left(\frac{1}{ab}y\right)\right]+ab(a+b)f\left(\frac{1}{ab}x+\frac{1}{ab}y\right) \tag{1.8}$$

for fixed integers a, b with $a, b \neq 0$, $a \pm b \neq 0$, and they proved the following theorem.

Theorem 1.1 (see [7, Theorem 2.1]). *Let X and Y be real vector spaces. If a mapping $f : X \to Y$ satisfies the functional equation* (1.6)*, then f satisfies the functional equation* (1.8).

We will introduce linear operators which are constructed from the Euler-Lagrange-type cubic and the Pexiderization of the Euler-Lagrange-type cubic functional equations (1.7) and (1.8).

Definition 1.2. The operators $C_1^P, C_2^P : Z_\lambda^5 \to X_\lambda^2$ are defined by

$$\begin{aligned} C_1^P(f_1,\ldots,f_5)(x,y) &:= f_1(x+y)+f_2(x-y)-mf_3\left(\frac{1}{m}x+y\right)\\ &\quad -mf_4\left(\frac{1}{m}x-y\right)-2m(m^2-1)f_5\left(\frac{1}{m}x\right),\\ C_2^P(f_1,\ldots,f_5)(x,y) &:= f_1\left(\frac{1}{a}x+\frac{1}{b}y\right)+f_2\left(\frac{1}{b}x+\frac{1}{a}y\right)-(a+b)(a-b)^2\left[f_3\left(\frac{1}{ab}x\right)+f_4\left(\frac{1}{ab}y\right)\right]\\ &\quad -ab(a+b)f_5\left(\frac{1}{ab}x+\frac{1}{ab}y\right), \end{aligned} \tag{1.9}$$

where a, b, and m are fixed integers with $a,b\neq 0$, $a\pm b\neq 0$, and $m\neq 0,\pm 1$.

Definition 1.3. The operators $C_1, C_2 : X_\lambda \to X_\lambda^2$ are defined by

$$\begin{aligned} C_1(f)(x,y) &:= f(x+y)+f(x-y)-mf\left(\frac{1}{m}x+y\right)\\ &\quad -mf\left(\frac{1}{m}x-y\right)-2m(m^2-1)f\left(\frac{1}{m}x\right),\\ C_2(f)(x,y) &:= f\left(\frac{1}{a}x+\frac{1}{b}y\right)+f\left(\frac{1}{b}x+\frac{1}{a}y\right)\\ &\quad -(a+b)(a-b)^2\left[f\left(\frac{1}{ab}x\right)+f\left(\frac{1}{ab}y\right)\right]-ab(a+b)f\left(\frac{1}{ab}x+\frac{1}{ab}y\right), \end{aligned} \tag{1.10}$$

where a, b, and m are fixed integers with $a,b\neq 0$, $a\pm b\neq 0$, and $m\neq 0,\pm 1$.

In this paper, we will give the exact norms of the operators C_1^P, C_2^P on the function space Z_λ^5, and norms of the operators C_1, C_2 on the function space X_λ. The results extend the results of [4].

2. Main results

Throughout this section, a, b, and m are fixed integers with $a,b\neq 0$, $a\pm b\neq 0$, and $m\neq 0,\pm 1$.

The next theorems give us the exact norms of operators C_1^P, C_2^P, C_1, and C_2.

Theorem 2.1. *The operator $C_1^P : Z_\lambda^5 \to X_\lambda^2$ is a bounded linear operator with*

$$\|C_1^P\| = 2|m|^3+2. \tag{2.1}$$

Proof. First, we show that $\|C_1^P\| \leq 2|m|^3+2$. Since

$$\max\left\{\|x+y\|, \|x-y\|, \left\|\frac{1}{m}x+y\right\|, \left\|\frac{1}{m}x-y\right\|, \left\|\frac{1}{m}x\right\|\right\} \leq \|x\|+\|y\| \tag{2.2}$$

for all $x, y \in X$, we get

$$\begin{aligned}
\|C_1^P(f_1,\dots,f_5)\| &= \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)}\left\| f_1(x+y)+f_2(x-y)-mf_3\left(\frac{1}{m}x+y\right)\right.\\
&\qquad\left. -mf_4\left(\frac{1}{m}x-y\right)-2m(m^2-1)f_5\left(\frac{1}{m}x\right)\right\|\\
&\le \sup_{x,y\in X} e^{-\lambda\|x+y\|}\|f_1(x+y)\| + \sup_{x,y\in X} e^{-\lambda\|x-y\|}\|f_2(x-y)\|\\
&\quad + |m|\sup_{x,y\in X} e^{-\lambda\|(1/m)x+y\|}\left\|f_3\left(\frac{1}{m}x+y\right)\right\|\\
&\quad + |m|\sup_{x,y\in X} e^{-\lambda\|(1/m)x-y\|}\left\|f_4\left(\frac{1}{m}x-y\right)\right\|\\
&\quad + 2|m|(m^2-1)\sup_{x\in X} e^{-\lambda\|(1/m)x\|}\left\|f_5\left(\frac{1}{m}x\right)\right\|\\
&= \|f_1\|+\|f_2\|+|m|\|f_3\|+|m|\|f_4\|+2|m|(m^2-1)\|f_5\|\\
&\le (2|m|^3+2)\max\{\|f_1\|,\|f_2\|,\|f_3\|,\|f_4\|,\|f_5\|\}\\
&= (2|m|^3+2)\|(f_1,f_2,f_3,f_4,f_5)\|
\end{aligned} \tag{2.3}$$

for each $(f_1,\dots,f_5)\in Z_\lambda^5$. This implies that

$$\|C_1^P\| \le 2|m|^3+2. \tag{2.4}$$

Now, let $v\in Y$ be such that $\|v\|=1$ and let $\{\xi_n\}_n$ be a sequence of positive real numbers decreasing to 0. We define

$$f_n(x)=\begin{cases} e^{2\lambda\xi_n}v, & \text{if } \|x\|=2\xi_n \text{ or } \|x\|=0,\\ -\dfrac{|m|}{m}e^{2\lambda\xi_n}v, & \text{if } \|mx\|=|m+1|\xi_n,\ \|mx\|=|m-1|\xi_n \text{ or } \|mx\|=\xi_n,\\ 0, & \text{otherwise}\end{cases} \tag{2.5}$$

for all $x\in X$. Hence we have

$$e^{-\lambda\|x\|}\|f_n(x)\|=\begin{cases} e^{2\lambda\xi_n}, & \text{if } \|x\|=0,\\ 1, & \text{if } \|x\|=2\xi_n,\\ e^{(2-|(m+1)/m|)\lambda\xi_n}, & \text{if } \|mx\|=|m+1|\xi_n,\\ e^{(2-|(m-1)/m|)\lambda\xi_n}, & \text{if } \|mx\|=|m-1|\xi_n,\\ e^{(2-1/|m|)\lambda\xi_n}, & \text{if } \|mx\|=\xi_n,\\ 0, & \text{otherwise}\end{cases} \tag{2.6}$$

for all $x \in X$, so that $f_n \in X_\lambda$ for all positive integers n, with

$$\|f_n\| = e^{2\lambda\xi_n}. \tag{2.7}$$

Let $u \in X$ be such that $\|u\| = 1$ and take $x_0, y_0 \in X$ as $x_0 = y_0 = \xi_n u$. Then it follows from the definition of f_n that

$$\begin{aligned}\|C_1^P(f_n,\dots,f_n)\| &= \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)}\left\| f_n(x+y) + f_n(x-y) - mf_n\left(\frac{1}{m}x+y\right)\right.\\ &\qquad \left. - mf_n\left(\frac{1}{m}x-y\right) - 2m(m^2-1)f_n\left(\frac{1}{m}x\right)\right\| \\ &\ge e^{-2\lambda\xi_n}\|e^{2\lambda\xi_n}v + e^{2\lambda\xi_n}v + |m|e^{2\lambda\xi_n}v + |m|e^{2\lambda\xi_n}v + 2|m|(m^2-1)e^{2\lambda\xi_n}v\| \\ &= 2|m|^3+2.\end{aligned} \tag{2.8}$$

If on the contrary $\|C_1^P\| < 2|m|^3+2$, then there exists a $\delta > 0$ such that

$$\|C_1^P(f_n,\dots,f_n)\| \le (2|m|^3+2-\delta)\|(f_n,\dots,f_n)\| \tag{2.9}$$

for all positive integers n. So it follows from (2.7), (2.8), and (2.9) that

$$2|m|^3+2 \le \|C_1^P f_n,\dots,f_n)\| \le (2|m|^3+2-\delta)e^{2\lambda\xi_n} \tag{2.10}$$

for all positive integers n. Since $\lim_{n\to\infty} e^{2\lambda\xi_n} = 1$, the right-hand side of (2.10) tends to $2|m|^3 + 2 - \delta$ as $n\to\infty$, whence $2|m|^3 + 2 \le 2|m|^3 + 2 - \delta$, which is a contradiction. Hence we have $\|C_1^P\| = 2|m|^3 + 2$. □

Theorem 2.1 of [4] is a result of Theorem 2.1 for $m = 2$.

Corollary 2.2. *The operator $C_1 : X_\lambda \to X_\lambda^2$ is a bounded linear operator with*

$$\|C_1\| = 2|m|^3 + 2. \tag{2.11}$$

Proof. The result follows from the proof of Theorem 2.1. □

Theorem 2.3. *The operator $C_2^P : Z_\lambda^5 \to X_\lambda^2$ is a bounded linear operator with*

$$\|C_2^P\| = 2|a+b|(a-b)^2 + |ab(a+b)| + 2. \tag{2.12}$$

Proof. Since

$$\max\left\{\left\|\frac{1}{a}x+\frac{1}{b}y\right\|, \left\|\frac{1}{b}x+\frac{1}{a}y\right\|, \left\|\frac{1}{ab}x\right\|, \left\|\frac{1}{ab}y\right\|, \left\|\frac{1}{ab}x+\frac{1}{ab}y\right\|\right\} \le \|x\|+\|y\| \tag{2.13}$$

for all $x, y \in X$, we get

$$\begin{aligned}
\|C_2^P(f_1,\dots,f_5)\| = \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)} \Bigg\| & f_1\left(\frac{1}{a}x+\frac{1}{b}y\right)+f_2\left(\frac{1}{b}x+\frac{1}{a}y\right) \\
& -(a+b)(a-b)^2\left[f_3\left(\frac{1}{ab}x\right)+f_4\left(\frac{1}{ab}y\right)\right] \\
& -ab(a+b)f_5\left(\frac{1}{ab}x+\frac{1}{ab}y\right)\Bigg\| \\
\le \sup_{x,y\in X} & e^{-\lambda\|(1/a)x+(1/b)y\|}\left\|f_1\left(\frac{1}{a}x+\frac{1}{b}y\right)\right\| \\
+ \sup_{x,y\in X} & e^{-\lambda\|(1/b)x+(1/a)y\|}\left\|f_2\left(\frac{1}{b}x+\frac{1}{a}y\right)\right\| \\
+ |a+b|(a-b)^2 & \sup_{x\in X} e^{-\lambda(\|(1/ab)x\|)}\left\|f_3\left(\frac{1}{ab}x\right)\right\| \\
+ |a+b|(a-b)^2 & \sup_{y\in X} e^{-\lambda(\|(1/ab)y\|)}\left\|f_4\left(\frac{1}{ab}y\right)\right\| \\
+ |ab(a+b)| & \sup_{x,y\in X} e^{-\lambda\|(1/ab)x+(1/ab)y\|}\left\|f_5\left(\frac{1}{ab}x+\frac{1}{ab}y\right)\right\| \\
\le \|f_1\| + \|f_2\| & + |a+b|(a-b)^2(\|f_3\|+\|f_4\|)+|ab(a+b)|\|f_5\| \\
\le (2|a+b|(a-b)^2 & + |ab(a+b)|+2)\max\{\|f_1\|,\|f_2\|,\|f_3\|,\|f_4\|,\|f_5\|\} \\
= (2|a+b|(a-b)^2 & + |ab(a+b)|+2)\|(f_1,f_2,f_3,f_4,f_5)\|
\end{aligned} \tag{2.14}$$

for each $(f_1,\dots,f_5) \in Z_\lambda^5$. This implies that

$$\|C_2^P\| \le 2|a+b|(a-b)^2 + |ab(a+b)| + 2. \tag{2.15}$$

Let η be a real number such that

$$\eta \notin \left\{0, 1, \frac{1-a}{b}, \frac{1-b}{a}, \frac{a-1}{1-b}, \frac{b-1}{1-a}, \frac{a}{1-b}, \frac{b}{1-a}\right\}. \tag{2.16}$$

Now, let $u \in X, v \in Y$ be such that $\|u\| = \|v\| = 1$ and let $\{\xi_n\}_n$ be a sequence of positive real numbers decreasing to 0. We define

$$f_n(x) = \begin{cases} e^{\lambda(1+|\eta|)\xi_n}v, & \text{if } x = \left(\dfrac{1}{a}+\dfrac{\eta}{b}\right)\xi_n u, \text{ or } x = \left(\dfrac{1}{b}+\dfrac{\eta}{a}\right)\xi_n u, \\ -\dfrac{|a+b|}{a+b}e^{\lambda(1+|\eta|)\xi_n}v, & \text{if } x = \dfrac{1}{ab}\xi_n u, \text{ or } x = \dfrac{\eta}{ab}\xi_n u, \\ -\dfrac{|ab(a+b)|}{ab(a+b)}e^{\lambda(1+|\eta|)\xi_n}v, & \text{if } x = \dfrac{1+\eta}{ab}\xi_n u, \\ 0, & \text{otherwise} \end{cases} \tag{2.17}$$

for all $x \in X$. Hence we have

$$e^{-\lambda\|x\|}\|f_n(x)\| = \begin{cases} e^{(1+|\eta|-|1/a+\eta/b|)\lambda\xi_n}, & \text{if } x = \left(\dfrac{1}{a}+\dfrac{\eta}{b}\right)\xi_n u, \\ e^{(1+|\eta|-|1/b+\eta/a|)\lambda\xi_n}, & \text{if } x = \left(\dfrac{1}{b}+\dfrac{\eta}{a}\right)\xi_n u, \\ e^{(1+|\eta|-|1/ab|)\lambda\xi_n}, & \text{if } x = \dfrac{1}{ab}\xi_n u, \\ e^{(1+|\eta|-|\eta/ab|)\lambda\xi_n}, & \text{if } x = \dfrac{\eta}{ab}\xi_n u, \\ e^{(1+|\eta|-|(1+\eta)/ab|)\lambda\xi_n}, & \text{if } x = \dfrac{1+\eta}{ab}\xi_n u, \\ 0, & \text{otherwise} \end{cases} \tag{2.18}$$

for all $x \in X$, so that $f_n \in X_\lambda$ for all positive integers n, with

$$\begin{aligned} \|f_n\| = \max\{&e^{(1+|\eta|-|1/a+\eta/b|)\lambda\xi_n}, e^{(1+|\eta|-|1/b+\eta/a|)\lambda\xi_n}, \\ &e^{(1+|\eta|-|1/ab|)\lambda\xi_n}, e^{(1+|\eta|-|\eta/ab|)\lambda\xi_n}, e^{(1+|\eta|-|(1+\eta)/ab|)\lambda\xi_n}\}. \end{aligned} \tag{2.19}$$

Let $x_0, y_0 \in X$ be such that $x_0 = \xi_n u$ and $y_0 = \eta\xi_n u$. Then it follows from the definition of f_n that

$$\begin{aligned} \|C_2^P(f_n,\ldots,f_n)\| = \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)} \bigg\| &f_n\left(\frac{1}{a}x+\frac{1}{b}y\right) + f_n\left(\frac{1}{b}x+\frac{1}{a}y\right) \\ &- (a+b)(a-b)^2\left[f_n\left(\frac{1}{ab}x\right) + f_n\left(\frac{1}{ab}y\right)\right] \\ &- ab(a+b)f_n\left(\frac{1}{ab}x+\frac{1}{ab}y\right)\bigg\| \\ \geq e^{-\lambda(1+|\eta|)\xi_n}\|e^{\lambda(1+|\eta|)\xi_n} &+ e^{\lambda(1+|\eta|)\xi_n} + 2|a+b|(a-b)^2 e^{\lambda(1+|\eta|)\xi_n} + |ab(a+b)|e^{\lambda(1+|\eta|)\xi_n}\| \\ = 2|a+b|(a-b)^2 &+ |ab(a+b)| + 2, \end{aligned} \tag{2.20}$$

so that

$$\|C_2^P(f_n,\ldots,f_n)\| \geq 2|a+b|(a-b)^2 + |ab(a+b)| + 2. \tag{2.21}$$

If on the contrary $\|C_2^P\| < 2|a+b|(a-b)^2 + |ab(a+b)| + 2$, then there exists a $\delta > 0$ such that

$$\|C_2^P(f_n,\ldots,f_n)\| \leq (2|a+b|(a-b)^2 + |ab(a+b)| + 2 - \delta)\|(f_n,\ldots,f_n)\| \tag{2.22}$$

for all positive integers n. So it follows from (2.21) and (2.22) that

$$2|a+b|(a-b)^2+|ab(a+b)|+2 \leq \|C_2^P(f_n,\ldots,f_n)\| \leq (2|a+b|(a-b)^2+|ab(a+b)|+2-\delta)\|f_n\| \tag{2.23}$$

for all positive integers n. Since $\lim_{n\to\infty}\xi_n = 0$, it follows from (2.19) that $\lim_{n\to\infty}\|f_n\| = 1$, so the right-hand side of (2.23) tends to $2|a+b|(a-b)^2+|ab(a+b)|+2-\delta$ as $n\to\infty$, whence

$$2|a+b|(a-b)^2+|ab(a+b)|+2 \leq 2|a+b|(a-b)^2+|ab(a+b)|+2-\delta, \tag{2.24}$$

which is a contradiction. Hence we have $\|C_2^P\| = 2|a+b|(a-b)^2+|ab(a+b)|+2$. □

Corollary 2.4. *The operator $C_2 : X_\lambda \to X_\lambda^2$ is a bounded linear operator with*

$$\|C_2\| = 2|a+b|(a-b)^2+|ab(a+b)|+2. \tag{2.25}$$

Proof. The result follows from the proof of Theorem 2.3. □

Acknowledgment

The authors would like to thank the referee for his/her useful comments.

References

[1] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[2] S. Czerwik and K. Dlutek, “Cauchy and Pexider operators in some function spaces,” in *Functional Equations, Inequalities and Applications*, pp. 11–19, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[3] M. S. Moslehian, T. Riedel, and A. Saadatpour, “Norms of operators in X_λ spaces,” *Applied Mathematics Letters*, vol. 20, no. 10, pp. 1082–1087, 2007.

[4] S.-M. Jung, “Cubic operator norm on X_λ space,” *Bulletin of the Korean Mathematical Society*, vol. 44, no. 2, pp. 309–313, 2007.

[5] K.-W. Jun and H.-M. Kim, “The generalized Hyers-Ulam-Rassias stability of a cubic functional equation,” *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 867–878, 2002.

[6] K.-W. Jun, H.-M. Kim, and I.-S. Chang, “On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation,” *Journal of Computational Analysis and Applications*, vol. 7, no. 1, pp. 21–33, 2005.

[7] K.-W. Jun and H.-M. Kim, “On the stability of Euler-Lagrange type cubic mappings in quasi-Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1335–1350, 2007.