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Research Article

The Locally Uniform Nonsquare in Generalized Cesàro Sequence Spaces

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We show that the generalized Cesàro sequence spaces possess the locally uniform nonsquare and have the fixed point property, but they are not uniformly nonsquare. This result is related to the result of the paper by J. Falset et al. (2006) by giving the examples and the motivation to find the geometric properties that are weaker than uniformly nonsquare but still possess the fixed point property in any Banach spaces.

1. Introduction

In the whole paper, $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural numbers and of real numbers, respectively. The space of all real sequence $x = (x(i))_{i=1}^{\infty}$ is denoted by ℓ^0. For a real normed space $(X, \|\cdot\|)$, we denote by $S(X)$ the unit sphere of X. We now give some definitions and basic concepts which will be used in this paper.

A Banach space $(X, \|\cdot\|)$ which is a subspace of ℓ^0 is said to be a *Köthe sequence space*, if

(i) for any $x \in \ell^0$ and $y \in X$ such that $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, we have $x \in X$ and $\|x\| \leq \|y\|$;

(ii) there is $x \in X$ with $x(i) \neq 0$ for all $i \in \mathbb{N}$.

An element x from a Köthe sequence space X is called *order continuous* if for any sequence (x_n) in X_+ (the positive cone of X) such that $x_n \leq |x|$ for all $n \in \mathbb{N}$ and $x_n \to 0$coordinatewise , we have $\|x_n\| \to 0$. It is easy to see that x is order continuous if and only if $\|(0, 0, \ldots, 0, x(n+1), x(n+2), \ldots)\| \to 0$ as $n \to \infty$.

1.1. Modular spaces

For a real vector space X, a function $\varrho : X \to [0,\infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular ϱ on X, the space

$$X_\varrho = \{x \in X : \varrho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0^+\} \tag{1.1}$$

is called the *modular space*.

If ϱ is a convex modular, the function

$$\|x\| = \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\right\} \tag{1.2}$$

is a norm on X_ϱ, which is called the *Luxemburg norm* (see [1]).

A modular ϱ is said to satisfy the Δ_2-*condition* ($\varrho \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$$\varrho(2x) \leq K\varrho(x) + \varepsilon \tag{1.3}$$

for all $x \in X_\varrho$ with $\varrho(x) \leq a$.

If ϱ satisfies the Δ_2-condition for all $a > 0$ with $K \geq 2$ dependent on a, we say that ϱ satisfies the *strong* Δ_2-*condition* ($\varrho \in \Delta_2^s$).

Lemma 1.1. *If $\varrho \in \Delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|\varrho(u + v) - \varrho(u)| < \varepsilon, \tag{1.4}$$

whenever $u, v \in X_\varrho$ with $\varrho(u) \leq L$ and $\varrho(v) \leq \delta$.

Proof. See [2, Lemma 2.1]. □

Lemma 1.2. *If $\varrho \in \Delta_2^s$, then for any $x \in X_\varrho$, $\|x\| = 1$ if and only if $\varrho(x) = 1$.*

Proof. See [2, Corollary 2.2]. □

1.2. Generalized Cesàro sequence spaces

For $1 \le p < \infty$, the Cesàro sequence space (write ces_p, for short) is defined by

$$\text{ces}_p = \left\{ x \in \ell^0 : \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^p < \infty \right\}, \tag{1.5}$$

equipped with the norm

$$\|x\| = \left(\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^p \right)^{1/p}. \tag{1.6}$$

Cesàro sequence spaces ces_p appeared in 1968 as the problem in the Dutch Mathematical Society to find their duals (see [3, 4]). Regular investigation of these spaces was done by Shiue [5] in 1970, while Leibowitz [6] and Jagers [7] proved that $\text{ces}_1 = \{0\}$, ces_p are separable reflexive Banach spaces for $1 < p < \infty$ and ℓ^p spaces are in ces_p for $1 < p \le \infty$.

Let $p = (p_j)$ be a sequences of positive real numbers with $p_j \ge 1$ for all $j \in \mathbb{N}$, *the generalized Cesàro sequence space*, $\text{ces}_{(p)}$, is defined by

$$\text{ces}_{(p)} = \{ x \in l^0 : \rho(\lambda x) < \infty, \text{ for some } \lambda > 0 \}, \tag{1.7}$$

where

$$\rho(x) = \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p_j} \tag{1.8}$$

is a convex modular on $\text{ces}_{(p)}$ (see [8]). Observe that if the space $\text{ces}_{(p)}$ is nontrivial, then it belongs to the class of Köthe sequence spaces. We also have some observations on $\text{ces}_{(p)}$ as follows.

Remark 1.3. (i) In the case when $p_j = p$, $1 \le p < \infty$ for all $j \in \mathbb{N}$, the generalized Cesàro sequence space $\text{ces}_{(p)}$ is nothing but the Cesàro sequence space ces_p and the Luxemburg norm is expressed by the formula (1.6).

(ii) Condition $\lim_{j\to\infty} \inf p_j > 1$ is obviously sufficient for $\text{ces}_{(p)} \ne \{0\}$ but it is not a necessity condition, for example, when $p_j = 1 + 2(\ln \ln j / \ln j)$, $j \ge 2$. However, if $\text{ces}_{(p)} \ne \{0\}$, we have $\sum_{j=1}^{\infty} (1/j)^{p_j} < \infty$.

(iii) It is easy to see that if $\lim_{j\to\infty} \sup p_j < \infty$ then $\rho \in \Delta_2^s$, and $A_{(p)} = \text{ces}_{(p)}$, where $A_{(p)} = \{ x \in l^0 : \rho(\lambda x) < \infty \text{ for all } \lambda > 0 \}$, but unfortunately we do not know whether it is a necessity condition for $A_{(p)} = \text{ces}_{(p)}$.

(iv) For each $x \in A_{(p)}$, where $A_{(p)}$ is defined as in (ii), we have x is an order continuous element. Indeed, for given $\varepsilon > 0$ by $x \in A_{(p)}$, we can find a natural number i_0 such that $\rho((x - x^i)/\varepsilon) < 1 - \varepsilon$ for all $i > i_0$. This implies that $\|x - x^i\|_p < \varepsilon$ for all $i > i_0$.

Other investigations to generalized Cesàro spaces can be found in [7, 9–14].

1.3. Nonsquareness

Now, we give the basic definitions related to the nonsquareness in Banach space.

Definition 1.4. A Banach space $(X, \|\cdot\|)$ is said to be

(i) *uniformly nonsquare in the sense of James* or *uniformly non-l_n^1* (write UN-l_n^1), $n \in \mathbb{N}$, $n \geq 2$, if there is $\delta > 0$ such that for any $x_1, x_2, \dots, x_n \in S(X)$,

$$\min\{\|x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n\| : \varepsilon_i = \pm 1,\ i = 2, \dots, n\} \leq n - \delta; \tag{1.9}$$

(ii) *locally uniform nonsquare in the sense of James* or *locally uniform non-l_n^1* (write LUN-l_n^1), $n \in \mathbb{N}$, $n \geq 2$, if for every $x \in S(X)$ there exists $\delta > 0$ such that for any $x_1, x_2, \dots, x_n \in S(X)$,

$$\min\{\|x + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n\| : \varepsilon_i = \pm 1,\ i = 2, \dots, n\} \leq n - \delta; \tag{1.10}$$

(iii) *nonsquare in the sense of James* or *non-l_n^1* (write N-l_n^1), $n \in \mathbb{N}$, $n \geq 2$, if for every $x \in S(X)$ there exists $\delta > 0$ such that for any $x_1, x_2, \dots, x_n \in S(X)$,

$$\min\{\|x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n\| : \varepsilon_i = \pm 1,\ i = 2, \dots, n\} < n. \tag{1.11}$$

The spaces UN-l_n^1, LUN-l_n^1, and N-l_n^1 were considered by many authors (see [15–18]).

On the other hand, in 1976, Schäffer (see [19]) introduced the other definitions of various kind nonsquareness.

Definition 1.5. A Banach space X is said to be

(i) *nonsquare* (write N-S) if for any $x, y \in S(X)$,

$$\max\{\|x + y\|, \|x - y\|\} > 1; \tag{1.12}$$

(ii) *locally uniform nonsquare* (write LUN-S) or if for any $x \in S(X)$ there exists $\delta > 0$ such that for all $y \in S(X)$,

$$\max\{\|x + y\|, \|x - y\|\} > 1 + \delta \tag{1.13}$$

for any $x, y \in S(X)$;

(iii) *uniformly nonsquare* (write UN-S) if there exists $\delta > 0$ such that

$$\max\{\|x+y\|, \|x-y\|\} > 1+\delta \tag{1.14}$$

for any $x, y \in S(X)$.

Remark 1.6. It is well known that N-S $\Leftrightarrow$ N-l_2^1 and UN-S $\Leftrightarrow$ UN-l_2^1 but LUN-S is not equivalent to LUN-l_2^1 (see [15] or [18]).

Recall that the Banach space X is said to be *strictly convex* if for any $x, y \in S(X)$ with $x \neq y$ we must have $\|(x+y)/2\| < 1$. The next results can be found in [20], but for the sake of completeness we present here a proof.

Theorem 1.7. *If X is a strictly convex Banach space then X is nonsquare in the sense of Schäffer.*

Proof. Suppose that X is not nonsquare in the sense of Schäffer. Then there exist $x, y \in S(X)$ such that $\|x \pm y\| = 1$. Put $z = ((x+y)+(x-y))/2$ then $z \in S(X)$ but it is not an extreme point. Hence, X is not strictly convex. □

Also, let us recall that the Banach space X is said to be *locally uniform convex* if $\{x_n\}$ and $\{y_n\}$ are any sequences in $S(X)$ such that $\lim_{n\to\infty}\|x_n + y_n\| = 2$ we must have $\lim_{n\to\infty}\|x_n - y_n\| = 0$. The next theorem shows the relation between locally uniform convex and locally uniform nonsquare in any Banach spaces.

Theorem 1.8. *If X is a locally uniform convex Banach space then X is locally uniform nonsquare in the sense of James.*

Proof. Suppose that X is not locally uniform nonsquare in the sense of James. Then there exists $x \in S(X)$ and $\{x_n\} \subset S(X)$ such that $\|x_n \pm x\| \to 2$. On the other hand, by X is locally uniform convex, we must have

$$\lim_{n\to\infty}\|x_n - x\| = 0, \tag{1.15}$$

which is a contradiction. □

In [21], it was showed that every uniformly nonsquare Banach space must have the fixed point property (that is for any nonempty closed, convex, and bounded subset A of X and any nonexpansive mapping P from A into itself has a fixed point z in A). On the other hand, under some suitable conditions, it is well known that the generalized Cesàro sequence spaces have the fixed point property, however, they are not uniform nonsquare (see [14, 22]). The main purpose of this this paper is to find the conditions for locally uniform nonsquare in the sense of James and Schäffer in these spaces. As consequently, we have the examples of Banach spaces that agree with a weaker geometric property than uniformly nonsquare but still possess the fixed point property.

In the sequel, we will assume that $\lim_{j\to\infty}\sup p_j < \infty$, say $p_j \leq \beta$ for all $j \in \mathbb{N}$. The following result is quite useful for our purpose.

Theorem 1.9. $\mathrm{ces}_{(p)}$ *is locally uniform convex.*

Proof. See [8]. □

2. Main results

We begin by obtaining the first main result.

Theorem 2.1. $\mathrm{ces}_{(p)}$ *is locally uniform nonsquare in the sense of James.*

Proof. The result is an immediate consequence of Theorems 1.8 and 1.9. □

Next, we will show that the space $\mathrm{ces}_{(p)}$ is locally uniform nonsquare in the sense of Schäffer. To do this, we need the following lemma.

Lemma 2.2. *A closed bounded set $K \subset \mathrm{ces}_{(p)}$ is compact if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\sum_{k=N}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)^{p_k} < \varepsilon \tag{2.1}$$

for any $y \in K$.

Proof. Let $\{x_n\}$ be a sequence in K. Define

$$\pi_k : \mathrm{ces}_{(p)} \longrightarrow \mathbb{R} \quad \text{by } \pi_k(x) = x(k), \tag{2.2}$$

we have π_k is a continuous function for each $k \in \mathbb{N}$. So, $\pi_k(\{x_n\})$ is a bounded subset of $\mathbb{R}$. Then, by using the orthogonal method, a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and $x \in \ell^0$ can be found such that $x_{n_j}(k) \to x(k)$ as $j \to \infty$ for all $k \in \mathbb{N}$. We claim that $x \in K$ and $x_{n_j} \to x$ as $j \to \infty$. Indeed, by our hypothesis, for each $\varepsilon \in (0,1)$ there exists $N = N_\varepsilon \in \mathbb{N}$ such that for all $j \in \mathbb{N}$, we have

$$\sum_{k=N}^{M}\left(\frac{1}{k}\sum_{i=1}^{k}|x_{n_j}(i)|\right)^{p_k} < \varepsilon^{\beta}, \tag{2.3}$$

when $M > N$. Letting $j \to \infty$ and $M \to \infty$, we get

$$\sum_{k=N}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} < \varepsilon^{\beta}, \tag{2.4}$$

which implies that $x \in \mathrm{ces}_{(p)}$. Moreover, since there exists $\beta \in \mathbb{R}$ which $1 \le p_k \le \beta$ for all $k \in \mathbb{N}$, (2.3) gives

$$\rho\left(\frac{x_{n_j} - x_{n_j}^N}{\varepsilon}\right) = \sum_{k=N+1}^{\infty}\left(\frac{(1/k)\sum_{i=1}^{k}|x_{n_j}(i)|}{\varepsilon}\right)^{p_k} < \frac{1}{\varepsilon^{\beta}}\sum_{k=N+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_{n_j}(i)|\right)^{p_k} < 1, \tag{2.5}$$

that is, for each $j \in \mathbb{N}$

$$\left\|x_{n_j} - x_{n_j}^N\right\|_p < \varepsilon, \tag{2.6}$$

and similarly, (2.4) gives

$$\left\|x - x^N\right\|_p < \varepsilon. \tag{2.7}$$

Next, since for each $k \in \mathbb{N}$ we have $x_{n_j}(k) \rightarrow x(k)$ as $j \rightarrow \infty$, there exists $j_o \in \mathbb{N}$ such that

$$\left|x_{n_j}(k) - x(k)\right| < \frac{\varepsilon^{\beta+1}}{\delta N^3} \tag{2.8}$$

for all $j > j_o$ and $k \in \{1, 2, 3, \ldots, N\}$, where $\delta = \max\{1, \sum_{j=N+1}^{\infty}(1/j^{p_j})\}$. Therefore,

$$\begin{aligned} \rho\left(\frac{x_{n_j}^N - x^N}{\varepsilon}\right) &= \sum_{k=1}^{N}\left(\frac{(1/k)\sum_{i=1}^{k}\left|x_{n_j}(i) - x(i)\right|}{\varepsilon}\right)^{p_k} + \sum_{k=N+1}^{\infty}\left(\frac{(1/k)\sum_{i=1}^{N}\left|x_{n_j}(i) - x(i)\right|}{\varepsilon}\right)^{p_k} \\ &< \frac{1}{\varepsilon^{\beta}}\left(\frac{\varepsilon^{\beta+1}}{\delta N^3} \cdot \frac{N^3}{2} + \frac{\varepsilon^{\beta+1}}{\delta N^3} \cdot N \cdot \delta\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned} \tag{2.9}$$

for all $j > j_o$, that is,

$$\left\|x_{n_j}^N - x^N\right\|_p < \varepsilon \tag{2.10}$$

for all $j > j_o$. Hence, by (2.6), (2.7), and (2.10) we can conclude that $x_{n_j} \rightarrow x$ as $j \rightarrow \infty$. Finally, since K is closed, we must have $x \in K$. □

Now, we are in position to prove the other main result.

Theorem 2.3. $\mathrm{ces}_{(p)}$ *is locally uniform nonsquare in the sense of Schäffer.*

Proof. Assume on the contrary that $\mathrm{ces}_{(p)}$ is not locally uniform nonsquare in the sense of Schäffer. Then, there are $x \in S(\mathrm{ces}_{(p)})$ and $\{y_n\} \subset S(\mathrm{ces}_{(p)})$ such that

$$\left\|x \pm y_n\right\| \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \tag{2.11}$$

First, we claim that there is $\varepsilon_o > 0$ such that for any $j \in \mathbb{N}$ there exists $n_j > j$ for which

$$\sum_{k=j}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|y_{n_j}(i)|\right)^{p_k} \geq \varepsilon_o. \tag{2.12}$$

Otherwise, by Lemma 2.2 the set $\{y_n\}$ is compact. So there is a subsequence $\{y_{n_k}\} \subset \{y_n\}$ and $y \in S(\mathrm{ces}_{(p)})$ such that

$$\lim_{k\to\infty}\|y_{n_k} - y\| = 0. \tag{2.13}$$

Therefore, $\|x \pm y\| = 1$, that is, $\mathrm{ces}_{(p)}$ is not nonsquare. This contradicts to Theorems 1.9 and 1.7, respectively. Hence, the claim holds true.

Note that by Lemma 1.1, a real number $\delta \in (0, \varepsilon_o/4)$ can be found such that

$$|\rho(u+v) - \rho(u)| < \frac{\varepsilon_o}{4}, \tag{2.14}$$

whenever $u, v \in \mathrm{ces}_{(p)}$ with $\rho(u) \leq 1$ and $\rho(v) \leq \delta$. Using this positive real number δ, since x is an order continuous element (see Remark 1.3(iii) and (iv)), there exists $i_o \in \mathbb{N}$ such that

$$\sum_{k=j_o+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} \leq \delta. \tag{2.15}$$

Hence, by Lemma 1.2, we get

$$1 - \frac{\varepsilon_o}{4} < \rho(x) - \sum_{k=j_o+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} = \sum_{k=1}^{j_o}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}. \tag{2.16}$$

Now, by (2.11), for a real number $\varepsilon_1 \in (0, \varepsilon_o)$ which satisfies the inequality $(1+\varepsilon_1)^{\beta} < 1+\varepsilon_o/2$, a number $n_o \in \mathbb{N}$ can be found such that

$$\max\{\|x \pm y_n\|\} < 1 + \varepsilon_1 \tag{2.17}$$

for all $n > n_o$. Taking $j_1 = \max\{j_o, n_o\}$, so there exists $n_{j_1} > j_1 + 1$ satisfying (2.12). Thus, by (2.14) and (2.16), we would have

$$
\begin{aligned}
\max\left\{\rho\left(\frac{x \pm y_{n_{j_1}}}{1+\varepsilon_1}\right)\right\} &= \max\left\{\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\left|\frac{x(i) \pm y_{n_{j_1}}(i)}{1+\varepsilon_1}\right|\right)^{p_k}\right\} \\
&\geq \frac{1}{(1+\varepsilon_1)^{\beta}} \max\left\{\sum_{k=1}^{j_1}\left(\frac{1}{k}\sum_{i=1}^{k}\left|x(i) \pm y_{n_{j_1}}(i)\right|\right)^{p_k}\right. \\
&\qquad \left. + \sum_{k=j_1+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\left|x(i) \pm y_{n_{j_1}}(i)\right|\right)^{p_k}\right\} \\
&\geq \frac{1}{(1+\varepsilon_1)^{\beta}} \max\left\{\sum_{k=1}^{j_1}\left(\frac{1}{k}\sum_{i=1}^{k}\left|x(i) \pm y_{n_{j_1}}(i)\right|\right)^{p_k}\right. \\
&\qquad \left. + \sum_{k=j_1+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\left|y_{n_{j_1}}(i)\right|\right)^{p_k} - \frac{\varepsilon_o}{4}\right\} \\
&\geq \frac{1}{(1+\varepsilon_1)^{\beta}} \max\left\{\sum_{k=1}^{j_1}\left(\frac{1}{k}\sum_{i=1}^{k}\left|x(i) \pm y_{n_{j_1}}(i)\right|\right)^{p_k} + \frac{3\varepsilon_o}{4}\right\} \\
&\geq \frac{1}{(1+\varepsilon_1)^{\beta}} \left\{\sum_{k=1}^{j_1}\left[\frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}\left|x(i) + y_{n_{j_1}}(i)\right|\right)^{p_k}\right.\right. \\
&\qquad \left.\left. + \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}\left|x(i) - y_{n_{j_1}}(i)\right|\right)^{p_k}\right] + \frac{3\varepsilon_o}{4}\right\} \\
&\geq \frac{1}{(1+\varepsilon_1)^{\beta}} \left\{\sum_{k=1}^{j_1}\left[\frac{1}{2k}\sum_{i=1}^{k}\left(\left|x(i) + y_{n_{j_1}}(i)\right| + \left|x(i) - y_{n_{j_1}}(i)\right|\right)\right]^{p_k} + \frac{3\varepsilon_o}{4}\right\} \\
&\geq \frac{1}{(1+\varepsilon_1)^{\beta}} \left\{\sum_{k=1}^{j_1}\left(\frac{1}{k}\sum_{i=1}^{k}\left|x(i)\right|\right)^{p_k} + \frac{3\varepsilon_o}{4}\right\} \\
&> \frac{1+\varepsilon_0/2}{(1+\varepsilon_1)^{\beta}} > 1.
\end{aligned}
\tag{2.18}
$$

This implies that $\max\{\|x \pm y_{n_{j_1}}\|\} > 1 + \varepsilon_1$, which is a contradiction to (2.17). Hence, $\mathrm{ces}_{(p)}$ is locally uniform nonsquare in the sense of Schäffer. □

Remark 2.4. In Theorems 2.1 and 2.3, we have shown that, under some suitable conditions, the generalized Cesàro sequence spaces are locally uniform nonsquare in the sense of James and Schäffer. This result gives the motivation to consider the geometric properties that are weaker than uniform nonsquare but still possess the fixed point property in any Banach spaces.

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