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Research Article

On Generalized Strong Vector Variational-Like Inequalities in Banach Spaces

Lu-Chuan Ceng, Yen-Cherng Lin, and Jen-Chih Yao

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The purpose of this paper is to study the solvability for a class of generalized strong vector variational-like inequalities in reflexive Banach spaces. Firstly, utilizing Brouwer's fixed point theorem, we prove the solvability for this class of generalized strong vector variational-like inequalities without monotonicity assumption under some quite mild conditions. Secondly, we introduce the new concept of pseudomonotonicity for vector multifunctions, and prove the solvability for this class of generalized strong vector variational-like inequalities for pseudomonotone vector multifunctions by using Fan's lemma and Nadler's theorem. Our results give an affirmative answer to an open problem proposed by Chen and Hou in 2000, and also extend and improve the corresponding results of Fang and Huang (2006).

1. Introduction and preliminaries

In 1980, Giannessi [3] initially introduced and considered a vector variational inequality in a finite-dimensional Euclidean space, which is the vector-valued version of the variational inequality of Hartman and Stampacchia. Ever since then, vector variational inequalities have been extensively studied and generalized in infinite-dimensional spaces since they have played very important roles in many fields, such as mechanics, physics, optimization, control, nonlinear programming, economics and transportation equilibrium, engineering sciences, and so forth. On account of their very valuable applicability, the vector variational inequality theory has been widely developed throughout over last 20 years; see [1, 2, 4, 5, 7–14] and the references therein.

Let X and Y be two real Banach spaces, let $K \subseteq X$ be a nonempty, closed, and convex set, and let $C \subseteq Y$ be a closed, convex, and pointed cone with apex at the origin. Recall that C is said to be a closed, convex, and pointed cone with apex at the origin if and only if C is closed and the following conditions hold:

(i) $\lambda C \subseteq C$, for all $\lambda > 0$;

(ii) $C + C \subseteq C$;

(iii) $C \cap (-C) = \{0\}$.

Given a closed, convex, and pointed cone C with apex at the origin in Y, we can define relations "$\leq_C$" and "$\nleq_C$" as follows:

$$x \leq_C y \Longleftrightarrow y - x \in C, \qquad x \nleq_C y \Longleftrightarrow y - x \notin C. \tag{1.1}$$

Clearly "$\leq_C$" is a partial order. In this case, $(Y, \leq_C)$ is called an ordered Banach space ordered by C. Let $L(X,Y)$ denote the space of all continuous linear maps from X into Y. Let $T : K \to 2^{L(X,Y)}$ be a multifunction, where $2^{L(X,Y)}$ denotes the collection of all nonempty subsets of $L(X,Y)$. Let $A : L(X,Y) \to L(X,Y)$, $h : K \to Y$, and $\eta : K \times K \to X$ be three mappings.

Definition 1.1. (i) The generalized (weak) vector variational-like inequality (*GVVLI*) consists of finding a vector $x^* \in K$ such that

$$\langle As^*, \eta(y, x^*) \rangle + h(y) - h(x^*) \nleq_{\operatorname{int} C} 0, \quad \forall y \in K, \tag{1.2}$$

for some $s^* \in Tx^*$, where $\operatorname{int} C$ denotes the interior of C and $a \nleq_{\operatorname{int} C} b$ means that $b - a \notin \operatorname{int} C$.

(ii) The generalized strong vector variational-like inequality (*GSVVLI*) consists of finding a vector $x^* \in K$ such that

$$\langle As^*, \eta(y, x^*) \rangle + h(y) - h(x^*) \nleq_{C \setminus \{0\}} 0, \quad \forall y \in K, \tag{1.3}$$

for some $s^* \in Tx^*$, where $a \nleq_{C \setminus \{0\}} b$ means that $b - a \notin C \setminus \{0\}$.

If $h(x) = 0$, $\eta(y,x) = y - x$ for all $x, y \in K$, $A = I$ the identity mapping of $L(X,Y)$, and T is a single-valued mapping, then Definition 1.1 reduces to the following definition.

Definition 1.2 [7]. (i) The (weak) vector variational inequality (*VVI*) consists of finding a vector $x^* \in K$ such that

$$\langle Tx^*, y - x^* \rangle \nleq_{\operatorname{int} C} 0, \quad \forall y \in K. \tag{1.4}$$

(ii) The strong vector variational inequality (*SVVI*) consists of finding a vector $x^* \in K$ such that

$$\langle Tx^*, y - x^* \rangle \nleq_{C \setminus \{0\}} 0, \quad \forall y \in K. \tag{1.5}$$

The concept of *VVI* was first introduced by Giannessi [3] in a finite-dimensional Euclidean space. In 2000, Chen and Hou [1] reviewed and summarized representative existence results of solutions for *VVI*, and pointed out that "most of the research results in this area touch upon a weak version of *VVI* and its generalizations. The existence of solutions for (strong) vector variational inequalities is still an open problem." Subsequently, Fang and Huang [14] obtained some existence results of solutions for a class of strong vector variational inequalities and partly answered the open problem proposed by Chen and Hou [1].

Definition 1.3. Let $A : L(X,Y) \to L(X,Y)$, $h : K \to Y$, and $\eta : K \times K \to X$ be three mappings. A nonempty compact-valued multifunction $T : K \to 2^{L(X,Y)}$ is said to be η-pseudomonotone with respect to A and h if for each $x,y \in K$, the existence of $s \in Tx$ such that

$$\langle As, \eta(y,x)\rangle + h(y) - h(x) \nleq_{C\setminus\{0\}} 0 \tag{1.6}$$

implies that

$$\langle At, \eta(y,x)\rangle + h(y) - h(x) \geq_C 0, \quad \forall t \in Ty. \tag{1.7}$$

If $h(x) = 0$, $\eta(y,x) = y - x$ for all $x,y \in K$, $A = I$ the identity mapping of $L(X,Y)$, and T is a single-valued mapping, then Definition 1.3 reduces to the following definition.

Definition 1.4 [7]. A mapping $T : K \to L(X,Y)$ is said to be pseudomonotone if for any $x,y \in K$,

$$\langle Tx, y - x\rangle \nleq_{C\setminus\{0\}} 0 \Longrightarrow \langle Ty, y - x\rangle \geq_C 0. \tag{1.8}$$

Definition 1.5. A map $h : K \to Y$ is said to be convex if

$$h(\lambda x + (1-\lambda)y) \leq_C \lambda h(x) + (1-\lambda)h(y), \quad \forall x,y \in K,\ \lambda \in [0,1]. \tag{1.9}$$

Lemma 1.6 (Nadler's theorem [15]). *Let $(X, \|\cdot\|)$ be a normed vector space and let H be the Hausdorff metric on the collection $CB(X)$ of all nonempty, closed, and bounded subsets of X, induced by a metric d in terms of $d(x,y) = \|x - y\|$, which is defined by*

$$H(U,V) = \max\left(\sup_{x\in U}\inf_{y\in V}\|x-y\|, \sup_{y\in V}\inf_{x\in U}\|x-y\|\right), \tag{1.10}$$

for U and V in $CB(X)$. If U and V are compact sets in X, then for each $x \in U$, there exists $y \in V$ such that

$$\|x - y\| \leq H(U,V). \tag{1.11}$$

Definition 1.7. (i) A mapping $T : K \to L(X,Y)$ is called hemicontinuous [7] if for any given $x,y \in K$, the mapping $\lambda \mapsto \langle T(x + \lambda(y - x)), y - x\rangle$ is continuous at 0^+.

(ii) A nonempty compact-valued multifunction $T : K \to 2^{L(X,Y)}$ is called H-uniformly continuous if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x,y \in K$ with $\|x - y\| < \delta$, there holds

$$H(Tx, Ty) < \varepsilon, \tag{1.12}$$

where H is the Hausdorff metric defined on $\mathrm{CB}(L(X,Y))$.

Recently, Fang and Huang [7] gave some more results related to the work [2] in the area of solvability for strong vector variational inequalities in Banach spaces. They established two existence theorems for solutions of the *SVVI* without monotonicity in Banach spaces by using Brouwer's fixed point theorem [6] and Browder's fixed point theorem [17], respectively.

Theorem 1.8. *Let K be a nonempty compact convex subset of a real Banach space X, let Y be a real Banach space ordered by a nonempty convex cone C with apex at the origin and* $\operatorname{int} C \neq \emptyset$. *Suppose that $T : K \to L(X,Y)$ is a nonlinear mapping such that for every $y \in K$, the set $\{x \in K : \langle Tx, y-x\rangle \leq_{C\setminus\{0\}} 0\}$ is open in K. Then problem SVVI is solvable.*

Theorem 1.9. *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X, let Y be a real Banach space ordered by a nonempty convex cone C with its apex at the origin, and* $\operatorname{int} C \neq \emptyset$. *Let $T : K \to L(X,Y)$ be a continuous nonlinear mapping. Suppose that there exist $x_0 \in K$ and $\varphi \in \{f \in Y^* : \langle f, c\rangle > 0, \text{ for all} c \in C\setminus\{0\}\}$ such that*

$$\frac{\langle \varphi\circ T(x) - \varphi\circ T(x_0), x - x_0\rangle}{\|x - x_0\|} \longrightarrow \infty \tag{1.13}$$

whenever $x \in K$ and $\|x\| \to \infty$. Then problem SVVI is solvable.

Moreover, they also derived the solvability for the *SVVI* with monotonicity by using Fan's lemma.

Theorem 1.10. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X and let Y be a real Banach space ordered by a pointed closed convex cone C with its apex at the origin and* $\operatorname{int} C \neq \emptyset$. *Let $T : K \to L(X,Y)$ be a hemicontinuous pseudomonotone mapping. Then problem SVVI is solvable.*

There is no doubt that Fang and Huang's results gave a positive answer to the open problem proposed by Chen and Hou [1].

In this paper, we study the solvability of the *GSVVLI* in reflexive Banach spaces. There is no doubt that the class of *GSVVLI* includes as a special case the one of *SVVI* considered by Fang and Huang [7]. First, the solvability of the *GSVVLI* without monotonicity is derived under some quite mild conditions by using Brouwer's fixed point theorem [6]. Second, we introduce the new concept of pseudomonotonicity for vector multifunctions, and prove the solvability of the *GSVVLI* for pseudomonotone vector multifunctions by using Fan's lemma [16] and Nadler's theorem [15]. Our results also give an affirmative answer to the open problem proposed by Chen and Hou [1], and extend and improve the corresponding results in Fang and Huang [7].

2. Solvability of the *GSVVLI* without monotonicity

In this section, we will derive the solvability of the *GSVVLI* without monotonicity assumption under some quite mild conditions by using Brouwer's fixed point theorem [6]. First, recall the following result.

LEMMA 2.1 (Brouwer's fixed point theorem [6]). *Let B be a nonempty, compact, and convex subset of a finite-dimensional space and let $g : B \to B$ be a continuous mapping. Then there exists $x \in B$ such that $g(x) = x$.*

THEOREM 2.2. *Let K be a nonempty, bounded, closed, and convex subset of a real reflexive Banach space X and let Y be a real Banach space ordered by a nonempty convex cone C with apex at the origin and* $\operatorname{int} C \neq \emptyset$. *Let $h : K \to Y$ and $A : L(X,Y) \to L(X,Y)$ be two mappings such that h is convex, and let $\eta : K \times K \to X$ be such that* (a) $\eta(x,z) = \eta(x,y) + \eta(y,z)$, *for all $x,y,z \in K$, and* (b) $\eta(\cdot,\cdot)$ *is affine in the first variable. Suppose that for given multifunction $T : K \to 2^{L(X,Y)}$, the set $\{x \in K : \langle As, \eta(y,x)\rangle + h(y) - h(x) \leq_{C\setminus\{0\}} 0$, for all $s \in Tx\}$ is weakly open in K for every $y \in K$. Then problem GSVVLI has a solution.*

Proof. First, observe that condition (a) implies that for each $x, y \in K$,

$$\eta(x,x) = 0, \qquad \eta(x,y) + \eta(y,x) = 0. \tag{2.1}$$

If problem *GSVVLI* does not have a solution, then for every $x_0 \in K$, there exists some $y \in K$ such that

$$\langle As_0, \eta(y,x_0)\rangle + h(y) - h(x_0) \leq_{C\setminus\{0\}} 0, \quad \forall s_0 \in Tx_0. \tag{2.2}$$

For every $y \in K$, define the set N_y as follows:

$$N_y = \{x \in K : \langle As, \eta(y,x)\rangle + h(y) - h(x) \leq_{C\setminus\{0\}} 0, \ \forall s \in Tx\}. \tag{2.3}$$

By the assumption, the set N_y is weakly open in K for every $y \in K$. It is easy to see that the family $\{N_y : y \in K\}$ is an open cover of K in the weak topology of X.

The weak compactness of K implies that there exists a finite set $\{y_1, y_2, \ldots, y_n\} \subseteq K$ such that

$$K = \bigcup_{i=1}^{n} N_{y_i}. \tag{2.4}$$

Hence there exists a continuous (in the weak topology of X) partition of unity $\{\beta_1, \beta_2, \ldots, \beta_n\}$ subordinated to $\{N_{y_1}, N_{y_2}, \ldots, N_{y_n}\}$ such that $\beta_j(x) \geq 0$, for all $x \in K$, $j = 1,2,\ldots,n$,

$$\sum_{j=1}^{n} \beta_j(x) = 1, \quad \forall x \in K, \tag{2.5}$$

$$\beta_j(x) \begin{cases} = 0 & \text{whenever } x \notin N_{y_j}, \\ > 0 & \text{whenever } x \in N_{y_j}. \end{cases} \tag{2.6}$$

Let $p : K \to X$ be defined as follows:

$$p(x) = \sum_{j=1}^{n} \beta_j(x) y_j, \quad \forall x \in K. \tag{2.7}$$

Since β_j is continuous in the weak topology of X for each j, p is continuous in the weak topology of X. Let $S = \operatorname{co}\{y_1, y_2, \ldots, y_n\}$ be the convex hull of $\{y_1, y_2, \ldots, y_n\}$ in K. Then S

is a simplex of a finite-dimensional space and p maps S into S. By Brouwer's fixed point theorem (Lemma 2.1), there exists some $x_0 \in S$ such that $p(x_0) = x_0$. Now for any given $x \in K$, let

$$k(x) = \{j : x \in N_{y_j}\} = \{j : \beta_j(x) > 0\}. \tag{2.8}$$

Obviously, $k(x) \neq \emptyset$.

Since $x_0 \in S \subseteq K$ is a fixed point of p, we have $p(x_0) = \sum_{j=1}^{n} \beta_j(x_0) y_j$ and hence from the definition of N_y and the convexity of h we derive for each $s_0 \in Tx_0$,

$$\begin{aligned} 0 &= \langle As_0, \eta(x_0, x_0)\rangle + h(x_0) - h(x_0) = \langle As_0, \eta(x_0, p(x_0))\rangle + h(x_0) - h(p(x_0)) \\ &= \left\langle As_0, \eta\left(x_0, \sum_{j=1}^{n} \beta_j(x_0) y_j\right)\right\rangle + h(x_0) - h\left(\sum_{j=1}^{n} \beta_j(x_0) y_j\right) \\ &= -\left\langle As_0, \eta\left(\sum_{j=1}^{n} \beta_j(x_0) y_j, x_0\right)\right\rangle + h(x_0) - h\left(\sum_{j=1}^{n} \beta_j(x_0) y_j\right) \\ &\geq_C -\sum_{j=1}^{n} \beta_j(x_0)\langle As_0, \eta(y_j, x_0)\rangle + h(x_0) - \sum_{j=1}^{n} \beta_j(x_0) h(y_j) \\ &= \sum_{j=1}^{n} \beta_j(x_0)\left[-\langle As_0, \eta(y_j, x_0)\rangle + h(x_0) - h(y_j)\right] \\ &= -\sum_{j=1}^{n} \beta_j(x_0)\left[\langle As_0, \eta(y_j, x_0)\rangle + h(y_j) - h(x_0)\right] \geq_{C\setminus\{0\}} 0, \end{aligned} \tag{2.9}$$

which leads to a contradiction. Therefore, there exist $x^* \in K$ and $s^* \in Tx^*$ such that

$$\langle As^*, \eta(y, x^*)\rangle + h(y) - h(x^*) \not\leq_{C\setminus\{0\}} 0, \quad \forall y \in K. \tag{2.10}$$

This completes the proof. □

Theorem 2.3. *Let K be a nonempty, closed, and convex subset of a real reflexive Banach space X with $0 \in K$ and let Y be a real Banach space ordered by a nonempty convex cone C with apex at the origin and $\operatorname{int} C \neq \emptyset$. Let $h : K \to Y$ and $A : L(X,Y) \to L(X,Y)$ be two mappings such that h is convex, and let $\eta : K \times K \to X$ be such that* (a) *$\eta(x,z) = \eta(x,y) + \eta(y,z)$, for all $x, y, z \in K$, and* (b) *$\eta(\cdot, y) : K \to X$ is affine for each $y \in K$. Suppose that for a given multifunction $T : K \to 2^{L(X,Y)}$, there exists some $r > 0$ such that the following conditions hold:*

(i) *for every $y \in K \cap B_r$, the set $\{x \in K \cap B_r : \langle As, \eta(y,x)\rangle + h(y) - h(x) \leq_{C\setminus\{0\}} 0$, for all $s \in Tx\}$ is weakly open in K, where $B_r = \{x \in X : \|x\| \leq r\}$;*

(ii) *$\langle At, \eta(y,0)\rangle + h(y) - h(0) \geq_{C\setminus\{0\}} 0$, for all $t \in Ty$, $y \in K$ with $\|y\| = r$.*

Then problem GSVVLI has a solution.

Proof. First, observe that condition (a) implies that for each $x, y \in K$,

$$\eta(x,x) = 0, \qquad \eta(x,y) + \eta(y,x) = 0. \tag{2.11}$$

Moreover, according to Theorem 2.2 there exist $x_r \in K \cap B_r$ and $s_r \in Tx_r$ such that

$$\langle As_r, \eta(y, x_r)\rangle + h(y) - h(x_r) \not\leq_{C\setminus\{0\}} 0, \quad \forall y \in K \cap B_r. \tag{2.12}$$

Putting $y = 0$ in the above inequality, one has

$$\langle As_r, \eta(0, x_r)\rangle + h(0) - h(x_r) \not\leq_{C\setminus\{0\}} 0, \tag{2.13}$$

which implies that

$$\langle As_r, \eta(x_r, 0)\rangle + h(x_r) - h(0) \not\geq_{C\setminus\{0\}} 0. \tag{2.14}$$

Combining condition (ii) with (2.14), we know that $\|x_r\| < r$. For any $z \in K$, choose $\lambda \in (0,1)$ enough small such that $(1-\lambda)x_r + \lambda z \in K \cap B_r$.

Putting $y = (1-\lambda)x_r + \lambda z$ in (2.12), one has

$$\langle As_r, \eta((1-\lambda)x_r + \lambda z, x_r)\rangle + h((1-\lambda)x_r + \lambda z) - h(x_r) \not\leq_{C\setminus\{0\}} 0. \tag{2.15}$$

Since h is convex and $\eta(\cdot,\cdot)$ is affine in the first variable, we have

$$\begin{aligned}
&\langle As_r, \eta((1-\lambda)x_r + \lambda z, x_r)\rangle + h((1-\lambda)x_r + \lambda z) - h(x_r) \\
&\quad \leq_C (1-\lambda)\langle As_r, \eta(x_r, x_r)\rangle + \lambda\langle As_r, \eta(z, x_r)\rangle + (1-\lambda)h(x_r) + \lambda h(z) - h(x_r) \\
&\quad = \lambda[\langle As_r, \eta(z, x_r)\rangle + h(z) - h(x_r)].
\end{aligned} \tag{2.16}$$

Now we claim that

$$\langle As_r, \eta(z, x_r)\rangle + h(z) - h(x_r) \not\leq_{C\setminus\{0\}} 0, \quad \forall z \in K. \tag{2.17}$$

Indeed, suppose to the contrary that

$$\langle As_r, \eta(z_0, x_r)\rangle + h(z_0) - h(x_r) \leq_{C\setminus\{0\}} 0 \tag{2.18}$$

for some $z_0 \in K$. Since $-(C \setminus \{0\})$ is a convex cone, we have

$$\lambda[\langle As_r, \eta(z_0, x_r)\rangle + h(z_0) - h(x_r)] \in -(C \setminus \{0\}). \tag{2.19}$$

Observe that

$$\begin{aligned}
&\langle As_r, \eta((1-\lambda)x_r + \lambda z_0, x_r)\rangle + h((1-\lambda)x_r + \lambda z_0) - h(x_r) \\
&\quad = \langle As_r, \eta((1-\lambda)x_r + \lambda z_0, x_r)\rangle + h((1-\lambda)x_r + \lambda z_0) - h(x_r) \\
&\qquad - \lambda[\langle As_r, \eta(z_0, x_r)\rangle + h(z_0) - h(x_r)] + \lambda[\langle As_r, \eta(z_0, x_r)\rangle + h(z_0) - h(x_r)] \\
&\quad \in -C - (C \setminus \{0\}) = -(C \setminus \{0\}),
\end{aligned} \tag{2.20}$$

which implies that

$$\langle As_r, \eta((1-\lambda)x_r + \lambda z_0, x_r)\rangle + h((1-\lambda)x_r + \lambda z_0) - h(x_r) \leq_{C\setminus\{0\}} 0. \tag{2.21}$$

This contradicts (2.15). Therefore, (2.17) holds, that is, x_r is a solution of problem *GSVVLI*. This completes the proof. □

3. Solvability of the *GSVVLI* with pseudomonotonicity

In this section, we will prove the solvability of the *GSVVLI* with pseudomonotonicity assumption by using Fan's lemma [16] and Nadler's theorem [15]. First we give some concepts and lemmas.

Let D be a nonempty subset of a topological vector space E. A multivalued map $G: D \to 2^E$ is called a KKM-map if for each finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq D$,

$$\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i), \tag{3.1}$$

where $\operatorname{co}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

Lemma 3.1 (Fan's lemma [16]). *Let D be an arbitrary nonempty subset of a Hausdorff topological vector space E. Let the multivalued mapping $G: D \to 2^E$ be a KKM-map such that $G(x)$ is closed for all $x \in D$ and is compact for at least one $x \in D$. Then*

$$\bigcap_{x \in D} G(x) \neq \emptyset. \tag{3.2}$$

Lemma 3.2. *Let K be a nonempty closed convex subset of a real Banach space X and let Y be a real Banach space ordered by a pointed closed convex cone C with its apex at the origin and* $\operatorname{int} C \neq \emptyset$. *Let $h: K \to Y$ be convex, and let $A: L(X,Y) \to L(X,Y)$ be continuous. Let $\eta: K \times K \to X$ be such that* (a) $\eta(x,x) = 0$, *for all $x \in K$, and* (b) $\eta(\cdot, y): K \to X$ *is affine for each $y \in K$. Let $T: K \to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction which is H-uniformly continuous and η-pseudomonotone with respect to A and h. Then the following are equivalent:*

(i) *there exist $x^* \in K$ and $s^* \in Tx^*$ such that*

$$\langle As^*, \eta(y, x^*) \rangle + h(y) - h(x^*) \not\leq_{C \setminus \{0\}} 0, \quad \forall y \in K; \tag{3.3}$$

(ii) *there exists $x^* \in K$ such that*

$$\langle At, \eta(y, x^*) \rangle + h(y) - h(x^*) \geq_C 0, \quad \forall y \in K,\ t \in Ty. \tag{3.4}$$

Proof. Suppose that there exist $x^* \in K$ and $s^* \in Tx^*$ such that

$$\langle As^*, \eta(y, x^*) \rangle + h(y) - h(x^*) \not\leq_{C \setminus \{0\}} 0, \quad \forall y \in K. \tag{3.5}$$

Since T is η-pseudomonotone with respect to A and h,

$$\langle At, \eta(y, x^*) \rangle + h(y) - h(x^*) \geq_C 0, \quad \forall y \in K,\ t \in Ty. \tag{3.6}$$

Conversely, suppose that there exists $x^* \in K$ such that

$$\langle At, \eta(y, x^*) \rangle + h(y) - h(x^*) \geq_C 0, \quad \forall y \in K,\ t \in Ty. \tag{3.7}$$

For any given $y \in K$, we know that $y_\lambda = \lambda y + (1-\lambda)x^* \in K$, for all $\lambda \in (0,1)$ since K is convex. Replacing y by y_λ in the above inequality, one derives for each $t_\lambda \in Ty_\lambda$,

$$\begin{aligned} 0 &\leq_C \langle At_\lambda, \eta(y_\lambda, x^*)\rangle + h(y_\lambda) - h(x^*) \\ &= \langle At_\lambda, \eta(\lambda y + (1-\lambda)x^*, x^*)\rangle + h(\lambda y + (1-\lambda)x^*) - h(x^*) \\ &\leq_C \lambda\langle At_\lambda, \eta(y, x^*)\rangle + (1-\lambda)\langle At_\lambda, \eta(x^*, x^*)\rangle \\ &\quad + \lambda h(y) + (1-\lambda)h(x^*) - h(x^*) \\ &= \lambda\big[\langle At_\lambda, \eta(y, x^*)\rangle + h(y) - h(x^*)\big]. \end{aligned} \tag{3.8}$$

Hence, we have

$$\langle At_\lambda, \eta(y, x^*)\rangle + h(y) - h(x^*) \geq_C 0, \quad \forall t_\lambda \in Ty_\lambda,\ \lambda \in (0,1). \tag{3.9}$$

Since Ty_λ and Tx^* are compact, from Lemma 1.6 it follows that for each fixed $t_\lambda \in Ty_\lambda$ there exists an $s_\lambda \in Tx^*$ such that

$$\|t_\lambda - s_\lambda\| \leq H(Ty_\lambda, Tx^*). \tag{3.10}$$

Since Tx^* is compact, without loss of generality, we may assume that $s_\lambda \to s^* \in Tx^*$ as $\lambda \to 0^+$. Since T is H-uniformly continuous and $\|y_\lambda - x^*\| = \lambda\|y - x^*\| \to 0$ as $\lambda \to 0^+$, so $H(Ty_\lambda, Tx^*) \to 0$ as $\lambda \to 0^+$. Thus one has

$$\|t_\lambda - s^*\| \leq \|t_\lambda - s_\lambda\| + \|s_\lambda - s^*\| \leq H(Ty_\lambda, Tx^*) + \|s_\lambda - s^*\| \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+. \tag{3.11}$$

Note that A is continuous. Therefore, letting $\lambda \to 0^+$, we obtain

$$\begin{aligned} \|\langle At_\lambda, \eta(y, x^*)\rangle - \langle As^*, \eta(y, x^*)\rangle\| &= \|\langle At_\lambda - As^*, \eta(y, x^*)\rangle\| \\ &\leq \|At_\lambda - As^*\|\|\eta(y, x^*)\| \longrightarrow 0. \end{aligned} \tag{3.12}$$

Also from (3.9) we deduce that $\langle At_\lambda, \eta(y, x^*)\rangle + h(y) - h(x^*) \in C$. Since C is closed, we have that $\langle As^*, \eta(y, x^*)\rangle + h(y) - h(x^*) \in C$, and hence

$$\langle As^*, \eta(y, x^*)\rangle + h(y) - h(x^*) \not\leq_{C\setminus\{0\}} 0. \tag{3.13}$$

Next, we claim that there holds

$$\langle As^*, \eta(z, x^*)\rangle + h(z) - h(x^*) \not\leq_{C\setminus\{0\}} 0, \quad \forall z \in K. \tag{3.14}$$

Indeed, let z be an arbitrary element in K and set $z_\lambda = \lambda z + (1-\lambda)x^*$ for each $\lambda \in (0,1)$. Then one has $\|y_\lambda - z_\lambda\| = \lambda\|y - z\| \to 0$ as $\lambda \to 0^+$. Hence from the H-uniform continuity of T it follows that $H(Ty_\lambda, Tz_\lambda) \to 0$ as $\lambda \to 0^+$. Let $\{t_\lambda\}_{\lambda\in(0,1)}$ be the net chosen as above such that $t_\lambda \to s^*$ as $\lambda \to 0^+$. Since Ty_λ and Tz_λ are compact, from Lemma 1.6 it follows that for each fixed $t_\lambda \in Ty_\lambda$ there exists an $r_\lambda \in Tz_\lambda$ such that

$$\|t_\lambda - r_\lambda\| \leq H(Ty_\lambda, Tz_\lambda). \tag{3.15}$$

Consequently, one has

$$||r_\lambda - s^*|| \le ||t_\lambda - r_\lambda|| + ||t_\lambda - s^*|| \le H(Ty_\lambda, Tz_\lambda) + ||t_\lambda - s^*|| \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+. \tag{3.16}$$

Note that A is continuous. Thus letting $\lambda \to 0^+$, we obtain

$$\begin{aligned} ||\langle Ar_\lambda, \eta(z,x^*)\rangle - \langle As^*, \eta(z,x^*)\rangle|| &= ||\langle Ar_\lambda - As^*, \eta(z,x^*)\rangle|| \\ &\le ||Ar_\lambda - As^*|| ||\eta(z,x^*)|| \longrightarrow 0. \end{aligned} \tag{3.17}$$

Replacing y, y_λ, and t_λ in (3.9) by z, z_λ, and r_λ, respectively, one deduces that

$$\langle Ar_\lambda, \eta(z,x^*)\rangle + h(z) - h(x^*) \ge_C 0, \quad \forall \lambda \in (0,1), \tag{3.18}$$

which hence implies that $\langle Ar_\lambda, \eta(z,x^*)\rangle + h(z) - h(x^*) \in C$. Since C is closed, one has that $\langle As^*, \eta(z,x^*)\rangle + h(z) - h(x^*) \in C$, and hence

$$\langle As^*, \eta(z,x^*)\rangle + h(z) - h(x^*) \not\le_{C\setminus\{0\}} 0. \tag{3.19}$$

Therefore, according to the arbitrariness of z the assertion is valid. This completes the proof. □

Theorem 3.3. *Let K be a nonempty, bounded, closed, and convex subset of a real reflexive Banach space X and let Y be a real Banach space ordered by a pointed closed convex cone C with its apex at the origin and $\operatorname{int} C \ne \emptyset$. Let $h : K \to Y$ be convex and continuous from the weak topology of X to the strong topology of Y, and let $A : L(X,Y) \to L(X,Y)$ be continuous. Let $\eta : K \times K \to X$ be such that* (a) *$\eta(x,z) = \eta(x,y) + \eta(y,z)$, for all $x,y,z \in K$, and* (b) *for each $y \in K$, $\eta(\cdot,y) : K \to X$ is affine and continuous from the weak topology of X to the strong topology of X. Let $T : K \to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction which is H-uniformly continuous and η-pseudomonotone with respect to A and h. Then the GSVVLI has a solution.*

Proof. We define two multivalued maps $F, G : K \to 2^K$ as follows:

$$\begin{aligned} F(y) &= \{x \in K : \langle As, \eta(y,x)\rangle + h(y) - h(x) \not\le_{C\setminus\{0\}} 0 \text{ for some } s \in Tx\}, \quad \forall y \in K, \\ G(y) &= \{x \in K : \langle At, \eta(y,x)\rangle + h(y) - h(x) \ge_C 0, \ \forall t \in Ty\}, \quad \forall y \in K. \end{aligned} \tag{3.20}$$

Obviously, both $F(y)$ and $G(y)$ are nonempty since $y \in F(y) \cap G(y)$ for all $y \in K$. We claim that F is a KKM mapping. If this is false, then there exist a finite set $\{y_1, y_2, \dots, y_n\} \subseteq K$ and $\alpha_i \ge 0$, $i = 1,2,\dots,n$, with $\sum_{i=1}^n \alpha_i = 1$ such that

$$\overline{y} = \sum_{i=1}^n \alpha_i y_i \notin \bigcup_{i=1}^n F(y_i). \tag{3.21}$$

Hence for any $\overline{t} \in T\overline{y}$ one has

$$\langle A\overline{t}, \eta(y_i, \overline{y})\rangle + h(y_i) - h(\overline{y}) \le_{C\setminus\{0\}} 0, \quad i = 1,2,\dots,n. \tag{3.22}$$

Since $\eta(\cdot,\cdot)$ is affine in the first variable and h is convex, it follows that

$$\begin{aligned}
0 &= \langle A\bar{t},\eta(\bar{y},\bar{y})\rangle + h(\bar{y}) - h(\bar{y}) = -\left\langle A\bar{t},\eta\left(\sum_{i=1}^{n}\alpha_i y_i,\bar{y}\right)\right\rangle + h(\bar{y}) - h\left(\sum_{i=1}^{n}\alpha_i y_i\right) \\
&\geq_C -\sum_{i=1}^{n}\alpha_i\langle A\bar{t},\eta(y_i,\bar{y})\rangle + h(\bar{y}) - \sum_{i=1}^{n}\alpha_i h(y_i) \\
&= -\sum_{i=1}^{n}\alpha_i\big[\langle A\bar{t},\eta(y_i,\bar{y})\rangle + h(y_i) - h(\bar{y})\big] \geq_{C\setminus\{0\}} 0,
\end{aligned} \tag{3.23}$$

which leads to a contradiction. So F is a KKM mapping. Furthermore, it is clear that $F(y)\subseteq G(y)$ for every $y\in K$ since T is η-pseudomonotone with respect to A and h. Thus, G is also a KKM mapping. Now we claim that $G(y)\subseteq K$ is weakly closed for each $y\in K$. Indeed, suppose $\{x_n\}\subseteq G(y)$ is a sequence such that x_n converges weakly to $\bar{x}\in K$. Then we derive for each $t\in Ty$,

$$-\big[\langle At,\eta(x_n,y)\rangle + h(x_n) - h(y)\big] = \langle At,\eta(y,x_n)\rangle + h(y) - h(x_n) \geq_C 0, \quad \forall n. \tag{3.24}$$

Since $A: L(X,Y)\to L(X,Y)$ is continuous, let $h: K\to Y$ be convex and continuous from the weak topology of X to the strong topology of Y, and for each $y\in K$, $\eta(\cdot,y): K\to X$ is continuous from the weak topology of X to the strong topology of X, hence we have

$$-\big[\langle At,\eta(x_n,y)\rangle + h(x_n) - h(y)\big] \longrightarrow -\big[\langle At,\eta(\bar{x},y)\rangle + h(\bar{x}) - h(y)\big] \quad \text{as } n\longrightarrow\infty. \tag{3.25}$$

Also, since C is closed,

$$-\big[\langle At,\eta(\bar{x},y)\rangle + h(\bar{x}) - h(y)\big] \in C. \tag{3.26}$$

Thus we get

$$\langle At,\eta(y,\bar{x})\rangle + h(y) - h(\bar{x}) \geq_C 0, \quad \forall t\in Ty, \tag{3.27}$$

and so $\bar{x}\in G(y)$. This shows that $G(y)$ is weakly closed for each $y\in K$. Since X is reflexive and $K\subseteq X$ is nonempty, bounded, closed, and convex, K is a weakly compact subset of X and so $G(y)$ is also weakly compact. According to Lemma 3.1,

$$\bigcap_{y\in K} G(y) \neq \emptyset. \tag{3.28}$$

This implies that there exists $x^*\in K$ such that

$$\langle At,\eta(y,x^*)\rangle + h(y) - h(x^*) \geq_C 0, \quad \forall y\in K,\ t\in Ty. \tag{3.29}$$

Therefore, by Lemma 3.2 we know that the *GSVVLI* has a solution. □

THEOREM 3.4. *Let K be a nonempty, unbounded, closed, and convex subset of a real reflexive Banach space X with $0\in K$ and let Y be a real Banach space ordered by a pointed closed convex cone C with its apex at the origin and* $\operatorname{int} C\neq\emptyset$. *Let $h: K\to Y$ be convex and continuous* *from the weak topology of* X *to the strong topology of* Y*, and let* $A : L(X,Y) \to L(X,Y)$ *be continuous. Let* $\eta : K \times K \to X$ *be such that* (a) $\eta(x,z) = \eta(x,y) + \eta(y,z)$*, for all* $x,y,z \in K$*, and* (b) *for each* $y \in K$*,* $\eta(\cdot,y) : K \to X$ *is affine and continuous from the weak topology of* X *to the strong topology of* X*. Let* $T : K \to 2^{L(X,Y)}$ *be a nonempty compact-valued multifunction which is* H*-uniformly continuous and* η*-pseudomonotone with respect to* A *and* h*. If there exists some* $r > 0$ *such that*

$$\langle At, \eta(y,0)\rangle + h(y) - h(0) \geq_{C\setminus\{0\}} 0, \quad \forall t \in Ty,\ y \in K \text{ with } \|y\| = r, \tag{3.30}$$

then the GSVVLI is solvable.

Proof. According to Theorem 3.3 there exist $x_r \in K \cap B_r$ and $s_r \in Tx_r$ such that

$$\langle As_r, \eta(y,x_r)\rangle + h(y) - h(x_r) \not\leq_{C\setminus\{0\}} 0, \quad \forall y \in K \cap B_r. \tag{3.31}$$

Since the remainder of the proof is similar to that of Theorem 2.3, we omit it. This completes the proof. □

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Lu-Chuan Ceng: Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
Email address: zenglc@hotmail.com

Yen-Cherng Lin: General Education Center, China Medical University, Taichung 404, Taiwan
Email address: yclin@mail.cmu.edu.tw

Jen-Chih Yao: Department of Applied Mathematics, National Sun Yat-sen University,
Kaohsiung 804, Taiwan
Email address: yaojc@math.nsysu.edu.tw