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*Research Article*
# Stability of Cubic Functional Equation in the Spaces of Generalized Functions

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In this paper, we reformulate and prove the Hyers-Ulam-Rassias stability theorem of the cubic functional equation $f(ax+y)+f(ax-y)=af(x+y)+af(x-y)+2a(a^2-1)f(x)$ for fixed integer $a$ with $a\neq 0,\pm 1$ in the spaces of Schwartz tempered distributions and Fourier hyperfunctions.



## 1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of group homomorphisms:

"*Let $f$ be a mapping from a group $G_1$ to a metric group $G_2$ with metric $d(\cdot,\cdot)$ such that*

$$d\big(f(xy),f(x)f(y)\big)\leq \varepsilon. \tag{1.1}$$

*Then does there exist a group homomorphism $L: G_1\to G_2$ and $\delta_\epsilon>0$ such that*

$$d\big(f(x),L(x)\big)\leq \delta_\epsilon \tag{1.2}$$

*for all $x\in G_1$?*"

The case of approximately additive mappings was solved by Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, Rassias [3] firstly generalized Hyers' result to the unbounded Cauchy difference. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4–12]). The terminology *Hyers-Ulam-Rassias stability* originates from these historical backgrounds and this terminology is also applied to the case of other functional equations.

Let both $E_1$ and $E_2$ be real vector spaces. Jun and Kim [13] proved that a function $f : E_1 \to E_2$ satisfies the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

if and only if there exists a mapping $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x,x,x)$ for all $x \in E_1$, where $B$ is symmetric for each fixed one variable and additive for each fixed two variables. The mapping $B$ is given by

$$B(x,y,z)=\frac{1}{24}\left[f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)\right] \tag{1.4}$$

for all $x,y,z \in E_1$. It is natural that (1.3) is called a cubic functional equation because the mapping $f(x) = ax^3$ satisfies (1.3). Also Jun et al. generalized cubic functional equation, which is equivalent to (1.3),

$$f(ax+y)+f(ax-y)=af(x+y)+af(x-y)+2a(a^2-1)f(x) \tag{1.5}$$

for fixed integer $a$ with $a \neq 0, \pm 1$ (see [14]).

In this paper, we consider the general solution of (1.5) and prove the stability theorem of this equation in the space $\mathcal{S}'(\mathbb{R}^n)$ of Schwartz tempered distributions and the space $\mathcal{F}'(\mathbb{R}^n)$ of Fourier hyperfunctions. Following the notations as in [15, 16] we reformulate (1.5) and related inequality as

$$u \circ A_1 + u \circ A_2 = au \circ B_1 + au \circ B_2 + 2a(a^2-1)u \circ P, \tag{1.6}$$

$$\|u \circ A_1 + u \circ A_2 - au \circ B_1 - au \circ B_2 - 2a(a^2-1)u \circ P\| \le \epsilon(|x|^p + |y|^q), \tag{1.7}$$

respectively, where $A_1$, $A_2$, $B_1$, $B_2$, and $P$ are the functions defined by

$$\begin{gathered} A_1(x,y)=ax+y, \qquad A_2(x,y)=ax-y, \\ B_1(x,y)=x+y, \qquad B_2(x,y)=x-y, \qquad P(x,y)=x, \end{gathered} \tag{1.8}$$

and $p$, $q$ are nonnegative real numbers with $p,q \neq 3$. We note that $p$ need not be equal to $q$. Here $u \circ A_1$, $u \circ A_2$, $u \circ B_1$, $u \circ B_2$, and $u \circ P$ are the pullbacks of $u$ in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ by $A_1$, $A_2$, $B_1$, $B_2$, and $P$, respectively. Also $|\cdot|$ denotes the Euclidean norm, and the inequality $\|v\| \le \psi(x,y)$ in (1.7) means that $|\langle v,\varphi\rangle| \le \|\psi\varphi\|_{L^1}$ for all test functions $\varphi(x,y)$ defined on $\mathbb{R}^{2n}$.

If $p<0$ or $q<0$, the right-hand side of (1.7) does not define a distribution and so inequality (1.7) makes no sense. If $p,q=3$, it is not guaranteed whether Hyers-Ulam-Rassias stability of (1.5) is hold even in classical case (see [13, 14]). Thus we consider only the case $0 \le p$, $q<3$, or $p,q>3$.

We prove as results that every solution $u$ in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ of inequality (1.7) can be written uniquely in the form

$$u=\sum_{1\le i\le j\le k\le n} a_{ijk}x_ix_jx_k + h(x), \quad a_{ijk} \in \mathbb{C}, \tag{1.9}$$

where $h(x)$ is a measurable function such that

$$|h(x)| \le \frac{\epsilon}{2\,\big||a|^3 - |a|^p\big|}|x|^p. \tag{1.10}$$

## 2. Preliminaries

We first introduce briefly spaces of some generalized functions such as Schwartz tempered distributions and Fourier hyperfunctions. Here we use the multi-index notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

*Definition 2.1* [17, 18]. Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions $\varphi$ in $\mathbb{R}^n$ satisfying

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. A linear form $u$ on $\mathcal{S}(\mathbb{R}^n)$ is said to be *Schwartz tempered distribution* if there is a constant $C \ge 0$ and a nonnegative integer $N$ such that

$$|\langle u, \varphi\rangle| \le C \sum_{|\alpha|,|\beta| \le N} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| \tag{2.2}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all Schwartz tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Imposing growth conditions on $\|\cdot\|_{\alpha,\beta}$ in (2.1), Sato and Kawai introduced the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions.

*Definition 2.2* [19]. Denote by $\mathcal{F}(\mathbb{R}^n)$ the Sato space of all infinitely differentiable functions $\varphi$ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty \tag{2.3}$$

for some positive constants $A$, $B$ depending only on $\varphi$. We say that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{A,B} \to 0$ as $j \to \infty$ for some $A, B > 0$, and denote by $\mathcal{F}'(\mathbb{R}^n)$ the strong dual of $\mathcal{F}(\mathbb{R}^n)$ and call its elements *Fourier hyperfunctions*.

It can be verified that the seminorms (2.3) are equivalent to

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n, \alpha\in\mathbb{N}_0^n} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|} \alpha!} < \infty \tag{2.4}$$

for some constants $h, k > 0$. It is easy to see the following topological inclusion:

$$\mathcal{F}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n), \qquad \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{F}'(\mathbb{R}^n). \tag{2.5}$$

In order to solve (1.6), we employ the $n$-dimensional heat kernel, that is, the fundamental solution $E_t(x)$ of the heat operator $\partial_t - \triangle_x$ in $\mathbb{R}^n_x \times \mathbb{R}^+_t$ given by

$$E_t(x) = \begin{cases} (4\pi t)^{-n/2} \exp\left( - \dfrac{|x|^2}{4t} \right), & t > 0, \\ 0, & t \le 0. \end{cases} \tag{2.6}$$

Since for each $t > 0$, $E_t(\cdot)$ belongs to $\mathcal{S}(\mathbb{R}^n)$, the convolution

$$\tilde{u}(x,t) = (u * E_t)(x) = \langle u_y, E_t(x-y) \rangle, \quad x \in \mathbb{R}^n,\ t > 0, \tag{2.7}$$

is well defined for each $u \in \mathcal{S}'(\mathbb{R}^n)$ and $u \in \mathcal{F}'(\mathbb{R}^n)$, which is called the *Gauss transform* of $u$. Also we use the following result which is called the *heat kernel method* (see [20]).

Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then its Gauss transform $\tilde{u}(x,t)$ is a $C^\infty$-solution of the heat equation

$$\left( \frac{\partial}{\partial t} - \Delta \right) \tilde{u}(x,t) = 0 \tag{2.8}$$

satisfying the following.

(i) There exist positive constants $C$, $M$, and $N$ such that

$$|\tilde{u}(x,t)| \le C t^{-M} (1 + |x|)^N \quad \text{in } \mathbb{R}^n \times (0,\delta). \tag{2.9}$$

(ii) $\tilde{u}(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t) \varphi(x) dx. \tag{2.10}$$

Conversely, every $C^\infty$-solution $U(x,t)$ of the heat equation satisfying the growth condition (2.9) can be uniquely expressed as $U(x,t) = \tilde{u}(x,t)$ for some $u \in \mathcal{S}'(\mathbb{R}^n)$.

Similarly, we can represent Fourier hyperfunctions as initial values of solutions of the heat equation as a special case of the results (see [21]). In this case, the estimate (2.9) is replaced by the following.

For every $\varepsilon > 0$ there exists a positive constant $C_\varepsilon$ such that

$$|\tilde{u}(x,t)| \le C_\varepsilon \exp\left( \varepsilon \left( |x| + \frac{1}{t} \right) \right) \quad \text{in } \mathbb{R}^n \times (0,\delta). \tag{2.11}$$

We refer to [17, Chapter VI] for pullbacks and to [16, 18, 20] for more details of $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$.

## 3. General solution in $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$

Jun and Kim (see [22]) showed that every continuous solution of (1.5) in $\mathbb{R}$ is a cubic function $f(x) = f(1)x^3$ for all $x \in \mathbb{R}$. Using induction argument on the dimension $n$, it is easy to see that every continuous solution of (1.5) in $\mathbb{R}^n$ is a cubic form

$$f(x) = \sum_{1 \le i \le j \le k \le n} a_{ijk} x_i x_j x_k, \quad a_{ijk} \in \mathbb{C}. \tag{3.1}$$

In this section, we consider the general solution of the cubic functional equation in the spaces of $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$. It is well known that the *semigroup property* of the heat kernel

$$(E_t * E_s)(x) = E_{t+s}(x) \tag{3.2}$$

holds for convolution. Semigroup property will be useful to convert (1.6) into the classical functional equation defined on upper-half plane.

Convolving the tensor product $E_t(\xi)E_s(\eta)$ of $n$-dimensional heat kernels in both sides of (1.6), we have

$$\begin{aligned}
&[(u\circ A_1) * (E_t(\xi)E_s(\eta))](x,y)\\
&\quad= \langle u\circ A_1, E_t(x-\xi)E_s(y-\eta)\rangle = \Big\langle u_\xi, a^{-n}\int E_t\Big(x-\frac{\xi-\eta}{a}\Big)E_s(y-\eta)d\eta\Big\rangle\\
&\quad= \Big\langle u_\xi, a^{-n}\int E_t\Big(\frac{ax+y-\xi-\eta}{a}\Big)E_s(\eta)d\eta\Big\rangle = \Big\langle u_\xi, \int E_{a^2t}(ax+y-\xi-\eta)E_s(\eta)d\eta\Big\rangle\\
&\quad= \langle u_\xi, (E_{a^2t} * E_s)(ax+y-\xi)\rangle = \langle u_\xi, E_{a^2t+s}(ax+y-\xi)\rangle = \tilde{u}(ax+y, a^2t+s),
\end{aligned} \tag{3.3}$$

and similarly we get

$$\begin{aligned}
[(u\circ A_2) * (E_t(\xi)E_s(\eta))](x,y) &= \tilde{u}(ax-y, a^2t+s),\\
[(u\circ B_1) * (E_t(\xi)E_s(\eta))](x,y) &= \tilde{u}(x+y, t+s),\\
[(u\circ B_2) * (E_t(\xi)E_s(\eta))](x,y) &= \tilde{u}(x-y, t+s),\\
[(u\circ P) * (E_t(\xi)E_s(\eta))](x,y) &= \tilde{u}(x,t).
\end{aligned} \tag{3.4}$$

Thus (1.6) is converted into the classical functional equation

$$\begin{aligned}
&\tilde{u}(ax+y, a^2t+s) + \tilde{u}(ax-y, a^2t+s)\\
&\qquad= a\tilde{u}(x+y, t+s) + a\tilde{u}(x-y, t+s) + 2a(a^2-1)\tilde{u}(x,t)
\end{aligned} \tag{3.5}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$.

Lemma 3.1. *Let $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be a continuous function satisfying*

$$\begin{aligned}
&f(ax+y, a^2t+s) + f(ax-y, a^2t+s)\\
&\qquad= af(x+y, t+s) + af(x-y, t+s) + 2a(a^2-1)f(x,t)
\end{aligned} \tag{3.6}$$

*for fixed integer $a$ with $a \neq 0, \pm 1$. Then the solution is of the form*

$$f(x,t) = \sum_{1\le i\le j\le k\le n} a_{ijk}x_ix_jx_k + t\sum_{1\le i\le n} b_ix_i, \quad a_{ijk}, b_i \in \mathbb{C}. \tag{3.7}$$

*Proof.* In view of (3.6) and given the continuity, $f(x,0^+) := \lim_{t\to 0^+} f(x,t)$ exists. Define $h(x,t) := f(x,t) - f(x,0^+)$, then $h(x,0^+) = 0$ and

$$\begin{aligned} &h(ax+y,a^2t+s)+h(ax-y,a^2t+s) \\ &\quad = ah(x+y,t+s)+ah(x-y,t+s)+2a(a^2-1)h(x,t) \end{aligned} \tag{3.8}$$

for all $x,y \in \mathbb{R}^n, t,s > 0$. Setting $y = 0,\ s \to 0^+$ in (3.8), we have

$$h(ax,a^2t) = a^3h(x,t). \tag{3.9}$$

Putting $y = 0,\ s = a^2s$ in (3.8), and using (3.9), we get

$$a^2h(x,t+s) = h(x,t+a^2s)+(a^2-1)h(x,t). \tag{3.10}$$

Letting $t \to 0^+$ in (3.10), we obtain

$$a^2h(x,s) = h(x,a^2s). \tag{3.11}$$

Replacing $t$ by $a^2t$ in (3.10) and using (3.11), we have

$$h(x,a^2t+s) = h(x,t+s)+(a^2-1)h(x,t). \tag{3.12}$$

Switching $t$ with $s$ in (3.12), we get

$$h(x,t+a^2s) = h(x,t+s)+(a^2-1)h(x,s). \tag{3.13}$$

Adding (3.10) to (3.13), we obtain

$$h(x,t+s) = h(x,t)+h(x,s), \tag{3.14}$$

which shows that

$$h(x,t) = h(x,1)t. \tag{3.15}$$

Letting $t \to 0^+,\ s = 1$ in (3.8), we have

$$h(ax+y,1)+h(ax-y,1) = ah(x+y,1)+ah(x-y,1). \tag{3.16}$$

Also letting $t = 1,\ s \to 0^+$ in (3.8), and using (3.11), we get

$$a^2h(ax+y,1)+a^2h(ax-y,1) = ah(x+y,1)+ah(x-y,1)+2a(a^2-1)h(x,1). \tag{3.17}$$

Now taking (3.16) into (3.17), we obtain

$$h(x+y,1)+h(x-y,1) = 2h(x,1). \tag{3.18}$$

Replacing $x,\ y$ by $(x+y)/2,\ y=(x-y)/2$ in (3.18), respectively, we see that $h(x,1)$ satisfies Jensen functional equation

$$2h\left(\frac{x+y}{2},1\right)=h(x,1)+h(y,1). \tag{3.19}$$

Putting $x=y=0$ in (3.16), we get $h(0,1)=0$. This shows that $h(x,1)$ is additive.

On the other hand, letting $t=s\to 0^+$ in (3.6), we can see that $f(x,0^+)$ satisfies (1.5). Given the continuity, the solution $f(x,t)$ is of the form

$$f(x,t)=\sum_{1\le i\le j\le k\le n}a_{ijk}x_ix_jx_k+t\sum_{1\le i\le n}b_ix_i,\quad a_{ijk},b_i\in\mathbb{C}, \tag{3.20}$$

which completes the proof. □

As a direct consequence of the above lemma, we present the general solution of the cubic functional equation in the spaces of $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$.

Theorem 3.2. *Suppose that $u$ in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ satisfies the equation*

$$u\circ A_1+u\circ A_2=au\circ B_1+au\circ B_2+2a(a^2-1)u\circ P \tag{3.21}$$

*for fixed integer $a$ with $a\neq 0,\pm 1$. Then the solution is the cubic form*

$$u=\sum_{1\le i\le j\le k\le n}a_{ijk}x_ix_jx_k,\quad a_{ijk}\in\mathbb{C}. \tag{3.22}$$

*Proof.* Convolving the tensor product $E_t(\xi)E_s(\eta)$ of $n$-dimensional heat kernels in both sides of (3.21), we have the classical functional equation

$$\begin{aligned}&\tilde{u}(ax+y,a^2t+s)+\tilde{u}(ax-y,a^2t+s)\\&\quad=a\tilde{u}(x+y,t+s)+a\tilde{u}(x-y,t+s)+2a(a^2-1)\tilde{u}(x,t)\end{aligned} \tag{3.23}$$

for all $x,y\in\mathbb{R}^n, t,s>0$, where $\tilde{u}$ is the Gauss transform of $u$. By Lemma 3.1, the solution $\tilde{u}$ is of the form

$$\tilde{u}(x,t)=\sum_{1\le i\le j\le k\le n}a_{ijk}x_ix_jx_k+t\sum_{1\le i\le n}b_ix_i,\quad a_{ijk},b_i\in\mathbb{C}. \tag{3.24}$$

Thus we get

$$\langle\tilde{u},\varphi\rangle=\left\langle\sum_{1\le i\le j\le k\le n}a_{ijk}x_ix_jx_k+t\sum_{1\le i\le n}b_ix_i,\varphi\right\rangle \tag{3.25}$$

for all test functions $\varphi$. Now letting $t\to 0^+$, it follows from the heat kernel method that

$$\langle u,\varphi\rangle=\left\langle\sum_{1\le i\le j\le k\le n}a_{ijk}x_ix_jx_k,\varphi\right\rangle \tag{3.26}$$

for all test functions $\varphi$. This completes the proof. □

## 4. Stability in $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$

We are going to prove the stability theorem of the cubic functional equation in the spaces of $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$.

We note that the Gauss transform

$$\psi_p(x,t) := \int |\xi|^p E_t(x-\xi)\,d\xi \tag{4.1}$$

is well defined and $\psi_p(x,t) \to |x|^p$ locally uniformly as $t \to 0^+$. Also $\psi_p(x,t)$ satisfies *semi-homogeneous property*

$$\psi_p(rx, r^2t) = r^p \psi_p(x,t) \tag{4.2}$$

for all $r \geq 0$.

We are now in a position to state and prove the main result of this paper.

Theorem 4.1. *Let $a$ be fixed integer with $a \neq 0, \pm 1$ and let $\epsilon$, $p$, $q$ be real numbers such that $\epsilon \geq 0$ and $0 \leq p$, $q < 3$, or $p,q > 3$. Suppose that $u$ in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ satisfy the inequality*

$$\|u \circ A_1 - u \circ A_2 - au \circ B_1 - au \circ B_2 - 2a(a^2-1)u \circ P\| \leq \epsilon(|x|^p + |y|^q). \tag{4.3}$$

*Then there exists a unique cubic form*

$$c(x) = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k \tag{4.4}$$

*such that*

$$\|u - c(x)\| \leq \frac{\epsilon}{2\big||a|^3 - |a|^p\big|} |x|^p. \tag{4.5}$$

*Proof.* Let $v := u \circ A_1 - u \circ A_2 - au \circ B_1 - au \circ B_2 - 2a(a^2-1)u \circ P$. Convolving the tensor product $E_t(\xi)E_s(\eta)$ of $n$-dimensional heat kernels in $v$, we have

$$\begin{aligned} |[v * (E_t(\xi)E_s(\eta))](x,y)| &= |\langle v, E_t(x-\xi)E_s(y-\eta)\rangle| \\ &\leq \epsilon \|(|\xi|^p + |\eta|^q)E_t(x-\xi)E_s(y-\eta)\|_{L^1} \\ &= \epsilon(\psi_p(x,t) + \psi_q(y,s)). \end{aligned} \tag{4.6}$$

Also we see that, as in Theorem 3.2,

$$\begin{aligned} [v * (E_t(\xi)E_s(\eta))](x,y) = {}& \widetilde{u}(ax+y, a^2t+s) + \widetilde{u}(ax-y, a^2t+s) \\ & - a\widetilde{u}(x+y, t+s) - a\widetilde{u}(x-y, t+s) - 2a(a^2-1)\widetilde{u}(x,t), \end{aligned} \tag{4.7}$$

where $\widetilde{u}$ is the Gauss transform of $u$. Thus inequality (4.3) is converted into the classical functional inequality

$$\begin{aligned}&|\widetilde{u}(ax+y,a^2t+s)+\widetilde{u}(ax-y,a^2t+s)-a\widetilde{u}(x+y,t+s)-a\widetilde{u}(x-y,t+s)-2a(a^2-1)\widetilde{u}(x,t)|\\&\quad\leq\epsilon\left(\psi_p(x,t)+\psi_q(y,s)\right)\end{aligned}\tag{4.8}$$

for all $x,y\in\mathbb{R}^n, t,s>0$.

We first prove for $0\leq p,\ q<3$. Letting $y=0$, $s\to 0^+$ in (4.8) and dividing the result by $2|a|^3$, we get

$$\left|\frac{\widetilde{u}(ax,a^2t)}{a^3}-\widetilde{u}(x,t)\right|\leq\frac{\epsilon}{2|a|^3}\psi_p(x,t).\tag{4.9}$$

By virtue of the semihomogeneous property of $\psi_p$, substituting $x,\ t$ by $ax,\ a^2t$, respectively, in (4.9) and dividing the result by $|a|^3$, we obtain

$$\left|\frac{\widetilde{u}(a^2x,a^4t)}{a^6}-\frac{\widetilde{u}(ax,a^2t)}{a^3}\right|\leq\frac{\epsilon}{2|a|^3}|a|^{p-3}\psi_p(x,t).\tag{4.10}$$

Using induction argument and triangle inequality, we have

$$\left|\frac{\widetilde{u}(a^nx,a^{2n}t)}{a^{3n}}-\widetilde{u}(x,t)\right|\leq\frac{\epsilon}{2|a|^3}\psi_p(x,t)\sum_{j=0}^{n-1}|a|^{(p-3)j}\tag{4.11}$$

for all $n\in\mathbb{N}$, $x\in\mathbb{R}^n$, $t>0$. Let us prove the sequence $\{a^{-3n}\widetilde{u}(a^nx,a^{2n}t)\}$ is convergent for all $x\in\mathbb{R}^n$, $t>0$. Replacing $x,\ t$ by $a^mx,\ a^{2m}t$, respectively, in (4.11) and dividing the result by $|a|^{3m,}$ we see that

$$\left|\frac{\widetilde{u}(a^{m+n}x,a^{2(m+n)}t)}{a^{3(m+n)}}-\frac{\widetilde{u}(a^mx,a^{2m}t)}{a^{3m}}\right|\leq\frac{\epsilon}{2|a|^3}\psi_p(x,t)\sum_{j=m}^{n-1}|a|^{(p-3)j}.\tag{4.12}$$

Letting $m\to\infty$, we have $\{a^{-3n}\widetilde{u}(a^nx,a^{2n}t)\}$ is a Cauchy sequence. Therefore, we may define

$$G(x,t)=\lim_{n\to\infty}a^{-3n}\widetilde{u}(a^nx,a^{2n}t)\tag{4.13}$$

for all $x\in\mathbb{R}^n$, $t>0$.

Now we verify that the given mapping $G$ satisfies (3.6). Replacing $x,\ y,\ t,\ s$ by $a^nx$, $a^ny$, $a^{2n}t$, $a^{2n}s$ in (4.8), respectively, and then dividing the result by $|a|^{3n}$, we get

$$\begin{aligned}&|a|^{-3n}|\widetilde{u}(a^n(ax+y),a^{2n}(a^2t+s))+\widetilde{u}(a^n(ax-y),a^{2n}(a^2t+s))\\&\qquad-a\widetilde{u}(a^n(x+y),a^{2n}(t+s))-a\widetilde{u}(a^n(x+y),a^{2n}(t+s))-2a(a^2-1)\widetilde{u}(a^nx,a^{2n}t)|\\&\quad\leq|a|^{-3n}\left(\psi_p(a^nx,a^{2n}t)+\psi_q(a^ny,a^{2n}s)\right)\\&\quad=\left(|a|^{(p-3)n}\psi_p(x,t)+|a|^{(q-3)n}\psi_q(y,s)\right).\end{aligned}\tag{4.14}$$

Now letting $n \to \infty$, we see by definition of $G$ that $G$ satisfies

$$G(ax+y, a^2t+s) + G(ax-y, a^2t+s) \\ = aG(x+y, t+s) + aG(x-y, t+s) + 2a(a^2-1)G(x,t) \tag{4.15}$$

for all $x, y \in \mathbb{R}^n, t, s > 0$. By Lemma 3.1, $G(x,t)$ is of the form

$$G(x,t) = \sum_{1 \le i \le j \le k \le n} a_{ijk}x_ix_jx_k + t\sum_{1 \le i \le n} b_ix_i, \quad a_{ijk}, b_i \in \mathbb{C}. \tag{4.16}$$

Letting $n \to \infty$ in (4.11) yields

$$\left| G(x,t) - \tilde{u}(x,t) \right| \le \frac{\epsilon}{2(|a|^3 - |a|^p)} \psi_p(x,t). \tag{4.17}$$

To prove the uniqueness of $G(x,t)$, we assume that $H(x,t)$ is another function satisfying (4.15) and (4.17). Setting $y = 0$ and $s \to 0^+$ in (4.15), we have

$$G(ax, a^2t) = a^3G(x,t). \tag{4.18}$$

Then it follows from (4.15), (4.17), and (4.18) that

$$\begin{aligned} &\left| G(x,t) - H(x,t) \right| \\ &\quad = |a|^{-3n} \left| G(a^nx, a^{2n}t) - H(a^nx, a^{2n}t) \right| \le |a|^{-3n} \left| G(a^nx, a^{2n}t) - \tilde{u}(a^nx, a^{2n}t) \right| \\ &\qquad + |a|^{-3n} \left| \tilde{u}(a^nx, a^{2n}t) - H(a^nx, a^{2n}t) \right| \le \frac{\epsilon}{|a|^{3n}(|a|^3 - |a|^p)} \psi_p(x,t) \end{aligned} \tag{4.19}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t > 0$. Letting $n \to \infty$, we have $G(x,t) = H(x,t)$ for all $x \in \mathbb{R}^n$, $t > 0$. This proves the uniqueness.

It follows from the inequality (4.17) that

$$\left| \langle G(x,t) - \tilde{u}(x,t), \varphi \rangle \right| \le \frac{\epsilon}{2(|a|^3 - |a|^p)} \langle \psi_p(x,t), \varphi \rangle \tag{4.20}$$

for all test functions $\varphi$. Letting $t \to 0^+$, we have the inequality

$$\left\| u - \sum_{1 \le i \le j \le k \le n} a_{ijk}x_ix_jx_k \right\| \le \frac{\epsilon}{2\left| |a|^3 - |a|^p \right|}. \tag{4.21}$$

Now we consider the case $p, q > 3$. For this case, replacing $x$, $y$, $t$ by $x/a$, $0$, $t/a^2$ in (4.8), respectively, and letting $s \to 0^+$ and then multiplying the result by $|a|^3$, we have

$$\left| \tilde{u}(x,t) - a^3\tilde{u}\left(\frac{x}{a}, \frac{t}{a^2}\right) \right| \le \frac{\epsilon}{2|a|^3} |a|^{3-p} \psi_p(x,t). \tag{4.22}$$

Substituting $x$, $t$ by $x/a$, $t/a^2$, respectively, in (4.22) and multiplying the result by $|a|^3$ we get

$$\left| a^3\tilde{u}\left(\frac{x}{a}, \frac{t}{a^2}\right) - a^6\tilde{u}\left(\frac{x}{a^2}, \frac{t}{a^4}\right) \right| \le \frac{\epsilon}{2|a|^3} |a|^{2(3-p)} \psi_p(x,t). \tag{4.23}$$

Using induction argument and triangle inequality, we obtain

$$\left|\tilde{u}(x,t) - a^{3n}\tilde{u}\left(\frac{x}{a^n}, \frac{t}{a^{2n}}\right)\right| \le \frac{\epsilon}{2|a|^3}\psi_p(x,t)\sum_{j=1}^{n}|a|^{(3-p)j} \tag{4.24}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t > 0$. Following the same method as in the case $0 \le p$, $q < 3$, we see that

$$G(x,t) := \lim_{n\to\infty} a^{3n}\tilde{u}\left(\frac{x}{a^n}, \frac{t}{a^{2n}}\right) \tag{4.25}$$

is the unique function satisfying (4.15). Letting $n \to \infty$ in (4.24), we get

$$|\tilde{u}(x,t) - C(x,t)| \le \frac{\epsilon}{2(|a|^p - |a|^3)}\psi_p(x,t). \tag{4.26}$$

Now letting $t \to 0^+$ in (4.26), we have the inequality

$$\left\|u - \sum_{1\le i\le j\le k\le n} a_{ijk}x_ix_jx_k\right\| \le \frac{\epsilon}{2\,|\,|a|^p - |a|^3\,|}. \tag{4.27}$$

This completes the proof. □

*Remark 4.2.* The above norm inequality

$$||u - c(x)|| \le \frac{\epsilon}{2\,|\,|a|^p - |a|^3\,|}|x|^p \tag{4.28}$$

implies that $u - c(x)$ is a measurable function. Thus all the solution u in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ can be written uniquely in the form

$$u = c(x) + h(x), \tag{4.29}$$

where $|h(x)| \le (\epsilon/(2||a|^p - |a|^3|))|x|^p$.

Corollary 4.3. *Let a be fixed integer with $a \neq 0, \pm 1$ and $\epsilon \ge 0$. Suppose that u in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ satisfy the inequality*

$$||u \circ A_1 - u \circ A_2 - au \circ B_1 - au \circ B_2 - 2a(a^2 - 1)u \circ P|| \le \epsilon. \tag{4.30}$$

*Then there exists a unique cubic form*

$$c(x) = \sum_{1\le i\le j\le k\le n} a_{ijk}x_ix_jx_k \tag{4.31}$$

*such that*

$$||u - c(x)|| \le \frac{\epsilon}{2(a^3 - 1)}. \tag{4.32}$$

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## References

[1] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, John Wiley & Sons, New York, NY, USA, 1964.
[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.
[3] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[4] J. Chmieliński, "Stability of the orthogonality preserving property in finite-dimensional inner product spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 2, pp. 433–443, 2006.
[5] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[6] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.
[7] K.-W. Jun, G. H. Kim, and Y. W. Lee, "Stability of generalized gamma and beta functional equations," *Aequationes Mathematicae*, vol. 60, no. 1-2, pp. 15–24, 2000.
[8] S.-M. Jung, "Hyers-Ulam stability of Butler-Rassias functional equation," *Journal of Inequalities and Applications*, vol. 2005, no. 1, pp. 41–47, 2005.
[9] T. Miura, S.-E. Takahasi, and G. Hirasawa, "Hyers-Ulam-Rassias stability of Jordan homomorphisms on Banach algebras," *Journal of Inequalities and Applications*, vol. 2005, no. 4, pp. 435–441, 2005.
[10] J. M. Rassias, "Solution of the Ulam stability problem for Euler-Lagrange quadratic mappings," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 2, pp. 613–639, 1998.
[11] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.
[12] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
[13] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 867–878, 2002.
[14] K.-W. Jun, H.-M. Kim, and I.-S. Chang, "On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation," *Journal of Computational Analysis and Applications*, vol. 7, no. 1, pp. 21–33, 2005.
[15] J. Chung, "Stability of functional equations in the spaces of distributions and hyperfunctions," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 1, pp. 177–186, 2003.
[16] J. Chung, S.-Y. Chung, and D. Kim, "The stability of Cauchy equations in the space of Schwartz distributions," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 107–114, 2004.
[17] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I: Distribution Theory and Fourier Analysis*, vol. 256 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1983.
[18] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, France, 1966.
[19] J. Chung, S.-Y. Chung, and D. Kim, "A characterization for Fourier hyperfunctions," *Publications of the Research Institute for Mathematical Sciences*, vol. 30, no. 2, pp. 203–208, 1994.

[20] T. Matsuzawa, "A calculus approach to hyperfunctions—III," *Nagoya Mathematical Journal*, vol. 118, pp. 133–153, 1990.

[21] K. W. Kim, S.-Y. Chung, and D. Kim, "Fourier hyperfunctions as the boundary values of smooth solutions of heat equations," *Publications of the Research Institute for Mathematical Sciences*, vol. 29, no. 2, pp. 289–300, 1993.

[22] K.-W. Jun and H.-M. Kim, "On the stability of Euler-Lagrange type cubic mappings in quasi-Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1335–1350, 2007.

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