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Research Article

Rearrangement and Convergence in Spaces of Measurable Functions

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We prove that the convergence of a sequence of functions in the space L_0 of measurable functions, with respect to the topology of convergence in measure, implies the convergence μ-almost everywhere (μ denotes the Lebesgue measure) of the sequence of rearrangements. We obtain nonexpansivity of rearrangement on the space L_∞, and also on Orlicz spaces L_N with respect to a finitely additive extended real-valued set function. In the space L_∞ and in the space E_Φ, of finite elements of an Orlicz space L_Φ of a σ-additive set function, we introduce some parameters which estimate the Hausdorff measure of noncompactness. We obtain some relations involving these parameters when passing from a bounded set of L_∞, or L_Φ, to the set of rearrangements.

1. Introduction

The notion of rearrangement of a real-valued μ-measurable function was introduced by Hardy et al. in [1]. It has been studied by many authors and leads to interesting results in Lebesgue spaces and, more generally, in Orlicz spaces (see, e.g., [2–5]). The space L_0 is a space of real-valued *measurable functions*, defined on a nonempty set Ω, in which we can give a natural generalization of the topology of convergence in measure using a group pseudonorm which depends on a submeasure defined on the power set $\mathcal{P}(\Omega)$ of Ω (see [6, 7] and the references given there). In the second section of this note we study rearrangements of functions of the space L_0. The rearrangements belong to the space $T_0([0,+\infty))$ of all real-valued totally μ-measurable functions defined on $[0,+\infty)$. We extend to this setting some convergence results (see, e.g., [3, 5]). Precisely, we prove that the convergence in the space L_0 implies the convergence μ-almost everywhere of rearrangements. Moreover, by the convergence in L_0 of a nondecreasing sequence of nonnegative functions, we obtain the convergence in measure of the corresponding nondecreasing sequence of rearrangements. In the third section we introduce, in a natural manner, the space L_∞ as the closure of the subspace of all simple functions of L_0 with respect to the essentially supremum norm. The space L_∞ so defined is contained in L_0, and we prove nonexpansivity of rearrangement on this space. In the last section we obtain nonexpansivity of rearrangement on Orlicz spaces L_N of a finitely additive extended real-valued set function.

We recall (see [8]) that for a bounded subset Y of a normed space $(X,\|\cdot\|)$ the *Hausdorff measure of noncompactness* $\gamma_X(Y)$ of Y is defined by

$$\gamma_X(Y)=\inf\{\varepsilon>0:\ \text{there is a finite subset } F \text{ of } X \text{ such that } Y\subseteq\cup_{f\in F}B_X(f,\varepsilon)\},\tag{1.1}$$

where $B_X(f,\varepsilon)=\{g\in X:\|f-g\|\le\varepsilon\}$. In sections 3 and 4 we introduce, respectively, a parameter ω_{L_∞} in L_∞ and a parameter ω_{E_Φ} in the space E_Φ of finite elements of a classical Orlicz space L_Φ of a σ-additive set function. By means of these parameters, we derive an exact formula in L_∞ and an estimate in E_Φ for the Hausdorff measure of noncompactness. Then as a consequence of nonexpansivity of rearrangement we obtain inequalities involving such parameters, when passing from a set of functions in L_∞, or in L_Φ, to the set of rearrangements. We denote by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ the set of all natural, rational, and real numbers, respectively.,

2. Rearrangements of functions and convergence in the space L_0

Let Ω be a nonempty set and $\mathbb{R}^\Omega$ the set of all real-valued functions on Ω with its natural Riesz space structure. Let $\mathcal{A}$ be an algebra in the power set $\mathcal{P}(\Omega)$ of Ω and let $\eta:\mathcal{P}(\Omega)\to[0,+\infty]$ be a submeasure (i.e., a monotone, subadditive function with $\eta(\emptyset)=0$). Then

$$\|f\|_0=\inf\{a>0:\eta(\{|f|>a\})<a\},\tag{2.1}$$

where $\{|f|>a\}=\{x\in\Omega:|f(x)|>a\}$ and where $\inf\emptyset=+\infty$ defines a group pseudonorm on $\mathbb{R}^\Omega$ (i.e., $\|0\|_0=0$, $\|f\|_0=\|-f\|_0$ and $\|f+g\|_0\le\|f\|_0+\|g\|_0$ for all $f,g\in\mathbb{R}^\Omega$). We denote by

$$S(\Omega,\mathcal{A})=\left\{\sum_{i=1}^{n}a_i\chi_{A_i}:n\in\mathbb{N},\ a_i\in\mathbb{R},\ A_i\in\mathcal{A}\right\}\tag{2.2}$$

the space of all real-valued $\mathcal{A}$-simple functions on Ω; hereby χ_A denotes the characteristic function of A defined on Ω. By $L_0:=L_0(\Omega,\mathcal{A},\eta)$ we denote the closure of the space $S(\Omega,\mathcal{A})$ in $(\mathbb{R}^\Omega,\|\cdot\|_0)$.

For each function $f\in\mathbb{R}^\Omega$, set $|f|_\infty=\sup_\Omega|f|$ and denote by $B(\Omega,\mathcal{A})$ the closure of the space $S(\Omega,\mathcal{A})$ in $(\mathbb{R}^\Omega,|\cdot|_\infty)$. As $\|f\|_0\le|f|_\infty$, we have $B(\Omega,\mathcal{A})\subseteq L_0$. If for $M\in\mathcal{P}(\Omega)$ we set $\eta(M)=0$ if $M=\emptyset$ and $\eta(M)=+\infty$ if $M\neq\emptyset$, then $(L_0,\|\cdot\|_0)=(B(\Omega,\mathcal{A}),|\cdot|_\infty)$. We point out that the space $B(\Omega,\mathcal{P}(\Omega))$ coincides with the space of all real-valued bounded functions defined on Ω, and clearly $B(\Omega,\mathcal{A})\subseteq B(\Omega,\mathcal{P}(\Omega))$.

Throughout this note, given a finitely additive set function $\nu : \mathcal{A} \to [0,+\infty]$, we denote by $\nu^* : \mathcal{P}(\Omega) \to [0,+\infty]$ the submeasure defined by $\nu^*(E) = \inf\{\nu(A) : A \in \mathcal{A} \text{ and } E \subseteq A\}$. Moreover, whenever Ω is a Lebesgue measurable subset of $\mathbb{R}^n$, we denote by μ the Lebesgue measure on the σ-algebra of all Lebesgue measurable subsets of Ω, we write μ-a.e. for μ-almost everywhere.

Example 2.1 (see [9, Chapter III]). Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$, $\mathcal{A}$ the σ-algebra of all Lebesgue measurable subsets of Ω and $\eta = \mu^*$. If $\eta(\Omega) < +\infty$, then L_0 coincides with the space $M(\Omega)$ of all real-valued μ-measurable functions defined on Ω. If $\eta(\Omega) = +\infty$, then L_0 coincides with the space $T_0(\Omega)$ of all real-valued totally μ-measurable functions defined on Ω.

The following definitions are adapted from [10, Chapter 4].

Definition 2.2.
(i) A subset A of Ω is said to be an *η-null set* if $\eta(A) = 0$.
(ii) A function $f \in \mathbb{R}^\Omega$ is said to be an *η-null function* if $\eta(\{|f| > a\}) = 0$ for every $a > 0$.
(iii) Two functions $f,g \in \mathbb{R}^\Omega$ are said to be *equal η-almost everywhere*, and is used the notation $f = g$ η-a.e. if $f - g$ is an η-null function.
(iv) A function $f \in \mathbb{R}^\Omega$ is said to be *dominated η-almost everywhere* by a function g, and is used the notation $f \le g$ η-a.e. if there exists an η-null function $h \in \mathbb{R}^\Omega$ such that $f \le g + h$.

Observe that a function $f \in \mathbb{R}^\Omega$ is an η-null function if and only if $\|f\|_0 = 0$.
The *distribution function η_f* of a function $f \in L_0$ is defined by

$$\eta_f(\lambda) = \eta(\{|f| > \lambda\}) \quad (\lambda \ge 0). \tag{2.3}$$

Observe that $\eta_f = \eta_{|f|}$ and η_f may assume the value $+\infty$. In the next proposition, we state some elementary properties of the distribution function η_f (see [2, Chapter 2]).

Proposition 2.3. *Let $f,g \in L_0$ and $a \ne 0$. Then the distribution function η_f of f is nonnegative and decreasing. Moreover,*
(i) *$\eta_{af}(\lambda) = \eta_f(\lambda/|a|)$ for each $\lambda \ge 0$,*
(ii) *$\eta_{f+g}(\lambda_1 + \lambda_2) \le \eta_f(\lambda_1) + \eta_g(\lambda_2)$ for each $\lambda_1, \lambda_2 \ge 0$.*

Proposition 2.4. *Let $f,g \in L_0$. If $\|f - g\|_0 = 0$ then $\eta_f = \eta_g$ μ-a.e.*

Proof. Let $f,g \in L_0$ and $h \in L_0$ be an η-null function such that $g = f + h$. Let I and J denote the intervals $\{\lambda \ge 0 : \eta_f(\lambda) = +\infty\}$ and $\{\lambda \ge 0 : \eta_g(\lambda) = +\infty\}$, respectively. We start by proving that $\mu(I) = \mu(J)$. Assume $\mu(I) \ne \mu(J)$ and $\mu(I) < \mu(J)$. Then $I \subset J$ and $\mu(J \backslash I) > 0$. Denoted by $\text{int}(J \backslash I)$ the interior of the interval $J \backslash I$, we have $\eta_g(\lambda) = +\infty$ and $\eta_f(\lambda) < +\infty$ for each $\lambda \in \text{int}(J \backslash I)$. Fix $\lambda_1 \in \text{int}(J \backslash I)$ and $\lambda_2 > 0$ such that $\lambda_1 + \lambda_2 \in \text{int}(J \backslash I)$. By property (ii) of Proposition 2.3, we have

$$+\infty = \eta_g(\lambda_1 + \lambda_2) = \eta_{f+h}(\lambda_1 + \lambda_2) \le \eta_f(\lambda_1) + \eta_h(\lambda_2) = \eta_f(\lambda_1) < +\infty, \tag{2.4}$$

that is a contradiction. Set $\overline{\lambda} = \sup I = \sup J$ and let $\lambda_0 \in [\overline{\lambda}, +\infty)$ be a point of continuity of both the functions η_f and η_g. By property (ii) of Proposition 2.3, it follows that

$$\eta_f(\lambda_0) = \lim_n \eta_{g-h}\left(\lambda_0 + \frac{1}{n}\right) \leq \eta_g(\lambda_0) + \lim_n \eta_h\left(\frac{1}{n}\right) = \eta_g(\lambda_0). \tag{2.5}$$

Similarly, we find $\eta_g(\lambda_0) \leq \eta_f(\lambda_0)$. Hence $\eta_f = \eta_g$ μ-a.e. □

Proposition 2.5. *Let $f, g \in L_0$. If $|f| \leq |g|$ η-a.e., then $\eta_f \leq \eta_g$ μ-a.e.*

Proof. Let $h \in L_0$ be an η-null function such that $|f| \leq |g| + h$. Then $\eta_{|f|} \leq \eta_{|g|+h}$ and, by Proposition 2.4, $\eta_{|g|} = \eta_{|g|+h}$ μ-a.e. Hence $\eta_{|f|} \leq \eta_{|g|}$ μ-a.e., which gives the assert. □

Observe that, when $(\Omega, \mathcal{A}, \nu)$ is a totally σ-finite measure space and $\eta = \nu^*$, the distribution function η_f of $f \in L_0$ is right continuous (see [2]). In our setting this is not true anymore, as the following example shows.

Example 2.6 (see [9, Chapter III, page 103]). Let $\Omega = [0,1)$ and let $\mathcal{A}$ be the algebra of all finite unions of right-open intervals contained in Ω. Denote again by μ the Lebesgue measure μ restricted to $\mathcal{A}$. Let $\eta = \mu^*$. Consider the function $f : [0,1) \to \mathbb{R}$ defined as $f(x) = 0$, if $x \in [0,1) \setminus \mathbb{Q}$, and as $f(x) = 1/q$, if $x = p/q \in [0,1) \cap \mathbb{Q}$ in lowest terms. Then $\|f\|_0 = 0$ and so f is an η-null function but f is not null μ-a.e. since $\eta(\{|f| > 0\}) = 1$. Moreover, $\eta_f(\lambda) = 0$ if $\lambda > 0$ and $\eta_f(0) = 1$. Then η_f is not right continuous in 0.

Throughout, without loss of generality, we will assume that the distribution function η_f of a function $f \in L_0$ is right continuous, which together with Proposition 2.4 yields $\eta_f = \eta_g$ whenever $f, g \in L_0$ and $\|f - g\|_0 = 0$.

The *decreasing rearrangement* f^* of a function $f \in L_0$ is defined by

$$f^*(t) = \inf\{\lambda \geq 0 : \eta_f(\lambda) \leq t\} \quad (t \geq 0). \tag{2.6}$$

Clearly, by the above assumption on η_f, $f^* = g^*$ if $f, g \in L_0$ with $\|f - g\|_0 = 0$.

Proposition 2.7. *Let $f \in L_0$. If $f^*(t) = +\infty$, then $t = 0$.*

Proof. Assume that $f^*(t) = +\infty$. Then $\eta_f(\lambda) > t$ for all $\lambda \geq 0$. Since $\|f\|_0 < +\infty$, for some $\overline{\lambda} \geq 0$ we have $\eta_f(\overline{\lambda}) < +\infty$. Hence, as η_f is decreasing, there exists finite $\lim_{\lambda \to +\infty} \eta_f(\lambda) = l \geq 0$. The thesis follows by proving that $l = 0$. Assume $l > 0$ and choose a function $s \in S(\Omega, \mathcal{A})$ such that $\|f - s\|_0 \leq l/2$.

Fix $\lambda > l + \max_\Omega |s|$ and put $A = \{|f| > \lambda\}$, then $\eta(A) = \eta_f(\lambda) \geq l$ and

$$|f(x) - s(x)| \geq \big| |f(x)| - |s(x)| \big| \geq l \tag{2.7}$$

for each $x \in A$. So that $\|f - s\|_0 \geq l$. So we obtain $l \leq \|f - s\|_0 \leq l/2$: a contradiction. □

The following proposition contains some properties of rearrangements of functions of L_0. The proofs of (i)–(iv) (except some slight modifications) are identical to that of [2] for rearrangements of functions of a Banach function space, and we omit them.

Proposition 2.8. *Let $f, g \in L_0$ and $a \in \mathbb{R}$. Then f^* is nonnegative, decreasing, and right continuous. Moreover,*

(i) $(af)^* = |a| f^*$;

(ii) $f^*(\eta_f(\lambda)) \leq \lambda$, $(\eta_f(\lambda) < +\infty)$ *and* $\eta_f(f^*(t)) \leq t$, $(f^*(t) < +\infty)$;

(iii) $(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2)$ *for each* $t_1, t_2 \geq 0$;

(iv) *if* $|f| \leq |g|$ η*-a.e., then* $f^* \leq g^*$ μ*-a.e.*

Proof. Clearly f^* is nonnegative and decreasing. We prove that f^* is right continuous. Fix $t_0 \geq 0$ and assume that $\lim_{t \to t_0^+} f^*(t) = a < f^*(t_0) < +\infty$. Choose $b \in (a, f^*(t_0))$. Observe that, since $b < f^*(t_0)$, we have that $\eta_f(b) > t_0$ by the definition of f^*. Moreover, since $\lim_{t \to t_0^+} f^*(t) = a$, there exists $t_1 > 0$ such that $t_0 < t_1 < \eta_f(b)$ and $f^*(t_1) < b$. From the definition of f^* we obtain that $\eta_f(b) \leq t_1$. It follows that $t_1 < \eta_f(b) \leq t_1$ which is a contradiction. Then $\lim_{t \to t_0^+} f^*(t) = f^*(t_0)$.

To complete the proof, suppose that $f^*(0) = +\infty$, and assume that $\lim_{t \to 0^+} f^*(t) = a < +\infty$. Choose $b > a$. Then $\eta_f(b) > 0$ and since $\lim_{t \to 0^+} f^*(t) = a$ we have that there exists $t_2 > 0$ such that $t_2 < \eta_f(b)$ and $f^*(t_2) < b$. From the definition of f^* we obtain that $\eta_f(b) \leq t_2$. It follows that $t_2 < \eta_f(b) \leq t_2$ which is contradiction. Hence $\lim_{t \to 0^+} f^*(t_2) = +\infty$. □

Now we show that the rearrangement of a function of L_0 is a function of the space $T_0([0,+\infty))$ of all real-valued totally μ-measurable functions defined on $[0,+\infty)$, introduced in [9, Chapter III, Definition 10] (see also Example 2.1). In $T_0([0,+\infty))$, we write $|\cdot|_0$ instead of $\|\cdot\|_0$.

Theorem 2.9. *Let $f \in L_0$. Then*

(i) f *and* f^* *are equimeasurable, that is,* $\eta_f(\lambda) = \mu_{f^*}(\lambda)$ *for all* $\lambda \geq 0$;

(ii) $f^* \in T_0([0,+\infty))$ *and* $|f^*|_0 = \|f\|_0$.

Proof. (i) Fixed $\lambda \geq 0$ such that $\eta_f(\lambda) < +\infty$, by the first inequality of property (ii) of Proposition 2.8, we have that $f^*(\eta_f(\lambda)) \leq \lambda$. Moreover, since f^* is decreasing, we have $f^*(t) \leq \lambda$ for each t such that $\eta_f(\lambda) < t$. It follows that $\mu_{f^*}(\lambda) = \sup\{f^* > \lambda\} \leq \eta_f(\lambda)$. It remains to prove that $\eta_f(\lambda) \leq \mu_{f^*}(\lambda)$. Suppose that $f^*(0) = +\infty$. Then $\mu_{f^*}(\lambda) = \sup\{f^* > \lambda\}$ for all $\lambda \geq 0$. Assume that there exists $\lambda_0 \geq 0$ such that $\eta_f(\lambda_0) > \mu_{f^*}(\lambda_0)$. Fixed $t \in (\mu_{f^*}(\lambda_0), \eta_f(\lambda_0))$, we have that $f^*(t) \leq \lambda_0$ since $t > \mu_{f^*}(\lambda_0) = \sup\{f^* > \lambda_0\}$. On the other hand, since $t < \eta_f(\lambda_0)$, by the definition of f^*, we obtain $f^*(t) > \lambda_0$ which is a contradiction. The same proof breaks down if $f^*(0) < +\infty$ and $\lambda < f^*(0)$. If $f^*(0) < +\infty$ and $\lambda \geq f^*(0)$ then $\mu_{f^*}(\lambda) = 0$. Moreover, by the second part of the property (ii) of Proposition 2.8, it follows that $\eta_f(f^*(0)) = 0$ and then $\eta_f(\lambda) = 0$ for all $\lambda \geq f^*(0)$. This completes the proof.

(ii) is an immediate consequence of (i). □

The next theorem states two well-known convergence results (see, e.g., [5, Lemma 1.1] and [3, Lemma 2], resp.).

Theorem 2.10. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$, and let $\{f_n\}$ be a sequence of elements of the space $T_0(\Omega)$ of all real-valued totally μ-measurable functions defined on Ω.*

(i) *If* $\{f_n\}$ *converges in measure to* f, *then* $f_n^*(t)$ *converges to* $f^*(t)$ *in each point* t *of continuity of* f^*.

(ii) *If* $\{f_n\}$ *is a nondecreasing sequence of nonnegative functions convergent to* f μ*-a.e, then* f_n^* *is a nondecreasing sequence convergent to* f^* *pointwise.*

The remainder of this section will be devoted to extend these convergence results to the general setting of the space L_0. We need the following lemma.

Lemma 2.11. *Let* $f_n, f \in L_0$ $(n = 1,2,\dots)$ *be such that* $\|f_n - f\|_0 \to 0$. *Then* $\eta_{f_n}(\lambda) \to \eta_f(\lambda)$ *for each point* λ *of continuity of* η_f. *Moreover, if* $\lim_{\lambda\to\lambda_0^+} \eta_f(\lambda) = +\infty$ *then* $\lim_{n\to+\infty} \eta_{f_n}(\lambda_0) = +\infty$.

Proof. Let $\lambda > 0$ be a point of continuity of η_f and assume $\eta_{f_n}(\lambda) \nrightarrow \eta_f(\lambda)$. Then there are $\varepsilon_0 > 0$ and a subsequence $(\eta_{f_{n_k}})$ of (η_{f_n}) such that $|\eta_{f_{n_k}}(\lambda) - \eta_f(\lambda)| > \varepsilon_0$ for each $k \in \mathbb{N}$. Put

$$I_1 = \{k \in \mathbb{N} : \eta_{f_{n_k}}(\lambda) > \eta_f(\lambda) + \varepsilon_0\}, \qquad I_2 = \{k \in \mathbb{N} : \eta_{f_{n_k}}(\lambda) < \eta_f(\lambda) - \varepsilon_0\}. \tag{2.8}$$

Either I_1 or I_2 is infinite. Let $h > 0$ such that

$$\eta_f(\lambda - h) < \eta_f(\lambda) + \frac{\varepsilon_0}{2}, \qquad \eta_f(\lambda + h) > \eta_f(\lambda) - \frac{\varepsilon_0}{2}. \tag{2.9}$$

Suppose I_1 is infinite and let $k \in I_1$. Consider the sets

$$\begin{aligned} A_{\lambda-h} &= \{x \in \Omega : |f(x)| > \lambda - h\}, \\ A_{n_k,\lambda} &= \{x \in \Omega : |f_{n_k}(x)| > \lambda\}. \end{aligned} \tag{2.10}$$

Then $\eta(A_{\lambda-h}) = \eta_f(\lambda - h)$ and $\eta(A_{n_k,\lambda}) = \eta_{f n_k}(\lambda)$. We have that $\eta_{f_{n_k}}(\lambda) - \eta_f(\lambda - h) > \varepsilon_0/2$. Moreover,

$$\eta(A_{n_k,\lambda} \backslash A_{\lambda-h}) \geq \eta(A_{n_k,\lambda}) - \eta(A_{\lambda-h}) > \frac{\varepsilon_0}{2}. \tag{2.11}$$

Let $x \in A_{n_k,\lambda} \backslash A_{\lambda-h}$. Then $|f(x)| \leq \lambda - h$ and $|f_{n_k}(x)| > \lambda$. Therefore $|f_{n_k}(x)| - |f(x)| > h$. Hence

$$\begin{aligned} \eta(\{x \in \Omega : |f_{n_k}(x) - f(x)| > h\}) &\geq \eta(\{x \in \Omega : |f_{n_k}(x)| - |f(x)| > h\}) \\ &\geq \eta(A_{n_k,\lambda} \backslash A_{\lambda-h}) > \frac{\varepsilon_0}{2}, \end{aligned} \tag{2.12}$$

and this is a contradiction since $\|f_n - f\|_0 \to 0$. The proof is similar in the case the set I_2 is infinite. The second part of the proposition follows analogously. □

Theorem 2.12. *Let* $f_n, f \in L_0$ $(n=1,2,\dots)$ *be such that* $\|f_n - f\|_0 \to 0$. *Then* $f_n^*(t) \to f^*(t)$ *for each point* t *of continuity of* f^*. *Moreover, if* $\lim_{t\to0^+} f^*(t) = +\infty$ *then* $\lim_{n\to+\infty} f_n^*(0) = +\infty$.

Proof. Let $t_0 > 0$ be a point of continuity of f^* and assume $f_n^*(t_0) \nrightarrow f^*(t_0)$. Then there are $\varepsilon_0 > 0$ and a subsequence $(f_{n_k}^*)$ of (f_n^*) such that $|f_{n_k}^*(t_0) - f^*(t_0)| > \varepsilon_0$ for each $k \in \mathbb{N}$.

Put

$$I_1 = \{k \in \mathbb{N} : f^*_{n_k}(t_0) > f^*(t_0) + \varepsilon_0\}, \qquad I_2 = \{k \in \mathbb{N} : f^*_{n_k}(t_0) < f^*(t_0) - \varepsilon_0\}. \tag{2.13}$$

Either I_1 or I_2 is infinite. Let $h > 0$ such that

$$f^*(t_0 - h) < f^*(t_0) + \frac{\varepsilon_0}{2}, \qquad f^*(t_0 + h) > f^*(t_0) - \frac{\varepsilon_0}{2}. \tag{2.14}$$

Suppose I_1 is infinite . Fix $k \in I_1$, $t \in [t_0 - h, t_0]$ and $\sigma \in [f^*(t_0) + \varepsilon_0/2, f^*(t_0) + \varepsilon_0]$. Then

$$\begin{aligned} f^*(t) &\le f^*(t_0) + \frac{\varepsilon_0}{2} \le \sigma, \\ f^*_{n_k}(t) &> f^*(t_0) + \varepsilon_0 \ge \sigma. \end{aligned} \tag{2.15}$$

Hence $\eta_f(\sigma) \le t_0 - h < t_0$ and $\eta_{f_k}(\sigma) \ge t_0$. This shows that $\eta_{f_n}(\sigma) \nrightarrow \eta_f(\sigma)$ for all $k \in I_1$ and $\sigma \in [f^*(t_0) + \varepsilon_0/2, f^*(t_0) + \varepsilon_0]$ which by Lemma 2.11 is a contradiction. The second implication follows similarly. □

Lemma 2.13. *Let $f_n, f \in L_0$ $(n = 1, 2, \dots)$ be such that $\{f_n\}$ is a nondecreasing sequence of nonnegative functions and $\|f_n - f\|_0 \to 0$. Then $|\eta_{f_n} - \eta_f|_0 \to 0$.*

Proof. Assume by contradiction $|\eta_{f_n} - \eta_f|_0 \nrightarrow 0$. Since $\eta_{f_n} \le \eta_{f_{n+1}} \le \eta_f$, we find $\varepsilon_0 > 0$, $\sigma_0 > 0$ and $\overline{n} \in \mathbb{N}$ such that

$$\mu(\{\lambda \ge 0 : \eta_f(\lambda) - \eta_{f_n}(\lambda) > \varepsilon_0\}) > \sigma_0 \tag{2.16}$$

for all $n \in \mathbb{N}$ with $n \ge \overline{n}$. Set $B_n = \{\lambda \ge 0 : \eta_f(\lambda) - \eta_{f_n}(\lambda) > \varepsilon_0\}$, then $\cap_{n \ge \overline{n}} B_n$ is nonempty, and for $\lambda_0 \in \cap_{n \ge \overline{n}} B_n$ we have

$$\sup_{n \ge \overline{n}} \eta_{f_n}(\lambda_0) \le \eta_f(\lambda_0) - \varepsilon_0. \tag{2.17}$$

Then we choose $h > 0$ such that

$$\eta_f(\lambda_1) - \eta_{f_n}(\lambda_2) \ge \frac{\varepsilon_0}{2} \tag{2.18}$$

for all $\lambda_1, \lambda_2 \in [\lambda_0, \lambda_0 + h]$ and all $n \ge \overline{n}$. In particular, we have

$$\eta_f(\lambda_0 + h) - \eta_{f_n}(\lambda_0) \ge \frac{\varepsilon_0}{2}. \tag{2.19}$$

Then using the same notations and considerations similar to that of Lemma 2.11, we find

$$\begin{aligned} \{x \in \Omega : f(x) - f_n(x) > h\} &\supseteq A_{\lambda_0 + h} \setminus A_{n, \lambda_0}, \\ \eta(A_{\lambda_0 + h} \setminus A_{n, \lambda_0}) &\ge \eta_f(\lambda_0 + h) - \eta_{f_n}(\lambda_0) \ge \frac{\varepsilon_0}{2} \end{aligned} \tag{2.20}$$

which is a contradiction since $\|f_n - f\|_0 \to 0$. □

Theorem 2.14. *Let $f_n, f \in L_0$ $(n = 1, 2, \dots)$ be such that $\{f_n\}$ is a nondecreasing sequence of nonnegative functions and $\|f_n - f\|_0 \to 0$. Then $|f^*_n - f^*|_0 \to 0$.*

Proof. The proof, using Lemma 2.13, is analogous to the proof of Theorem 2.12. □

We remark that if $\{f_n\}$ is a sequence of elements of the space $T_0(\Omega)$, Theorem 2.14 yields (ii) of Theorem 2.10.

3. Nonexpansivity of rearrangement in the space L_∞

We introduce the notion of essentially boundedness, following [10]. For $f \in \mathbb{R}^\Omega$, set

$$\|f\|_\infty = \inf_{A \subseteq \Omega,\, \eta(A)=0} \sup_{\Omega \setminus A} |f|, \tag{3.1}$$

then $\|\cdot\|_\infty$ defines a group pseudonorm on $\mathbb{R}^\Omega$, for each submeasure η on $\mathcal{P}(\Omega)$.

We recall that, if ν is a finitely additive extended real-valued set function on an algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ and $\eta = \nu^*$, the space $\mathfrak{L}_\infty(\Omega,\mathcal{A},\nu)$ of all real-valued essentially bounded functions introduced in [10] is defined by

$$\mathfrak{L}_\infty(\Omega,\mathcal{A},\nu) = \{f \in \mathbb{R}^\Omega : \|f\|_\infty < +\infty\}. \tag{3.2}$$

In our setting it is natural to define a space $L_\infty(\Omega,\mathcal{A},\eta)$ of all real-valued essentially bounded functions as follows.

Definition 3.1. The space $L_\infty := L_\infty(\Omega,\mathcal{A},\eta)$ is the closure of the space $S(\Omega,\mathcal{A})$ in $(\mathbb{R}^\Omega, \|\cdot\|_\infty)$.

Let $f \in \mathbb{R}^\Omega$. Since $\|f\|_0 \le \|f\|_\infty$, we have $L_\infty \subseteq L_0$. Moreover, $\|f\|_0 = 0$ if and only if $\|f\|_\infty = 0$. In the remainder part of this note we will identify functions $f,g \in \mathbb{R}^\Omega$ for which $\|f-g\|_0 = 0$. Then $(L_0, \|\cdot\|_0)$ and $(L_\infty, \|\cdot\|_\infty)$ become an F-normed space (in the sense of [11]) and a normed space, respectively.

Proposition 3.2. *Let ν be a finitely additive extended real-valued set function on an algebra $\mathcal{A}$ in $\mathcal{P}(\Omega)$ and $\eta = \nu^*$. Then the space $\mathfrak{L}_\infty(\Omega,\mathcal{A},\nu)$ coincides with the space $L_\infty(\Omega,\mathcal{P}(\Omega),\eta)$.*

Proof. Given $f \in L_\infty(\Omega,\mathcal{P}(\Omega),\eta)$, find a simple function $s \in S(\Omega,\mathcal{P}(\Omega))$ such that $\|f - s\|_\infty < +\infty$. From $\|f\|_\infty \le \|f-s\|_\infty + \|s\|_\infty$, we get $f \in \mathfrak{L}_\infty(\Omega,\mathcal{A},\nu)$. On the other hand, if $f \in \mathfrak{L}_\infty(\Omega,\mathcal{A},\nu)$ then there exists $A \subseteq \Omega$ such that $\eta(A) = 0$ and such that $\sup_{\Omega\setminus A}|f| < +\infty$. Consider the real function g on Ω defined by $g = f$ on $\Omega\setminus A$ and by $g = 0$ on A. Of course $g \in \mathfrak{L}_\infty(\Omega,\mathcal{A},\nu)$ and $\|f-g\|_\infty = 0$. Moreover, $g \in B(\Omega,\mathcal{P}(\Omega)) \subseteq L_\infty(\Omega,\mathcal{P}(\Omega),\eta)$. Then there exists a sequence (s_n) in $S(\Omega,\mathcal{P}(\Omega))$ such that $|g - s_n|_\infty \to 0$. Since $\|f - s_n\|_\infty \le \|f-g\|_\infty + \|g - s_n\|_\infty = |g-s_n|_\infty$, we have that $f \in L_\infty(\Omega,\mathcal{P}(\Omega),\eta)$. □

We write briefly $B([0,+\infty))$ instead of $B(\Omega,\mathcal{A})$, when $\Omega = [0,+\infty)$, $\mathcal{A}$ is the σ-algebra of all Lebesgue measurable subsets of Ω and $\eta = \mu^*$. The next proposition establishes that the rearrangement of a function of L_∞ is a function of $B([0,+\infty))$.

Proposition 3.3. *Let $f \in L_\infty$. Then $f^* \in B([0,+\infty))$ and $|f^*|_\infty = f^*(0) = \|f\|_\infty$.*

Proof. Let $\varepsilon > 0$. Then there is $A \subseteq \Omega$ such that $\eta(A) = 0$ and $\sup_{\Omega\setminus A}|f| < \|f\|_\infty + \varepsilon$. Hence $\{|f| > \|f\|_\infty + \varepsilon\} \subseteq A$, so that $\eta(\{|f| > \|f\|_\infty + \varepsilon\}) = 0$.

Therefore $|f^*|_\infty = f^*(0) \le \|f\|_\infty + \varepsilon$ so that $|f^*|_\infty \le \|f\|_\infty$. Now we have to prove that $\|f\|_\infty \le |f^*|_\infty$. Assume $|f^*|_\infty < c < \|f\|_\infty$. Then for each $A \subseteq \Omega$ such that $\eta(A) = 0$ we have $\sup_{\Omega\setminus A} |f| > c$ and $\eta_f(c) = \eta(\{|f| > c\}) > 0$. For $t \in [0, \eta_f(c))$, by the definition of the function f^*, we obtain $f^*(t) \ge c > |f^*|_\infty = f^*(0)$ which is a contradiction, since f^* is decreasing. □

Our next aim is to prove nonexpansivity of rearrangement on L_∞. We need the following two lemmas.

Lemma 3.4. *Let $s_1, s_2 \in S(\Omega, \mathcal{A})$. Then $|s_1^* - s_2^*|_\infty \le \|s_1 - s_2\|_\infty$.*

Proof. Let $s_1, s_2 \in S(\Omega, \mathcal{A})$ and put $\|s_1 - s_2\|_\infty = \varepsilon$. Let $\{A_1, \dots, A_n\}$ be a finite partition of Ω in $\mathcal{A}$ such that $s_1 = \sum_{i=1}^n a_i \chi_{A_i}$ and $s_2 = \sum_{i=1}^n b_i \chi_{A_i}$. Set

$$s = \sum_{i=1}^{n} \min\{|a_i|, |b_i|\} \chi_{A_i\setminus A}, \tag{3.3}$$

where $\eta(A) = 0$ and $|s_1(x) - s_2(x)| \le \varepsilon$ for all $x \in \Omega\setminus A$. It suffices to prove that

$$s(x) \le |s_1(x)| \le s_\varepsilon(x), \qquad s(x) \le |s_2(x)| \le s_\varepsilon(x), \tag{3.4}$$

for all $x \in \Omega\setminus A$, where $s_\varepsilon = |s| + \varepsilon$. In fact, from this and from property (iv) of Proposition 2.8, it follows that

$$s^* \le s_1^* \le s_\varepsilon^* \ \mu\text{-a.e.}, \qquad s^* \le s_2^* \le s_\varepsilon^* \ \mu\text{-a.e.}, \tag{3.5}$$

and thus $|s_1^* - s_2^*|_\infty \le |s_\varepsilon^* - s^*|_\infty = \varepsilon$. Fix $x \in \Omega\setminus A$ and let $i \in \{1, \dots, n\}$ such that $x \in A_i\setminus A$. Now, if $s(x) = |a_i|$ we have

$$s(x) = |s_1(x)| \le |a_i| + \varepsilon = s_\varepsilon(x). \tag{3.6}$$

If $s(x) = |b_i|$, since $\|s_1 - s_2\|_\infty = \varepsilon$ implies $0 \le |a_i| - |b_i| \le |a_i - b_i| \le \varepsilon$, we have

$$s(x) \le |a_i| = |s_1(x)| \le |b_i| + \varepsilon = s_\varepsilon(x). \tag{3.7}$$

Analogously we obtain $s(x) \le |s_2(x)| \le s_\varepsilon(x)$ for $x \in \Omega\setminus A$, and the lemma follows. □

Lemma 3.5. *Let $f \in L_\infty$. Then for each $\varepsilon > 0$ there exists a function $s \in S(\Omega, \mathcal{A})$ such that $\|f - s\|_\infty \le \varepsilon/2$ and $|f^* - s^*|_\infty \le \varepsilon$.*

Proof. Fix $\varepsilon > 0$. Then similar to [10, page 101] (see Theorem 3.10), we have that there is a finite partition $\{A_1, \dots, A_n\}$ of Ω in $\mathcal{A}$ and $A \subseteq \Omega$ with $\eta(A) = 0$ such that

$$\sup_{x,y\in A_i\setminus A} |f(x) - f(y)| \le \varepsilon \tag{3.8}$$

for each $i \in \{1, \dots, n\}$. Set

$$\lambda_i = \inf_{x\in A_i\setminus A} |f(x)|, \qquad \Lambda_i = \sup_{x\in A_i\setminus A} |f(x)|, \qquad a_i = \frac{\lambda_i + \Lambda_i}{2}, \tag{3.9}$$

for each $i \in \{1,\dots,n\}$. Define the simple function

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}. \tag{3.10}$$

Then for each $i \in \{1,\dots,n\}$ and for each $x \in A_i \setminus A$ we have $|f(x) - s(x)| \le \varepsilon/2$. Hence $\|f - s\|_\infty \le \varepsilon/2$. Now consider the simple function φ defined by

$$\varphi(x) = \begin{cases} \left| a_i + \dfrac{\varepsilon}{2} \right|, & \text{if } x \in A_i,\ a_i < -\dfrac{\varepsilon}{2}, \\ 0, & \text{if } x \in A_i,\ -\dfrac{\varepsilon}{2} \le a_i \le \dfrac{\varepsilon}{2}, \\ \left| a_i - \dfrac{\varepsilon}{2} \right|, & \text{if } x \in A_i,\ a_i > \dfrac{\varepsilon}{2}. \end{cases} \tag{3.11}$$

Then a direct computation shows that

$$\varphi(x) \le |f(x)| \le \varphi_\varepsilon(x), \qquad \varphi(x) \le |s(x)| \le \varphi_\varepsilon(x), \tag{3.12}$$

for all $x \in \Omega \setminus A$, where $\varphi_\varepsilon = |\varphi| + \varepsilon$. Put $h(x) = (\max|a_i|)\chi_A(x)$ and $k(x) = |f(x)|\chi_A(x)$. Then $\varphi \le |f| + h$ and $|f| \le \varphi_\varepsilon + k$. As h and k are both η-null functions, from the property (iv) of Proposition 2.8 it follows that $\varphi^* \le f^* \le \varphi_\varepsilon^*$ μ-a.e., and analogously $\varphi^* \le s^* \le \varphi_\varepsilon^*$ μ-a.e., hence $|f^* - s^*|_\infty \le |\varphi_\varepsilon^* - \varphi^*|_\infty = \varepsilon$. □

Theorem 3.6. *Let $f,g \in L_\infty$. Then $|f^* - g^*|_\infty \le \|f - g\|_\infty$.*

Proof. Let $\varepsilon > 0$. By Lemma 3.5 we can find $s,u \in S(\Omega,\mathcal{A})$ such that

$$\begin{gathered} \|f - s\|_\infty \le \frac{\varepsilon}{4}, \qquad \|g - u\|_\infty \le \frac{\varepsilon}{4}, \\ |f^* - s^*|_\infty \le \frac{\varepsilon}{2}, \qquad |g^* - u^*|_\infty \le \frac{\varepsilon}{2}. \end{gathered} \tag{3.13}$$

We have that

$$\|s - u\|_\infty \le \|f - s\|_\infty + \|f - g\|_\infty + \|g - u\|_\infty \le \|f - g\|_\infty + \frac{\varepsilon}{2}. \tag{3.14}$$

Then the last inequality and Lemma 3.4 imply $|s^* - u^*|_\infty \le \|f - g\|_\infty + \varepsilon/2$.

Consequently we have

$$|f^* - g^*|_\infty \le |f^* - s^*|_\infty + |s^* - u^*|_\infty + |g^* - u^*|_\infty \le \|f - g\|_\infty + \varepsilon, \tag{3.15}$$

and by the arbitrariness of ε the theorem follows. □

Remark 3.7. We observe that Theorem 3.6 does not hold in every space L_0. In fact, let $L_0 = M([0,1])$ (see Example 2.1) and set

$$s_n = \sum_{i=0}^{n-1} (n-i)\chi_{[i/n,(i+1)/n)}, \qquad t_n = \sum_{i=1}^{n-1} (n-i)\chi_{[i/n,(i+1)/n)}, \tag{3.16}$$

for $n = 2, 3, \ldots$. Then for each n we have $t_n = s_n \chi_{[1/n,1)}$, $s_n - t_n = n\chi_{[0,1/n)}$, and $|s_n - t_n|_0 = 1/n$. On the other hand, since $s_n^* = s_n$ and $t_n^* = \sum_{i=0}^{n-1}(n-1-i)\chi_{[i/n,(i+1)/n)}$, we have that $s_n^* - t_n^* = \chi_{[0,1)}$ and then $|s_n^* - t_n^*|_0 = 1$.

Throughout for a set M in L_0, we put $M^* = \{f^* : f \in M\}$. The following inequality between the Hausdorff measure of noncompactness of a bounded subset M of L_∞ and that of M^* is an immediate consequence of nonexpansivity of rearrangement on L_∞.

Corollary 3.8. *Let M be a bounded subset of L_∞. Then*

$$\gamma_{B([0,+\infty))}(M^*) \leq \gamma_{L_\infty}(M). \tag{3.17}$$

The following example shows that there is not any constant c such that $\gamma_{L_\infty}(M) \leq c\gamma_{B([0,+\infty))}(M^*)$.

Example 3.9. Let $M = \{\chi_I : I \subseteq [0,1],\ \mu(I) = 1/2\}$. Then $M^* = \{\chi_{[0,1/2)}\}$ and we have that $\gamma_{B([0,+\infty))}(M^*) = 0$ while $\gamma_{L_\infty}(M) > 0$.

In order to obtain a precise formula for the Hausdorff measure of noncompactness in the space L_∞, we consider for any bounded subset M of L_∞ the following parameter:

$$\begin{aligned}\omega_{L_\infty}(M) = \inf\Big\{\varepsilon > 0 :\ & \text{there exists a finite partition } \{A_1,\ldots,A_n\} \\ & \text{of } \Omega \text{ in } \mathcal{A} \text{ such that for all } f \in M \text{ there is } A_f \subseteq \Omega \\ & \text{with } \eta(A_f) = 0 \text{ and } \sup_{x,y\in A_i\setminus A_f} |f(x) - f(y)| \leq \varepsilon \text{ for all } i = 1,\ldots,n\Big\}.\end{aligned} \tag{3.18}$$

The proof of the following result is similar to that of [12, Theorem 2.1].

Theorem 3.10. *Let M be a bounded subset of L_∞. Then*

$$\gamma_{L_\infty}(M) = \frac{1}{2}\omega_{L_\infty}(M). \tag{3.19}$$

Proof. Fix $a > \gamma_{L_\infty}(M)$. Then we can find $s_1,\ldots,s_n \in S(\Omega,\mathcal{A})$ such that for each $f \in M$ there is $i \in \{1,\ldots,n\}$ with $\|f - s_i\|_\infty \leq a$. Let $\{A_1,\ldots,A_m\}$ be a partition of Ω in $\mathcal{A}$ such that the restriction $s_{i|A_j}$ is constant for all $i \in \{1,\ldots,n\}$ and for all $j \in \{1,\ldots,m\}$. Let $f \in M$, $i \in \{1,\ldots,n\}$, and $A_f \subseteq \Omega$ such that $\eta(A_f) = 0$ and $\sup_{\Omega\setminus A_f} |f - s_i| \leq a$. For each $j \in \{1,\ldots,m\}$, we have that

$$\sup_{x,y\in A_i\setminus A_f} |f(x) - f(y)| \leq 2a, \tag{3.20}$$

hence $\omega_{L_\infty}(M) \leq 2\gamma_{L_\infty}(M)$ and $(1/2)\omega_{L_\infty}(M) \leq \gamma_{L_\infty}(M)$.

Now fix $a > \omega_{L_\infty}(M)$ and let $c > 0$ such that $\|f\|_\infty \leq c$ for each $f \in M$. Then there is a finite partition $\{A_1,\ldots,A_n\}$ of Ω in $\mathcal{A}$ such that for all $f \in M$ there is $A_f \subseteq \Omega$ with $\eta(A_f) = 0$ and $\sup_{x,y\in A_i\setminus A_f} |f(x) - f(y)| \leq a$ for all $i = 1,\ldots,n$. Moreover, for all $f \in M$

there is $B_f \subseteq \Omega$ with $\eta(B_f) = 0$ such that $\sup_{\Omega\setminus B_f} |f| \le c$. Set $C_f = A_f \cup B_f$ for each $f \in M$. Fix $\varepsilon > 0$. Let $k, m \in \mathbb{N}$ such that $1/m < \varepsilon$ and $-c + k/m > c$. Set $X = \{-c + i/m : i = 0,\ldots,k\}$ and $F = \{\sum_{i=1}^{n} a_i\chi_{A_i} : a_i \in X\}$. Then for each $f \in M$ there is a function $s \in F$ such that $\sup_{\Omega\setminus C_f} |f - s| \le a/2 + 1/m \le a/2 + \varepsilon$. Since F is finite it follows that $\gamma_{L_\infty}(M) \le (1/2)\omega_{L_\infty}(M)$. This completes the proof. □

Observe that as a particular case of [12, Theorem 2.1], for a bounded subset T of $B([0,+\infty))$ we have

$$\gamma_{B([0,+\infty))}(T) = \frac{1}{2}\omega_{B([0,+\infty))}(T), \tag{3.21}$$

where

$$\omega_{B([0,+\infty))}(T) = \inf\Big\{\varepsilon > 0 : \text{ there exists a finite partition } \{A_1,\ldots,A_n\} \text{ of } [0,+\infty) \text{ of Lebesgue measurable sets such that for all } f \in T \ \sup_{x,y\in A_i\setminus A_f} |f(x) - f(y)| \le \varepsilon \text{ for all } i = 1,\ldots,n\Big\}. \tag{3.22}$$

In view of the formulas we have obtained, by Corollary 3.8 we have the following.

Corollary 3.11. *Let M be a bounded subset of L_∞. Then*

$$\omega_{B([0,+\infty))}(M^*) \le \omega_{L_\infty}(M). \tag{3.23}$$

4. Nonexpansivity of rearrangement in Orlicz spaces L_N

In this section, as a particular case of [6] (see also [13]), we consider *Orlicz spaces L_N* of finitely additive extended real-valued set functions defined on algebras of sets. The space L_N has been introduced in [6] in the same way as Dunford and Schwartz [9, page 112] define the space of integrable functions and the integral for integrable functions, and generalize the Orlicz spaces of σ-additive measures defined on σ-algebras of sets.

As in the previous sections, Ω is a nonempty set and $\mathcal{A}$ is an algebra in $\mathcal{P}(\Omega)$. Let $\nu : \mathcal{A} \to [0,+\infty]$ be a finitely additive set function. Throughout we assume that each simple function $s \in S(\Omega,\mathcal{A})$ is ν-integrable, that is, $s = \sum_{i=1}^{n} a_i\chi_{A_i}$ with $a_i \in \mathbb{R}$, $A_i \in \mathcal{A}$ and $a_i = 0$ if $\nu(A_i) = \infty$ (with $0 \cdot \infty = 0$). Denote by $(L_1(\Omega,\mathcal{A},\nu), \|\cdot\|_1)$ the Lebesgue space defined in [9], then $\|f\|_1 = \int_\Omega |f|d\nu$ is a Riesz pseudonorm in the sense of [14]. Let $\eta = \nu^*$ and $N : [0,+\infty) \to [0,+\infty)$ be a continuous, strictly increasing function such that $N(0) = 0$ and $N(s+t) \le k(N(s) + N(t))$ $(k \in \mathbb{N})$ for all $s,t \ge 0$. The latter condition holds if and only if N satisfies the Δ_2-condition, that is, there is a constant $c \in [0,+\infty[$ with $N(2t) \le cN(t)$ for all $t \ge 0$ (see [6, page 90]).

Then, for $s \in S(\Omega,\mathcal{A})$, $\|s\|_N$ is defined by $\|s\|_N = \|N \circ |s|\|_1$, and the space E_N is defined as follows.

Definition 4.1 (see [6, page 92]). The *space* $L_N := L_N(\Omega,\mathscr{A},\eta)$ is the space of all functions $f \in L_0$, for which there is a $\|\cdot\|_N$- Cauchy sequence (s_n) in $S(\Omega,\mathscr{A})$ converging to f with respect to $\|\cdot\|_0$, and $\|f\|_N = \lim_n \|s_n\|_N$, the sequence (s_n) is said to *determine* f.

Proposition 4.2 (see [6, Proposition 2.6 (c)]). *If (s_n) is a sequence in $S(\Omega,\mathscr{A})$ determining $f \in L_N$, then (s_n) converges to f with respect to $\|\cdot\|_N$.*

We will call *convergence in N-mean* the convergence with respect to $\|\cdot\|_N$.

Proposition 4.3 (see [6, Proposition 2.10 (b)]). *For all $f \in L_N$, $\|f\|_N = \|N\circ|f|\|_1$.*

In the following if $\Omega = [0,+\infty)$, $\mathscr{A}$ is the σ-algebra of all Lebesgue measurable subsets of $[0,+\infty)$ and $\eta = \mu^*$, we will write $L_N([0,+\infty))$ instead of L_N. For $f \in L_N([0,+\infty))$, we denote $\|f\|_N$ by $|f|_N$.

In order to consider rearrangements of functions of L_N to any function $s = \sum_{i=1}^n a_i\chi_{A_i}$ in $S(\Omega,\mathscr{A})$, we associate the simple function $\bar{s}:[0,+\infty)\to\mathbb{R}$ defined by

$$\bar{s} = \sum_{i=1}^{n} a_i \chi_{[\sum_{i=1}^{n-1}\nu(A_i),\sum_{i=1}^{n}\nu(A_i))}. \tag{4.1}$$

We immediately find $\|s\|_N = |\bar{s}|_N$ and $s^* = (\bar{s})^*$.

Lemma 4.4. *Let $s \in S(\Omega,\mathscr{A})$. Then $\|s\|_N = |s^*|_N$.*

Proof. An easy computation shows that $\int_{[0,+\infty)} N(|\bar{s}(t)|)d\mu = \int_{[0,+\infty)} N((\bar{s})^*(t))d\mu$. Therefore, we obtain

$$\|s\|_N = \int_{[0,+\infty)} N(|\bar{s}(t)|)d\mu = \int_{[0,+\infty)} N((\bar{s})^*(t))d\mu = \int_{[0,+\infty)} N(s^*(t))d\mu = |s^*|_N. \tag{4.2}$$

□

Lemma 4.5. *Let $s_1,s_2 \in S(\Omega,\mathscr{A})$. Then $|s_1^* - s_2^*|_N \le \|s_1 - s_2\|_N$.*

Proof. By [3, (6), page 24] we have

$$\int_{[0,+\infty)} N(|(\bar{s}_1)^*(t) - (\bar{s}_2)^*(t)|)d\mu \le \int_{[0,+\infty)} N(||\bar{s}_1(t)| - |\bar{s}_2(t)||)d\mu. \tag{4.3}$$

Since

$$\int_{[0,+\infty)} N(||\bar{s}_1(t)| - |\bar{s}_2(t)||)d\mu = \int_{\Omega} N(||s_1| - |s_2||)d\nu, \tag{4.4}$$

we get

$$|s_1^* - s_2^*|_N \le \int_{\Omega} N(||s_1| - |s_2||)d\nu \le \|s_1 - s_2\|_N. \tag{4.5}$$

□

Lemma 4.6. *Let (s_n) be a sequence in $S(\Omega,\mathscr{A})$ such that $\|s_n - f\|_N \to 0$. Then*

$$|s_n^* - f^*|_N \longrightarrow 0, \|f\|_N = |f^*|_N. \tag{4.6}$$

Proof. Since $\|s_n - f\|_N \to 0$ by [6, Theorem 2.7], we have $\|s_n - f\|_0 \to 0$. Then by Theorem 2.14 it follows that $|s_n^* - f|_0 \to 0$, and so we can choose a subsequence $(s_{n_k}^*)$ of (s_n^*) which converges to f^* μ-a.e. On the other hand by Lemma 4.4 since (s_n) is a $\|\cdot\|_N$-Cauchy sequence we have that (s_n^*) is a $|\cdot|_N$-Cauchy. Then there is a function $g \in L_N([0,+\infty))$ such that $|s_n^* - g|_N \to 0$. Therefore $|s_n^* - g|_0 \to 0$ and so we can find a subsequence $(s_{n_l}^*)$ of (s_n^*) which converges to g μ-a.e. Then $f^* = g$ μ-a.e. and $|s_n^* - f^*|_N \to 0$. Finally

$$\begin{aligned} \big|\, \|f\|_N - |f^*|_N \big| &\leq \big|\, \|f\|_N - \|s_n\|_N \big| + \big|\, \|s_n\|_N - |s_n^*|_N \big| + \big|\, |s_n^*|_N - |f^*|_N \big| \\ &\leq \|s_n - f\|_N + |s_n^* - f^*|_N. \end{aligned} \tag{4.7}$$

Hence $\|f\|_N = |f^*|_N$ and this proves the lemma. □

We omit the proof of nonexpansivity of rearrangement on L_N, which is analogous to that of Theorem 3.6, when we use the above lemma.

Theorem 4.7. *Let $f,g \in L_N$. Then $|f^* - g^*|_N \leq \|f - g\|_N$.*

Corollary 4.8. *Let M be a bounded set in L_N. Then*

$$\gamma_{L_N([0,+\infty))}(M^*) \leq \gamma_{L_N}(M). \tag{4.8}$$

Now let Ω be an open bounded subset of the n-dimensional Euclidean space $\mathbb{R}^n$ (with norm $\|\cdot\|_n$), and let $\mathcal{A}$ be the σ-algebra of all Lebesgue measurable subsets of Ω and $\eta = \mu^*$. Now we assume that Φ is a Young function and we consider the space E_Φ of finite elements of the Orlicz space L_Φ generated by Φ. In this situation, we introduce a parameter ω_{E_Φ} to estimate the Hausdorff measure of noncompactness.

Recall that Φ is a *Young function* if $\Phi(t) = \int_0^t \varphi(s)ds$ $(t \geq 0)$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is such that

(i) $\varphi(0) = 0$;
(ii) $\varphi(s) > 0$, $s > 0$;
(iii) φ is right continuous at any point $s \geq 0$;
(iv) φ is nondecreasing on $[0,+\infty)$;
(v) $\lim_{s\to+\infty} \varphi(s) = +\infty$.

In particular, Φ is continuous, nonnegative, strictly increasing, convex on $[0,+\infty)$ and $\Phi(0) = 0$.

By $L_\Phi(\Omega)$ we denote the Orlicz space generated by Φ, that is,

$$L_\Phi(\Omega) = \Big\{ f \in L_0 : \lim_{\lambda\to0^+} \|\Phi\circ(\lambda|f|)\|_1 = 0 \Big\}. \tag{4.9}$$

We equip $L_\Phi(\Omega)$ with the Luxemburg norm

$$|||f|||_\Phi = \inf\Big\{ k > 0 : \Big\| \Phi\circ\Big(\frac{|f|}{k}\Big) \Big\|_1 \leq 1 \Big\}. \tag{4.10}$$

By $E_\Phi(\Omega)$ we denote the space of finite elements, that is,

$$E_\Phi(\Omega) = \{ f \in L_0 : \|\Phi\circ(\lambda|f|)\|_1 < +\infty, \text{ for any } \lambda > 0 \}. \tag{4.11}$$

The space $E_\Phi(\Omega)$ is a closed subspace of $L_\Phi(\Omega)$ and $E_\Phi(\Omega) = L_\Phi(\Omega)$ if the Δ_2-condition holds. For details on Orlicz spaces see [15, 16].

We recall that the convergence with respect to the Luxemburg norm $|||\cdot|||_\Phi$ implies Φ-mean convergence, for $\Phi \in \Delta_2$ the two types of convergence are equivalent.

For $r > 0$, $x \in \mathbb{R}^n$, and $f \in L_\Phi(\Omega)$ let us put $f(y) = 0$ if $y \notin \Omega$. The so called *Steklow function* $S_r(f)$ corresponding to f is defined as follows:

$$S_r(f)(x) = \frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)d\mu = \frac{1}{\mu(B(x,r))}\int_{\|y\|_n<r} f(x+y)d\mu. \tag{4.12}$$

$S_r(f)$ is continuous on $\mathbb{R}^n$, has compact support and $|||S_r(f)|||_\Phi \le |||f|||_\Phi$ (cfr., [16, Theorem 9.10]).

Theorem 4.9 (see [15, (ii) page 173]). *Let M be a bounded subset of $L_\Phi(\Omega)$. Set $M_r = \{S_r(f) : f \in M\}$. Then*
(1) *$M_r \subset C_o^\infty(\mathbb{R}^n)$;*
(2) *M_r is relatively compact in $C(\overline{\Omega})$ with respect to $\|\cdot\|_\infty$.*

Now for any bounded subset M of $E_\Phi(\Omega)$, generalizing an analogous parameter defined in the case of Lebesgue spaces $L_p[0,1]$, we put

$$\omega_{E_\Phi}(M) = \lim_{\delta\to 0}\sup_{f\in M}\max_{0<r\le\delta} |||f - S_r(f)|||_\Phi. \tag{4.13}$$

The following theorem gives an estimate of the Hausdorff measure of noncompactness γ_{E_Φ} by means of the parameter ω_{E_Φ}. We observe that the theorem is an extension of the compactness criterion given in [15, Theorem 3.14.6], which is the analogous in $E_\Phi(\Omega)$ of the Kolmogorov compactness criterion in the Lebesgue spaces $L_p[0,1]$.

Theorem 4.10. *Let M be a bounded set of $E_\Phi(\Omega)$. Then*

$$\frac{1}{2}\omega_{E_\Phi}(M) \le \gamma_{E_\Phi}(M) \le \omega_{E_\Phi}(M). \tag{4.14}$$

Proof. Let $\alpha > \omega_{E_\Phi}(M)$. For some $0 < r \le \delta$ we have that $|||f - S_r(f)|||_\Phi \le \alpha$ for all $f \in M$. Since M_r is compact in $C(\overline{\Omega})$ with respect to $\|\cdot\|_\infty$, for all $\varepsilon > 0$ we can choose an ε-net $\{S_r(f_1), S_r(f_2), \dots, S_r(f_n)\}$ for M_r in M_r. Then for any $f \in M$ there exists $i \in \{1,\dots,n\}$ such that $|S_r(f)(t) - S_r(f_i)(t)| \le \varepsilon$ for all $t \in \overline{\Omega}$, so that $|||S_r(f) - S_r(f_i)|||_\Phi \le \varepsilon|||\chi_\Omega|||_\Phi$. Hence

$$|||f - S_r(f_i)|||_\Phi \le |||f - S_r(f)|||_\Phi + |||S_r(f) - S_r(f_i)|||_\Phi \le \alpha + \varepsilon|||\chi_\Omega|||_\Phi \tag{4.15}$$

and consequently $\gamma_{E_\Phi}(M) \le \omega_{E_\Phi}(M)$.

We now prove the left inequality. Let $\alpha > \gamma_{E_\Phi}(M)$. Fix an α-net $\{f_1, f_2, \dots, f_n\}$ for M in E_Φ. Since $M \subset E_\Phi$ we can assume that the functions f_i $(i = 1,2,\dots,n)$ are in $C(\overline{\Omega})$. By the uniform continuity of each f_i on $\overline{\Omega}$, there is some $\delta > 0$ such that $|f_i(t) - f_i(x)| \le \varepsilon$ holds for each $i \in \{1,\dots,n\}$ whenever $t, x \in \overline{\Omega}$ satisfy $\|x - t\|_n < \delta$. Then, if $0 < r < \delta$ we obtain $|f_i(t) - S_r(f_i)(t)| \le \varepsilon$ for all $t \in \overline{\Omega}$. The latter inequality implies $|||f_i - S_r(f_i)|||_\Phi \le \varepsilon|||\chi_\Omega|||_\Phi$. Moreover $|||S_r(f) - S_r(f_i)|||_\Phi = |||S_r(f - f_i)|||_\Phi \le |||f - f_i|||_\Phi$. Therefore

$$\begin{aligned} |||f - S_r(f)|||_\Phi &\le |||f - f_i|||_\Phi + |||f_i - S_r(f_i)|||_\Phi + |||S_r(f_i) - S_r(f)|||_\Phi \\ &\le 2|||f - f_i|||_\Phi + |||f_i - S_r(f_i)|||_\Phi \le 2\alpha + \varepsilon|||\chi_\Omega|||_\Phi \end{aligned} \tag{4.16}$$

holds for all $f \in M$ and $0 < r < \delta$. Hence $\omega_{E_\Phi}(M) \le 2\gamma_{E_\Phi}(M)$. □

From the last result and Corollary 4.8 we get the following.

Corollary 4.11. *Assume that the Young function Φ satisfies the Δ_2-condition, and let M be a bounded subset of $L_\Phi(\Omega)$. Then*

$$\omega_{L_\Phi[0,+\infty)}(M^*) \le 2\omega_{L_\Phi}(M). \tag{4.17}$$

Remark 4.12. We observe that in the Lebesgue space $L_p[0,1]$ $(1 \le p < \infty)$

$$\omega_p(f^*;\delta) \le 2\omega_p(f;\delta) \tag{4.18}$$

for $0 \le \delta \le 1/2$, where $\omega_p(f;\delta) = \sup_{0\le h\le\delta}(\int_{[0,1-h]} |f(x) - f(x+h)|^p d\mu)^{1/p}$ is the modulus of continuity of a given function $f \in L_p[0,1]$ (see [5, Theorem 3.1]).

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