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Research Article

Bessel's Differential Equation and Its Hyers-Ulam Stability

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We solve the inhomogeneous Bessel differential equation and apply this result to obtain a partial solution to the Hyers-Ulam stability problem for the Bessel differential equation.

1. Introduction

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin, in which he discussed a number of important unsolved problems (see [1]). Among those was the question concerning the stability of homomorphisms: let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\delta > 0$, does there exist an $\varepsilon > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \varepsilon$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \delta$ for all $x \in G_1$?

In the following year, Hyers [2] partially solved the Ulam problem for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been generalized by Rassias (see [3]). Since then, the stability problems of various functional equations have been investigated by many authors (see [4–6]).

We will now consider the Hyers-Ulam stability problem for the differential equations: assume that X is a normed space over a scalar field $\mathbb{K}$ and that I is an open interval, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Let $a_0, a_1, \dots, a_n : I \to \mathbb{K}$ be given continuous functions, let $g : I \to X$ be a given continuous function, and let $y : I \to X$ be an n times continuously differentiable function satisfying the inequality

$$\left\|a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + g(t)\right\| \le \varepsilon \tag{1.1}$$

for all $t \in I$ and for a given $\varepsilon > 0$. If there exists an n times continuously differentiable function $y_0 : I \to X$ satisfying

$$a_n(t)y_0^{(n)}(t) + a_{n-1}(t)y_0^{(n-1)}(t) + \cdots + a_1(t)y_0'(t) + a_0(t)y_0(t) + g(t) = 0 \tag{1.2}$$

and $\|y(t) - y_0(t)\| \le K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$, then we say that the above differential equation has the Hyers-Ulam stability. For more detailed definitions of the Hyers-Ulam stability, we refer the reader to [4–8].

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved in [9] that if a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \le \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \le 3\varepsilon$ for any $t \in I$.

This result of Alsina and Ger has been generalized by Takahasi et al. They proved in [10] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [11, 12]).

Moreover, Miura et al. [13] investigated the Hyers-Ulam stability of nth order linear differential equation with complex coefficients. They [14] also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function. Indeed, they dealt with the differential inequality $\|y'(t) + g(t)y(t)\| \le \varepsilon$ for some $\varepsilon > 0$.

Recently, Jung proved the Hyers-Ulam stability of various linear differential equations of first order (see [15–18]) and further investigated the general solution of the inhomogeneous Legendre differential equation and its Hyers-Ulam stability (see [14, 19]).

In Section 2 of this paper, by using the ideas from [19], we investigate the general solution of the inhomogeneous Bessel differential equation of the form

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = \sum_{m=0}^{\infty} a_m x^m, \tag{1.3}$$

where the parameter ν is a given positive nonintegral number. Section 3 will be devoted to a partial solution of the Hyers-Ulam stability problem for the Bessel differential equation (2.1) in a subclass of analytic functions.

2. Inhomogeneous Bessel equation

A function is called a Bessel function if it satisfies the Bessel differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0. \tag{2.1}$$

The Bessel equation plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary-value problems exhibiting cylindrical symmetries.

In this section, we define

$$c_m = -\sum_{i=0}^{[m/2]} a_{m-2i} \prod_{j=0}^{i} \frac{1}{\nu^2 - (m-2j)^2} \tag{2.2}$$

for each $m \in \{0,1,2,\dots\}$, where $[m/2]$ denotes the largest integer not exceeding $m/2$, and we refer to (1.3) for the a_m's. We can easily check that c_m's satisfy

$$\begin{aligned} a_0 = -\nu^2 c_0, \qquad a_1 = -(\nu^2-1)c_1, \\ a_{m+2} = c_m - \left(\nu^2 - (m+2)^2\right) c_{m+2} \end{aligned} \tag{2.3}$$

for any $m \in \{0,1,2,\dots\}$.

Lemma 1. (a) *If the power series $\sum_{m=0}^{\infty} a_m x^m$ converges for all $x \in (-\rho,\rho)$ with $\rho > 1$, then the power series $\sum_{m=0}^{\infty} c_m x^m$ with c_m's given in (2.2) satisfies the inequality $|\sum_{m=0}^{\infty} c_m x^m| \le C_1/(1-|x|)$ for some positive constant C_1 and for any $x \in (-1,1)$.*

(b) *If the power series $\sum_{m=0}^{\infty} a_m x^m$ converges for all $x \in (-\rho,\rho)$ with $\rho \le 1$, then for any positive $\rho_0 < \rho$, the power series $\sum_{m=0}^{\infty} c_m x^m$ with c_m's given in (2.2) satisfies the inequality $|\sum_{m=0}^{\infty} c_m x^m| \le C_2$ for any $x \in (-\rho_0,\rho_0)$ and for some positive constant C_2 which depends on ρ_0. Since ρ_0 is arbitrarily close to ρ, this means that $\sum_{m=0}^{\infty} c_m x^m$ is convergent for all $x \in (-\rho,\rho)$.*

Proof. (a) Since the power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on its interval of convergence, with $x = 1$, $\sum_{m=0}^{\infty} a_m$ converges absolutely, that is, $\sum_{m=0}^{\infty} |a_m| < M_1$ by some number M_1. Suppose that $p < \nu < p+1$ for some integer p. Then for any nonnegative integer q, $1/|\nu^2 - q^2| = 1/|\nu+q|1/|\nu-q|$ is less than 1 except, possibly, for $q = p$ and $q = p+1$. Therefore,

$$\prod_{j=0}^{i} \frac{1}{|\nu^2 - (m-2j)^2|} \le \max\left\{ \frac{1}{|\nu^2 - p^2|}, \frac{1}{|\nu^2 - (p+1)^2|} \right\} = M_2 \tag{2.4}$$

for any m and i. Now,

$$|c_m| \le \sum_{i=0}^{[m/2]} |a_{m-2i}| \prod_{j=0}^{i} \frac{1}{|\nu^2 - (m-2j)^2|} \le \sum_{i=0}^{[m/2]} |a_{m-2i}| M_2 \le M_1 M_2 = C_1 \tag{2.5}$$

and, therefore,

$$\left| \sum_{m=0}^{\infty} c_m x^m \right| \le \sum_{m=0}^{\infty} |c_m| \, |x^m| \le C_1 \sum_{m=0}^{\infty} |x^m| \le \frac{C_1}{1-|x|} \tag{2.6}$$

for $x \in (-1,1)$.

(b) The power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on its interval of convergence, and, therefore, for any given $\rho_0<\rho$, the series $\sum_{m=0}^{\infty}|a_m x^m|$ is convergent on $[-\rho_0,\rho_0]$ and

$$\sum_{m=0}^{\infty} |a_m||x|^m \le \sum_{m=0}^{\infty} |a_m| \rho_0^m = M_3 \tag{2.7}$$

for any $x \in [-\rho_0,\rho_0]$.

Also for $m \ge p+2$, if we let $M_2' = \max\{1, M_2\}$, then

$$\prod_{j=0}^{i} \frac{1}{|\nu^2-(m-2j)^2|} \le \frac{1}{|\nu^2-m^2|} M_2' \le \frac{1}{(m-p-1)^2} M_2'. \tag{2.8}$$

Now,

$$\begin{aligned} \left|\sum_{m=p+2}^{\infty} c_m x^m\right| &= \left| -\sum_{m=p+2}^{\infty} x^m \sum_{i=0}^{[m/2]} a_{m-2i} \prod_{j=0}^{i} \frac{1}{\nu^2-(m-2j)^2}\right| \\ &\le \sum_{m=p+2}^{\infty} \sum_{i=0}^{[m/2]} |a_{m-2i}| \rho_0^m \frac{1}{(m-p-1)^2} M_2' \\ &\le \sum_{m=p+2}^{\infty} \frac{1}{(m-p-1)^2} \sum_{i=0}^{[m/2]} |a_{m-2i}| \rho_0^{m-2i} M_2' \\ &\le \sum_{m=p+2}^{\infty} \frac{1}{(m-p-1)^2} M_3 M_2' \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2} M_3 M_2' \le 2M_3M_2', \end{aligned} \tag{2.9}$$

and, therefore, if $|\sum_{m=0}^{p+1} c_m x^m| \le \sum_{m=0}^{p+1} \sum_{i=0}^{[m/2]} |a_{m-2i}| \rho_0^{m-2i} M_2 \le (p+2) M_3 M_2$, then

$$\left|\sum_{m=0}^{\infty} c_m x^m\right| \le (p+2)M_2M_3 + 2M_2'M_3 = [(p+2)M_2 + 2M_2']M_3 = C_2 \tag{2.10}$$

for all $x \in (-\rho_0,\rho_0)$. □

Lemma 2. *Suppose that the power series $\sum_{m=0}^{\infty} a_m x^m$ converges for all $x \in (-\rho,\rho)$ with some positive ρ. Let $\rho_1 = \min\{1,\rho\}$. Then the power series $\sum_{m=0}^{\infty} c_m x^m$ with c_m's given in (2.2) is convergent for all $x \in (-\rho_1,\rho_1)$. Further, for any positive $\rho_0 < \rho_1$, $|\sum_{m=0}^{\infty} c_m x^m| \le C$ for any $x \in (-\rho_0,\rho_0)$ and for some positive constant C which depends on ρ_0.*

Proof. The first statement follows from the latter statement. Therefore, let us prove the latter statement. If $\rho \le 1$, then $\rho_1 = \rho$. By Lemma 1(b), for any positive $\rho_0 < \rho = \rho_1$, $|\sum_{m=0}^{\infty} c_m x^m| \le C_2$ for $x \in (-\rho_0,\rho_0)$ and for some positive constant C_2 which depends on ρ_0.

If $\rho > 1$, then by Lemma 1(a), for any positive $\rho_0 < 1 = \rho_1$,

$$\left|\sum_{m=0}^{\infty} c_m x^m\right| \le \frac{C_1}{1-|x|} < \frac{C_1}{1-\rho_0} = C \tag{2.11}$$

for $x \in (-\rho_0, \rho_0)$ and for some positive constant C which depends on ρ_0. □

Using these definitions and the lemmas above, we will show that $\sum_{m=0}^{\infty} c_m x^m$ is a particular solution of the inhomogeneous Bessel equation (1.3).

Theorem 2.1. *Assume that ν is a given positive nonintegral number and the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is ρ. Let $\rho_1 = \min\{1, \rho\}$. Then, every solution $y : (-\rho_1, \rho_1) \to \mathbb{C}$ of the differential equation (1.3) can be expressed by*

$$y(x) = y_h(x) + \sum_{m=0}^{\infty} c_m x^m, \tag{2.12}$$

where $y_h(x)$ is a Bessel function and c_m's are given by(2.2).

Proof. We show that $\sum_{m=0}^{\infty} c_m x^m$ satisfies (1.3). By Lemma 2, the power series $\sum_{m=0}^{\infty} c_m x^m$ is convergent for each $x \in (-\rho_1, \rho_1)$.

Substituting $\sum_{m=0}^{\infty} c_m x^m$ for $y(x)$ in (1.3) and collecting like powers together, we have

$$\begin{aligned} &x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) \\ &\quad = -\nu^2 c_0 - (\nu^2 - 1) c_1 x + \sum_{m=0}^{\infty} \left[c_m - \left(\nu^2 - (m+2)^2\right) c_{m+2}\right] x^{m+2} \\ &\quad = a_0 + a_1 x + \sum_{m=0}^{\infty} a_{m+2} x^{m+2} = \sum_{m=0}^{\infty} a_m x^m \end{aligned} \tag{2.13}$$

for all $x \in (-\rho_1, \rho_1)$ by (2.3).

Therefore, every solution $y : (-\rho_1, \rho_1) \to \mathbb{C}$ of the differential equation (1.3) can be expressed by

$$y(x) = y_h(x) + \sum_{m=0}^{\infty} c_m x^m, \tag{2.14}$$

where $y_h(x)$ is a Bessel function. □

3. Partial solution to Hyers-Ulam stability problem

In this section, we will investigate a property of the Bessel differential equation (2.1) concerning the Hyers-Ulam stability problem. That is, we will try to answer the question whether there exists a Bessel function near any approximate Bessel function.

Theorem 3.1. *Let $y : (-\rho, \rho) \to \mathbb{C}$ be a given analytic function which can be represented by a power-series expansion centered at $x = 0$. Suppose there exists a constant $\varepsilon > 0$ such that*

$$\left|x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x)\right| \le \varepsilon \tag{3.1}$$

for all $x \in (-\rho,\rho)$ *and for some positive nonintegral number* ν. *Let* $\rho_1 = \min\{1,\rho\}$. *Suppose, further, that* $x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = \sum_{m=0}^{\infty} a_m x^m$ *satisfies*

$$\sum_{m=0}^{\infty} |a_m x^m| \le K \left| \sum_{m=0}^{\infty} a_m x^m \right| \tag{3.2}$$

for all $x \in (-\rho,\rho)$ *and for some constant* K. *Then there exists a Bessel function* $y_h : (-\rho_1,\rho_1) \to \mathbb{C}$ *such that*

$$| y(x) - y_h(x) | \le C\varepsilon \tag{3.3}$$

for all $x \in (-\rho_0,\rho_0)$, *where* $\rho_0 < \rho_1$ *is any positive number and* C *is some constant which depends on* ρ_0.

Proof. We assumed that $y(x)$ can be represented by a power series and

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{3.4}$$

also satisfies

$$\sum_{m=0}^{\infty} |a_m x^m| \le K \left| \sum_{m=0}^{\infty} a_m x^m \right| \le K\varepsilon \tag{3.5}$$

for all $x \in (-\rho,\rho)$ from (3.1).

According to Theorem 2.1, y can be written as $y_h + \sum_{m=0}^{\infty} c_m x^m$ for $x \in (-\rho_1,\rho_1)$, where y_h is some Bessel function and c_m's are given by (2.2). Then by Lemmas 1 and 2 and their proofs (replace M_1 and M_3 with $K\varepsilon$ in Lemma 1),

$$| y(x) - y_h(x) | = \left| \sum_{m=0}^{\infty} c_m x^m \right| \le C\varepsilon \tag{3.6}$$

for all $x \in (-\rho_0,\rho_0)$, where $\rho_0 < \rho_1$ is any positive number and C is some constant which depends on ρ_0. This completes the proof of our theorem. □

4. Example

In this section, our task is to show that there certainly exist functions $y(x)$ which satisfy all the conditions given in Theorem 3.1.

Example 1. Let $y : (-1,1) \to \mathbb{R}$ be an analytic function given by

$$y(x) = J_{1/2}(x) + b(x^2 + x^4 + \cdots + x^{2n}), \tag{4.1}$$

where $J_{1/2}(x)$ is the Bessel function of the first kind of order 1/2, n is a given positive integer, and b is a constant satisfying

$$0 \le b \le \left[\frac{2}{3}n\left(2n^2 + 3n + \frac{17}{8}\right)\right]^{-1} \varepsilon \tag{4.2}$$

for some $\varepsilon \geq 0$. Since $J_{1/2}(x)$ is a particular solution of the Bessel differential equation (2.1) with $\nu = 1/2$, we then have

$$x^2 y''(x) + xy'(x) + \left(x^2 - \frac{1}{4}\right) y(x) = bx^{2n+2} + \sum_{m=2}^{n} \left[(2m)^2 + \frac{3}{4} \right] bx^{2m} + \frac{15}{4} bx^2. \tag{4.3}$$

If we set

$$a_m = \begin{cases} b & \text{for } m = 2n+2, \\ \left(m^2 + \frac{3}{4}\right) b & \text{for } m \in \{4, 6, \ldots, 2n\}, \\ \left(\frac{15}{4}\right) b & \text{for } m = 2, \\ 0 & \text{otherwise}, \end{cases} \tag{4.4}$$

then we obtain

$$x^2 y''(x) + xy'(x) + \left(x^2 - \frac{1}{4}\right) y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{4.5}$$

for all $x \in (-1, 1)$. It further follows from (4.2) and (4.4) that

$$\sum_{m=0}^{\infty} | a_m x^m | = \left| \sum_{m=0}^{\infty} a_m x^m \right| \leq \varepsilon \tag{4.6}$$

for any $x \in (-1, 1)$.

Indeed, if we choose the $J_{1/2}(x)$ as a Bessel function, then we have

$$| y(x) - J_{1/2}(x) | = b \, | x^2 + x^4 + \cdots + x^{2n} | \leq nb \leq n \left[\frac{2}{3} n \left(2n^2 + 3n + \frac{17}{8} \right) \right]^{-1} \varepsilon \tag{4.7}$$

for all $x \in (-1, 1)$, which is consistent with the assertion of Theorem 3.1.

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