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Research Article

Hermite-Hadamard-Type Inequalities for Increasing Positively Homogeneous Functions

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We study Hermite-Hadamard-type inequalities for increasing positively homogeneous functions. Some examples of such inequalities for functions defined on special domains are given.

1. Introduction

Recently, Hermite-Hadamard-type inequalities and their applications have attracted considerable interest, as shown in the book [1], for example. These inequalities have been studied for various classes of functions such as convex functions [1], quasiconvex functions [2–4], p-functions [3, 5], Godnova-Levin type functions [5], r-convex functions [6], increasing convex-along-rays functions [7], and increasing radiant functions [8], and it is shown that these inequalities are sharp.

For instance, if $f : [0,1] \to \mathbb{R}$ is an arbitrary nonnegative quasiconvex function, then for any $u \in (0,1)$ one has (see [3])

$$f(u) \leq \frac{1}{\min(u, 1-u)} \int_0^1 f(x)dx, \tag{1.1}$$

and the inequality (1.1) is sharp.

In this paper, we consider one generalization of Hermite-Hadamard-type inequalities for the class of increasing positively homogeneous of degree one functions defined on $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1,2,3,\dots,n\}$.

The structure of the paper is as follows: in Section 2, certain concepts of abstract convexity, definition of increasing positively homogeneous of degree one functions and its important properties are given. In Section 3, Hermite-Hadamard-type inequalities for the class of increasing positively homogeneous of degree one functions are considered. Some examples of such inequalities for functions defined on $\mathbb{R}^2_{++}$ are given in Section 4.

2. Preliminaries

First we recall some definitions from abstract convexity. Let $\mathbb{R}$ be a real line and $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$. Consider a set X and a set H of function $h : X \to \mathbb{R}$ defined on X. A function $f : X \to \mathbb{R}_{+\infty}$ is called abstract convex with respect to H (or H-convex) if there exists a set $U \subset H$ such that

$$f(x) = \sup\{h(x) : h \in U\} \quad \forall x \in X. \tag{2.1}$$

Clearly, f is H-convex if and only if

$$f(x) = \sup\{h(x) : h \leq f\} \quad \forall x \in X. \tag{2.2}$$

Let Y be a set of functions $f : X \to \mathbb{R}_{+\infty}$. A set $H \subset Y$ is called a supremal generator of the set Y, if each function $f \in Y$ is abstract convex with respect to H.

In some cases, the investigation of Hermite-Hadamard-type inequalities is based on the principle of preservation of inequalities [9].

Proposition 2.1 (principle of preservation of inequalities). *Let H be a supremal generator of Y and let Ψ be an increasing functional defined on Y. Then*

$$\big(h(u) \leq \Psi(h)\ \forall h \in H\big) \Longleftrightarrow \big(f(u) \leq \Psi(f)\ \forall f \in Y\big). \tag{2.3}$$

A function f defined on $\mathbb{R}^n_{++}$ is called increasing (with respect to the coordinate-wise order relation) if $x \geq y$ implies $f(x) \geq f(y)$.

The function f is positively homogeneous of degree one if $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n_{++}$ and $\lambda > 0$.

Let L be the set of all min-type functions defined on $\mathbb{R}^n_{++}$, that is, the set L consists of identical zero and all the functions of the form

$$l(x) = \langle l, x\rangle = \min_i \frac{x_i}{l_i}, \quad x \in \mathbb{R}^n_{++} \tag{2.4}$$

with all $l \in \mathbb{R}^n_{++}$.

One has (see [9]) that a function $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is L-convex if and only if f is increasing and positively homogeneous of degree one (shortly IPH).

Let us present the important property of IPH functions.

Proposition 2.2. *Let f be an IPH function defined on $\mathbb{R}^n_{++}$. Then the following inequality holds for all $x, l \in \mathbb{R}^n_{++}$:*

$$f(l)\langle l, x\rangle \leq f(x). \tag{2.5}$$

Proof. Since $\langle l,x\rangle = \min_{1\le i\le n}(x_i/l_i)$, then $\langle l,x\rangle l_i \le x_i$ is proved for all $i = 1,2,3,\dots,n$.

Consequently, we get $\langle l,x\rangle l \le x$. Because f is an IPH function,

$$f(x) \ge f(\langle l,x\rangle l) = \langle l,x\rangle f(l) \quad \forall l,x \in \mathbb{R}^n_{++}. \tag{2.6}$$

□

Let f be an IPH function defined on $\mathbb{R}^n_{++}$ and $D \subset \mathbb{R}^n_{++}$. It can be easily shown by Proposition 2.2 that the function

$$f_D(x) = \sup_{l\in D}\left(f(l)\langle l,x\rangle\right) \tag{2.7}$$

is IPH and it possesses the properties

$$f_D(x) \le f(x) \quad \forall x \in \mathbb{R}^n_{++}, \qquad f_D(x) = f(x) \quad \forall x \in D. \tag{2.8}$$

Let $D \subset \mathbb{R}^n_{++}$. A function $f : D \to [0,\infty]$ is called IPH on D if there exists an IPH function F defined on $\mathbb{R}^n_{++}$ such that $F|_D = f$, that is, $F(x) = f(x)$ for all $x \in D$.

Proposition 2.3. *Let $f : D \to [0,\infty]$ be a function on $D \subset \mathbb{R}^n_{++}$, then the following assertions are equivalent:*

(i) *f is abstract convex with respect to the set of functions $c\langle l,\cdot\rangle : D \to [0,\infty)$ with $l \in D$, $c \ge 0$;*

(ii) *f is IPH function on D;*

(iii) *$f(l)\langle l,x\rangle \le f(x)$ for all $l,x \in D$.*

Proof. (i)⇒(ii) It is obvious since any function $l(x) = c\langle l,x\rangle$ defined on D can be considered as elementary function $l(x) \in L$ defined on $\mathbb{R}^n_{++}$.

(ii)⇒(iii) By definition, there exists an IPH function $F : \mathbb{R}^n_{++} \to [0,\infty]$ such that $F(x) = f(x)$ for all $x \in D$. Then by (2.7) we have

$$f(x) = F_D(x) = \sup_{l\in D}\left(F(l)\langle l,x\rangle\right) = \sup_{l\in D}\left(f(l)\langle l,x\rangle\right) \tag{2.9}$$

for all $x \in D$, which implies the assertion (iii).

(iii)⇒(i) Consider the function f_D defined on D, $\sup_{l\in D}(f(l)\langle l,x\rangle) = f_D(x)$. It is clear that f_D is abstract convex with respect to the set of functions $\{c\langle l,\cdot\rangle : l \in D,\ c \ge 0\}$ defined on D. Further, using (iii) we get that for all $x \in D$,

$$f_D(x) \le f(x) = f(x)\langle x,x\rangle \le \sup_{l\in D}\left(f(l)\langle l,x\rangle\right) = f_D(x). \tag{2.10}$$

So, $f_D(x) = f(x)$ for all $x \in D$ and we have the defined statement (i). □

3. Hermite-Hadamard-type inequalities for IPH functions

Now, we will research to Hermite-Hadamard-type inequality for IPH functions.

Proposition 3.1. *Let $D \subset \mathbb{R}^n_{++}$, $f : D \to [0,\infty]$ is IPH function, and f is integrable on D. Then*

$$f(u)\int_D \langle u,x\rangle dx \le \int_D f(x)dx \tag{3.1}$$

for all $u \in D$.

Proof. It can be seen via Proposition 2.3. Since $f(l)\langle l,x\rangle \le f(x)$ for all $l,x \in D$, (3.1) is clear. □

Let us investigate Hermite-Hadamard-type inequalities via $Q(D)$ sets given in [7, 8].

Let $D \subset \mathbb{R}^n_{++}$ be a closed domain, that is, D is bounded set such that $cl\,\mathrm{int}\,D = D$. Denote by $Q(D)$ the set of all points $x^* \in D$ such that

$$\frac{1}{A(D)}\int_D \langle x^*,x\rangle dx = 1, \tag{3.2}$$

where $A(D) = \int_D dx$.

Proposition 3.2. *Let f be an IPH function defined on D. If the set $Q(D)$ is nonempty and f is integrable on D, then*

$$\sup_{x^*\in Q(D)} f(x^*) \le \frac{1}{A(D)}\int_D f(x)dx. \tag{3.3}$$

Proof. If we take $f(x^*) = +\infty$, by using the equality (2.5), it can be easily shown that f cannot be integrable. So $f(x^*) < +\infty$. According to Proposition 2.3,

$$f(x^*)\langle x^*,x\rangle \le f(x) \quad \forall x \in D. \tag{3.4}$$

Since $x^* \in Q(D)$, then by (3.2) we get

$$\begin{aligned} f(x^*) &= f(x^*)\frac{1}{A(D)}\int_D \langle x^*,x\rangle dx \\ &= \frac{1}{A(D)}\int_D \langle x^*,x\rangle f(x^*)dx \le \frac{1}{A(D)}\int_D f(x)dx. \end{aligned} \tag{3.5}$$

□

Remark 3.3. For each $x^* \in Q(D)$ we have also the following inequality, which is weaker than (3.3):

$$f(x^*) \le \frac{1}{A(D)}\int_D f(x)dx. \tag{3.6}$$

However, even the inequality (3.6) is sharp. For example, if $f(x) = \langle x^*,x\rangle$, then (3.6) holds as the equality.

Remark 3.4. Let $Q(D)$ be a nonempty set. We can define a set $Q_k(D)$ for every positive real number k such that $Q_k(D) = \{u \in D : u = k \cdot x^*,\ x^* \in Q(D)\}$. The set $Q_k(D)$ above can be easily defined as follows: $Q_k(D) = \{u \in D : (k/A(D)) \int_D \langle u,x\rangle dx = 1\}$.

Considering the property that an IPH function is positively homogeneous of degree one, we can generalize the inequality (3.3) as follows:

$$\sup_{u \in Q_k(D)} f(u) \le \frac{k}{A(D)} \int_D f(x)dx. \tag{3.7}$$

Let us try to derive inequalities similar to the right hand of the statement which is derived for convex functions (see [1]).

Let f be an IPH function defined on a closed domain $D \subset \mathbb{R}^n_{++}$, and f is integrable on D. Then $f(l)\langle l,x\rangle \le f(x)$ for all $l,x \in D$. Hence for all $l,x \in D$,

$$f(l) \le \frac{f(x)}{\langle l,x\rangle} = \langle x,l\rangle^+ f(x), \tag{3.8}$$

where $\langle x,l\rangle^+ = \max_{1\le i\le n} l_i/x_i$ is the so-called max-type function.

We have established the following result.

Proposition 3.5. *Let f be IPH and integrable function on D. Then*

$$\int_D f(x)dx \le \inf_{u\in D}\left[f(u)\int_D \langle u,x\rangle^+ dx\right]. \tag{3.9}$$

For every $u \in D$, inequality

$$\int_D f(x)dx \le f(u)\int_D \langle u,x\rangle^+ dx \tag{3.10}$$

is sharp.

4. Examples

On some special domains D of the cones $\mathbb{R}_{++}$ and $\mathbb{R}^2_{++}$, Hermite-Hadamard-type inequalities have been stated for ICAR and InR functions (see [7, 8]). Let us derive the set $Q(D)$ and the inequalities (3.1), (3.6), (3.9), for IPH functions, too.

Before the examples, for a region $D \subset \mathbb{R}^2_{++}$ and every $u \in D$, let us derive the computation formula of the integral $\int_D \langle u,x\rangle dx$.

Let $D \subset \mathbb{R}^2_{++}$ and $u = (u_1,u_2) \in D$. In order to calculate the integral, we represent the set D as $D_1(u) \cup D_2(u)$, where

$$D_1(u) = \left\{x \in D : \frac{x_2}{u_2} \le \frac{x_1}{u_1}\right\}, \qquad D_2(u) = \left\{x \in D : \frac{x_2}{u_2} \ge \frac{x_1}{u_1}\right\}. \tag{4.1}$$

Then

$$\int_D \langle u,x\rangle dx = \int_{D_1(u)} \langle u,x\rangle dx + \int_{D_2(u)} \langle u,x\rangle dx = \frac{1}{u_2}\int_{D_1(u)} x_2 dx_1 dx_2 + \frac{1}{u_1}\int_{D_2(u)} x_1 dx_1 dx_2. \tag{4.2}$$

Example 4.1. Consider the triangle D defined as

$$D = \{(x_1,x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \le a,\ 0 < x_2 \le v x_1\}. \tag{4.3}$$

Let $u \in D$. Assume that the $\mathbb{R}_u$ is ray defined by the equation $x_2 = (u_2/u_1)x_1$. Since $u \in D$, we get $0 < u_2/u_1 \le v$. Hence $\mathbb{R}_u$ intersects the set D and divides the set into two parts D_1 and D_2 given as

$$D_1(u) = \left\{(x_1,x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \le a,\ 0 < x_2 \le \frac{u_2}{u_1}x_1\right\} = \left\{(x_1,x_2) \in D : \frac{x_2}{u_2} \le \frac{x_1}{u_1}\right\},$$
$$D_2(u) = \left\{(x_1,x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \le a,\ \frac{u_2}{u_1}x_1 \le x_2 \le v x_1\right\} = \left\{(x_1,x_2) \in D : \frac{x_2}{u_2} \ge \frac{x_1}{u_1}\right\}. \tag{4.4}$$

By (4.2) we get

$$\begin{aligned}\int_D \langle u,x\rangle dx &= \frac{1}{u_2}\int_{D_1(u)} x_2 dx_1 dx_2 + \frac{1}{u_1}\int_{D_2(u)} x_1 dx_1 dx_2 \\ &= \frac{1}{u_2}\int_0^a \int_0^{(u_2/u_1)x_1} x_2 dx_2 dx_1 + \frac{1}{u_1}\int_0^a \int_{(u_2/u_1)x_1}^{v x_1} x_1 dx_2 dx_1 \\ &= \frac{a^3 u_2}{6u_1^2} + \frac{(u_1 v - u_2)a^3}{3u_1^2} = \frac{(2u_1 v - u_2)a^3}{6u_1^2}.\end{aligned} \tag{4.5}$$

Thus, for the given region D, the inequality (3.1) will be as follows:

$$f(u_1,u_2) \le \frac{6u_1^2}{a^3(2u_1 v - u_2)}\int_D f(x_1,x_2) dx_1 dx_2. \tag{4.6}$$

Since $A(D) = va^2/2$, then a point $x^* \in D$ belongs to $Q(D)$ if and only if

$$\frac{2}{va^2}\frac{(2x_1^* v - x_2^*)a^3}{6(x_1^*)^2} = 1 \Longleftrightarrow x_2^* = -\frac{3v}{a}(x_1^*)^2 + 2v x_1^*. \tag{4.7}$$

Consider now the inequality (3.9) for triangle D. Let us calculate the integral of the function $\langle u,x\rangle^+$ on D:

$$\begin{aligned}\int_D \langle u,x\rangle^+ dx &= \frac{1}{u_1}\int_{D_1(u)} x_1 dx_1 dx_2 + \frac{1}{u_2}\int_{D_2(u)} x_2 dx_1 dx_2 \\ &= \frac{1}{u_1}\int_0^a \int_0^{(u_2/u_1)x_1} x_1 dx_2 dx_1 + \frac{1}{u_2}\int_0^a \int_{(u_2/u_1)x_1}^{vx_1} x_2 dx_2 dx_1 \\ &= \frac{a^3}{6}\left(\frac{u_2}{u_1^2}+\frac{v^2}{u_2}\right).\end{aligned} \tag{4.8}$$

Therefore,

$$\int_D f(x_1,x_2)dx_1dx_2 \le \frac{a^3}{6}\inf_{u\in D}\left\{\left(\frac{u_2}{u_1^2}+\frac{v^2}{u_2}\right)f(u_1,u_2)\right\}. \tag{4.9}$$

Example 4.2. Let $D \subset \mathbb{R}^2_{++}$ be the triangle with vertices $(0,0)$, $(a,0)$ and $(0,b)$, that is

$$D = \left\{x \in \mathbb{R}^2_{++} : \frac{x_1}{a}+\frac{x_2}{b} \le 1\right\}. \tag{4.10}$$

If $u \in D$, then we get

$$\begin{aligned} D_1(u) &= \left\{x \in \mathbb{R}^2_{++} : 0 < x_2 < \frac{abu_2}{au_2+bu_1},\ \frac{u_1}{u_2}x_2 \le x_1 \le a - \frac{a}{b}x_2\right\} \\ D_2(u) &= \left\{x \in \mathbb{R}^2_{++} : 0 < x_1 < \frac{abu_1}{au_2+bu_1},\ \frac{u_2}{u_1}x_1 \le x_2 \le b - \frac{b}{a}x_1\right\}.\end{aligned} \tag{4.11}$$

By (4.2) we have

$$\begin{aligned}\int_D \langle u,x\rangle dx &= \frac{1}{u_2}\int_{D_1(u)} x_2 dx_1 dx_2 + \frac{1}{u_1}\int_{D_2(u)} x_1 dx_1 dx_2 \\ &= \frac{1}{u_2}\int_0^{abu_2/(au_2+bu_1)} \int_{(u_1/u_2)x_2}^{a-(a/b)x_2} x_2 dx_1 dx_2 + \frac{1}{u_1}\int_0^{abu_1/(au_2+bu_1)} \int_{(u_2/u_1)x_1}^{b-(b/a)x_1} x_1 dx_2 dx_1 \\ &= \frac{a^3b^2u_2}{6(au_2+bu_1)^2} + \frac{a^2b^3u_1}{6(au_2+bu_1)^2} = \frac{a^2b^2}{6(au_2+bu_1)} = \frac{ab}{6(u_1/a+u_2/b)}.\end{aligned} \tag{4.12}$$

In this triangular region D, the inequality (3.1) is as follows:

$$f(u_1,u_2) \le \frac{6}{ab}\left(\frac{u_1}{a}+\frac{u_2}{b}\right)\int_D f(x_1,x_2)dx_1dx_2. \tag{4.13}$$

Let us derive the set $Q(D)$ for the given triangular region D. Since $A(D) = ab/2$, then for $x^* \in D$,

$$x^* \in Q(D) \Longleftrightarrow \frac{x_1^*}{a}+\frac{x_2^*}{b} = \frac{1}{3}. \tag{4.14}$$

Therefore,

$$Q(D) = \left\{x^* \in D : \frac{x_1^*}{a} + \frac{x_2^*}{b} = \frac{1}{3}\right\}. \tag{4.15}$$

For the same region D, let us compute $\int_D \langle u,x\rangle^+ dx$ in order to derive the inequality (3.9):

$$\begin{aligned}
\int_D \langle u,x\rangle^+ dx &= \frac{1}{u_1}\int_{D_1(u)} x_1 dx_1 dx_2 + \frac{1}{u_2}\int_{D_2(u)} x_2 dx_1 dx_2 \\
&= \frac{1}{2u_1}\left[\frac{a^3 b u_2}{au_2+bu_1} - \frac{a^4 b u_2^2}{(au_2+bu_1)^2} + \left(\frac{a^2}{b^2} - \frac{u_1^2}{u_2^2}\right)\frac{a^3 b^3 u_2^3}{3(au_2+bu_1)^3}\right] \\
&\quad + \frac{1}{2u_2}\left[\frac{ab^3 u_1}{au_2+bu_1} - \frac{b^4 a u_1^2}{(au_2+bu_1)^2} + \left(\frac{b^2}{a^2} - \frac{u_2^2}{u_1^2}\right)\frac{a^3 b^3 u_1^3}{3(au_2+bu_1)^3}\right] \\
&= \frac{ab}{6}\left(\frac{au_2+bu_1}{u_1u_2} - \frac{1}{au_2+bu_1}\right).
\end{aligned} \tag{4.16}$$

Hence,

$$\int_D f(x_1,x_2)dx_1dx_2 \le \frac{ab}{6}\inf_{u\in D}\left\{\left(\frac{au_2+bu_1}{u_1u_2} - \frac{1}{au_2+bu_1}\right)f(u_1,u_2)\right\}. \tag{4.17}$$

Example 4.3. We will now consider the rectangle in $\mathbb{R}^2_{++}$. Let D be the rectangle defined as

$$D = \{x \in \mathbb{R}^2_{++} : x_1 \le a,\ x_2 \le b\}. \tag{4.18}$$

We consider two possible cases for $u \in D$.

(a) If $u_2/u_1 \le b/a$, then we have

$$\begin{aligned}
D_1(u) &= \left\{x \in \mathbb{R}^2_{++} : 0 < x_1 \le a,\ 0 < x_2 \le \frac{u_2}{u_1}x_1\right\}, \\
D_2(u) &= \left\{x \in \mathbb{R}^2_{++} : 0 < x_1 \le a,\ \frac{u_2}{u_1}x_1 \le x_2 \le b\right\}.
\end{aligned} \tag{4.19}$$

Therefore,

$$\begin{aligned}
\int_D \langle u,x\rangle dx &= \frac{1}{u_2}\int_{D_1(u)} x_2 dx_1 dx_2 + \frac{1}{u_1}\int_{D_2(u)} x_1 dx_1 dx_2 \\
&= \frac{1}{u_2}\int_0^a\int_0^{(u_2/u_1)x_1} x_2 dx_2 dx_1 + \frac{1}{u_1}\int_0^a\int_{(u_2/u_1)x_1}^b x_1 dx_2 dx_1 \\
&= \frac{1}{u_2}\frac{u_2^2 a^3}{6u_1^2} + \frac{1}{u_1}\left(\frac{ba^2}{2} - \frac{u_2}{u_1}\frac{a^3}{3}\right) = \frac{3ba^2u_1 - u_2a^3}{6u_1^2}.
\end{aligned} \tag{4.20}$$

By using the equality above, the inequality (3.1) will be as follows:

$$f(u_1,u_2) \le \frac{6u_1^2}{3ba^2u_1 - u_2a^3}\int_D f(x_1,x_2)\,dx_1dx_2. \tag{4.21}$$

Let us derive the set $Q(D)$. Since $A(D) = ab$, then we get the equation for $x^* \in Q(D)$,

$$\frac{1}{ab}\frac{3ba^2x_1^* - x_2^*a^3}{6(x_1^*)^2} = 1 \Longleftrightarrow x_2^* = -\frac{6b}{a^2}(x_1^*)^2 + \frac{3b}{a}x_1^*. \tag{4.22}$$

(b) If $u_2/u_1 \ge b/a$, then by analogy

$$\int_D \langle u,x\rangle dx = \frac{3b^2au_2 - u_1b^3}{6u_2^2}. \tag{4.23}$$

Hence,

$$f(u_1,u_2) \le \frac{6u_2^2}{3ab^2u_2 - u_1b^3}\int_D f(x_1,x_2)\,dx_1dx_2. \tag{4.24}$$

We get the symmetric equation for $x^* \in Q(D)$:

$$x_1^* = -\frac{6a}{b^2}(x_2^*)^2 + \frac{3a}{b}x_2^*. \tag{4.25}$$

By taking into account both cases, $Q(D)$ becomes as the following:

$$\begin{aligned} Q(D) = {}&\left\{x^* \in D : \frac{x_2^*}{x_1^*} \le \frac{b}{a},\ x_2^* = -\frac{6b}{a^2}(x_1^*)^2 + \frac{3b}{a}x_1^*\right\} \\ &\cup \left\{x^* \in D : \frac{x_2^*}{x_1^*} \ge \frac{b}{a},\ x_1^* = -\frac{6a}{b^2}(x_2^*)^2 + \frac{3a}{b}x_2^*\right\}. \end{aligned} \tag{4.26}$$

Consider now inequality (3.9). If $u_2/u_1 \le b/a$, then $D_1(u)$ and $D_2(u)$ are stated as similar to (4.19). Consequently,

$$\int_D \langle u,x\rangle^+ dx = \frac{1}{u_1}\int_{D_1(u)} x_1dx_1dx_2 + \frac{1}{u_2}\int_{D_2(u)} x_2dx_1dx_2 = \frac{u_2a^3}{6u_1^2} + \frac{ab^2}{2u_2}. \tag{4.27}$$

If $u_2/u_1 \ge b/a$, then by analogy

$$\int_D \langle u,x\rangle^+ dx = \frac{u_1b^3}{6u_2^2} + \frac{ba^2}{2u_1}. \tag{4.28}$$

That is,

$$\int_D \langle u,x\rangle^+ dx = \varphi(u) = \begin{cases} \dfrac{u_2a^3}{6u_1^2} + \dfrac{ab^2}{2u_2}, & \text{if } \dfrac{u_2}{u_1} \le \dfrac{b}{a}, \\[2ex] \dfrac{u_1b^3}{6u_2^2} + \dfrac{ba^2}{2u_1}, & \text{if } \dfrac{u_2}{u_1} \ge \dfrac{b}{a}. \end{cases} \tag{4.29}$$

Therefore

$$\int_D f(x_1,x_2)\,dx_1dx_2 \leq \inf_{u\in D}\{f(u_1,u_2)\varphi(u_1,u_2)\}. \tag{4.30}$$

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