A CHARACTERIZATION OF CHAOTIC ORDER

CHANGSEN YANG AND FUGEN GAO

Received 15 November 2005; Accepted 4 January 2006

The chaotic order $A \gg B$ among positive invertible operators $A, B > 0$ on a Hilbert space is introduced by $\log A \geq \log B$. Using Uchiyama's method and Furuta's Kantorovich-type inequality, we will point out that $A \gg B$ if and only if $\|B^p A^{-p/2} B^{-p/2}\| A^p \geq B^p$ holds for any $0 < p < p_0$, where p_0 is any fixed positive number. On the other hand, for any fixed $p_0 > 0$, we also show that there exist positive invertible operators A, B such that $\|B^p A^{-p/2} B^{-p/2}\| A^p \geq B^p$ holds for any $p \geq p_0$, but $A \gg B$ is not valid.

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive, in symbol $T \geq 0$ if $(Tx, x) \geq 0$ for all $x \in H$. In particular, we denote by $A > 0$ if $A \geq 0$ is invertible. By the operator monotonicity of the logarithmic function, we know that $A \geq B > 0$ implies the chaotic order $A \gg B$. For the chaotic order, several characterizations were shown by many authors, for example, [1–3, 6]. The following well-known results about chaotic order were obtained.

Theorem 1.1 [1, 2]. *Let A and B be positive invertible operators. Then the following properties are mutually equivalent:*

(i) $\log A \geq \log B$*;*

(ii) $(B^{p/2} A^p B^{p/2})^{1/2} \geq B^p$ *for all* $p \geq 0$*;*

(iii) $(B^{r/2} A^p B^{r/2})^{r/(p+r)} \geq B^r$ *for all* $p \geq 0$ *and* $r \geq 0$.

Theorem 1.2 Kantorovich type inequalities [3]. *Let $A > 0$ and for positive numbers M, m, $M \geq B \geq m > 0$. Then the following parallel statements hold. Moreover, (ii) can be derived from (i).*

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 79123, Pages 1–6
DOI 10.1155/JIA/2006/79123

(i) $A \geq B$ *implies* $((M^{p-1}+m^{p-1})^2/(4m^{p-1}M^{p-1}))A^p \geq B^p$ *for all* $p \geq 2$.
(ii) $\log A \geq \log B$ *implies* $((M^p+m^p)^2/(4m^pM^p))A^p \geq B^p$ *for all* $p \geq 0$.

Theorem 1.3 [6]. *Let A and B be positive invertible operators. Then* $A \geq B > 0$ *if and only if* $\|B^{p-1}A^{-(p-2)/2}B^{-p/2}\|A^{p-1} \geq B^{p-1}$ *for all* $p \geq 2$.

As a parallel statement of Theorem 1.3, we point out the following result on the chaotic order of two positive invertible operators.

Theorem 1.4. *Let A and B be positive invertible operators. Then for a fixed* $p_0 > 0$, *the following assertions are mutually equivalent:*
(i) $A \gg B$;
(ii) $\|B^pA^{-p/2}B^{-p/2}\|A^p \geq B^p$ *holds for all* $p > 0$;
(iii) $\|B^pA^{-p/2}B^{-p/2}\|A^p \geq B^p$ *holds for any* $p \in (0,p_0)$.

On the other hand, we will prove that the condition $p \in (0,p_0)$ in Theorem 1.4 is essential as follows.

Theorem 1.5. *For a fixed* $p_0 > 0$, *there exist positive invertible operators A, B such that* $\|B^pA^{-p/2}B^{-p/2}\|A^p \geq B^p$ *holds for any* $p \geq p_0$, *but* $A \gg B$ *is not valid.*

2. The proofs of the main results

To give a proof of Theorem 1.4, we also need the following well-known theorem used in [3] which is essentially the same as [5].

Theorem 2.1 [3, 5]. *Let* $X > 0$, *then* $\lim_{n\to\infty}(I+\log X/n)^n = X$.

Proof of Theorem 1.4. (i)⇒(ii) Suppose that $\log A \geq \log B$. Let $p > 0$, then for sufficiently large n, we have $I+\log A/n \geq I+\log B/n > 0$ and $np \geq 2$. Put $A_1 = I+\log A/n$ and $B_1 = I+\log B/n$. Then we have $A_1 \geq B_1 > 0$ and applying Theorem 1.3, the following inequality holds:

$$\left\|B_1^{n(p-1/n)}A_1^{n(-(p-2/n)/2)}B_1^{n(-p/2)}\right\|A_1^{n(p-1/n)} \geq B_1^{n(p-1/n)} \tag{2.1}$$

for all $np \geq 2$. By Theorem 2.1, we have $A_1^n \to A$ and $B_1^n \to B$ as $n \to \infty$. Hence let $n \to \infty$ in (2.1), then we obtain $\|B^pA^{-p/2}B^{-p/2}\|A^p \geq B^p$ holds for all $p > 0$;
(ii)⇒(iii) Obvious.
(iii)⇒(i) Let $0 < p < p_0$ and $\lambda_p = \|B^pA^{-p/2}B^{-p/2}\|$. Then $B^p \leq \lambda_pA^p$ by (iii). By L-H theorem, we also have $B^{p/2} \leq \lambda_p^{1/2}A^{p/2}$, thus $B^{3p/2} \leq \lambda_p^{1/2}B^{p/2}A^{p/2}B^{p/2}$. Now suppose that $0 < m \leq B \leq M$. So $0 < m^{3p/2} \leq B^{3p/2} \leq M^{3p/2}$. Applying (i) of Theorem 1.2, we obtain

$$B^{3p} \leq \frac{(M^{3p/2}+m^{3p/2})^2}{4M^{3p/2}m^{3p/2}}\lambda_p(B^{p/2}A^{p/2}B^{p/2})^2. \tag{2.2}$$

Hence

$$B^{2p} \le \frac{\left(M^{3p/2}+m^{3p/2}\right)^2}{4M^{3p/2}m^{3p/2}}\lambda_p A^{p/2}B^pA^{p/2}. \tag{2.3}$$

By (2.3) and $\lambda_p = \|B^{-p/2}A^{-p/2}B^{2p}A^{-p/2}B^{-p/2}\|^{1/2}$, we have

$$\lambda_p^2 \le \frac{\left(M^{3p/2}+m^{3p/2}\right)^2}{4M^{3p/2}m^{3p/2}}\lambda_p. \tag{2.4}$$

So

$$\lambda_p \le \frac{\left(M^{3p/2}+m^{3p/2}\right)^2}{4M^{3p/2}m^{3p/2}}. \tag{2.5}$$

Therefore

$$B^p \le \frac{\left(M^{3p/2}+m^{3p/2}\right)^2}{4M^{3p/2}m^{3p/2}}A^p. \tag{2.6}$$

By (2.6), we also have

$$\log B \le \frac{1}{p}\log\frac{\left(M^{3p/2}+m^{3p/2}\right)^2}{4M^{3p/2}m^{3p/2}} + \log A. \tag{2.7}$$

Let $p \to 0$, we obtain (i). □

To prove Theorem 1.5, we first cite the following simple inequalities.

Lemma 2.2. *Let a, b, d be three positive numbers, then*

(i) $b \le \|\left(\begin{smallmatrix} a & b \\ b & d\end{smallmatrix}\right)\|$,

(ii) $\left(\begin{smallmatrix} a & b \\ b & d\end{smallmatrix}\right) \le (a+b+d)I$.

Proof of Theorem 1.5. Suppose $p_0 > 0$. Let $A = \left(\begin{smallmatrix} 9/5 & -2/5 \\ -2/5 & 6/5\end{smallmatrix}\right)$, and $B = \left(\begin{smallmatrix} 2 & 0 \\ 0 & \varepsilon\end{smallmatrix}\right)$, where $\varepsilon \in (0,(1/2)[(2-2^{1-p_0/2})^2/(7+3\cdot 2^{-p_0})^4]^{1/p_0})$.

Note that $A = U^*\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1\end{smallmatrix}\right)U$, where $U = (1/\sqrt{5})\left(\begin{smallmatrix} -2 & 1 \\ 1 & 2\end{smallmatrix}\right)$ is a unitary operator, by a simple computation, we have

$$\begin{aligned} &B^{-p/2}A^{-p/2}B^{2p}A^{-p/2}B^{-p/2} \\ &= \frac{1}{25}\begin{pmatrix} \left(1+2^{2-p/2}\right)^2 2^p + 2^{-p}\varepsilon^{2p}\left(2-2^{1-p/2}\right)^2 & \left(2-2^{1-p/2}\right)\left[\dfrac{2^{3p/2}\left(1+2^{2-p/2}\right)}{\varepsilon^{p/2}} + \dfrac{\varepsilon^{3p/2}\left(4+2^{-p/2}\right)}{2^{p/2}}\right] \\ \left(2-2^{1-p/2}\right)\left[\dfrac{2^{3p/2}\left(1+2^{2-p/2}\right)}{\varepsilon^{p/2}} + \dfrac{\varepsilon^{3p/2}\left(4+2^{-p/2}\right)}{2^{p/2}}\right] & 2^{2p}\varepsilon^{-p}\left(2-2^{1-p/2}\right)^2 + \varepsilon^p\left(4+2^{-p/2}\right)^2 \end{pmatrix}. \end{aligned} \tag{2.8}$$

Applying (i) of Lemma 2.2, we obtain

$$\begin{aligned}
&\left\|B^{-p/2}A^{-p/2}B^{2p}A^{-p/2}B^{-p/2}\right\|^{1/2}\\
&\quad\geq \frac{1}{5}\left\{\left(2-2^{1-p/2}\right)\left[\frac{2^{3p/2}\left(1+2^{2-p/2}\right)}{\varepsilon^{p/2}}+\frac{\varepsilon^{3p/2}\left(4+2^{-p/2}\right)}{2^{p/2}}\right]\right\}^{1/2}\\
&\quad\geq \frac{\varepsilon^{-p/4}2^{3p/4}\left(2-2^{1-p/2}\right)^{1/2}}{5}\geq\frac{\varepsilon^{-p/4}2^{3p/4}\left(2-2^{1-p_0/2}\right)^{1/2}}{5}.
\end{aligned}\tag{2.9}$$

On the other hand, we can compute that

$$\begin{aligned}
&A^{-p/2}B^pA^{-p/2}\\
&\quad=\frac{1}{25}\begin{pmatrix}
2^p\left(1+4\cdot 2^{-p/2}\right)^2+4\varepsilon^p\left(1-2^{-p/2}\right)^2 & \begin{array}{r}\left(1-2^{-p/2}\right)\left[2^{p+1}\left(1+4\cdot 2^{-p/2}\right)\right.\\ \left.+2\varepsilon^p\left(4+2^{-p/2}\right)\right]\end{array}\\
\begin{array}{r}\left(1-2^{-p/2}\right)\left[2^{p+1}\left(1+4\cdot 2^{-p/2}\right)\right.\\ \left.+2\varepsilon^p\left(4+2^{-p/2}\right)\right]\end{array} & 4\cdot 2^p\left(1-2^{-p/2}\right)^2+\varepsilon^p\left(4+2^{-p/2}\right)^2
\end{pmatrix}.
\end{aligned}\tag{2.10}$$

Hence by Lemma 2.2 (ii), we have

$$\begin{aligned}
A^{-p/2}B^pA^{-p/2}&\leq\frac{1}{25}\left\{2^p\left(1+4\cdot 2^{-p/2}\right)^2+4\varepsilon^p\left(1-2^{-p/2}\right)^2\right.\\
&\qquad+\left(1-2^{-p/2}\right)\left[2^{p+1}\left(1+4\cdot 2^{-p/2}\right)+2\varepsilon^p\left(4+2^{-p/2}\right)\right]\\
&\qquad\left.+4\cdot 2^p\left(1-2^{-p/2}\right)^2+\varepsilon^p\left(4+2^{-p/2}\right)^2\right\}\\
&=\frac{2^p}{25}\left[7+6\cdot 2^{-p/2}+12\cdot 2^{-p}\right]+\frac{\varepsilon^p}{25}\left[28-6\cdot 2^{-p/2}+3\cdot 2^{-p}\right]\\
&\leq\frac{2^p}{25}\left[35+15\cdot 2^{-p}\right]\leq\frac{2^p}{5}\left[7+3\cdot 2^{-p_0}\right].
\end{aligned}\tag{2.11}$$

Because $0<(2\varepsilon)^{p_0/4}<(2-2^{1-p_0/2})^{1/2}/(7+3\cdot 2^{-p_0})<1$, so for $p>p_0$,

$$(2\varepsilon)^{p/4}<\frac{\left(2-2^{1-p_0/2}\right)^{1/2}}{7+3\cdot 2^{-p_0}}<1.\tag{2.12}$$

Therefore by (2.9), (2.11), and (2.12), we have

$$\begin{aligned}
\left\|B^{-p/2}A^{-p/2}B^{2p}A^{-p/2}B^{-p/2}\right\|^{1/2}&\geq\frac{\varepsilon^{-p/4}2^{3p/4}\left(2-2^{1-p_0/2}\right)^{1/2}}{5}\\
&\geq\frac{2^p}{5}\left[7+3\cdot 2^{-p_0}\right]\geq A^{-p/2}B^pA^{-p/2}.
\end{aligned}\tag{2.13}$$

To complete the proof of Theorem 1.5, we only prove that $(AB^2A)^{1/2} \nleq A^2$ for very small $\varepsilon > 0$ by Theorem 1.1. But by a simple computation, this is equivalent to prove

$$\begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix} \equiv \begin{pmatrix} 324+4\varepsilon^2 & -72-12\varepsilon^2 \\ -72-12\varepsilon^2 & 16+36\varepsilon^2 \end{pmatrix}^{1/2} \nleq \begin{pmatrix} 17 & -6 \\ -6 & 8 \end{pmatrix}. \tag{2.14}$$

Let $A_1 = 324+4\varepsilon^2$, $A_2 = 16+36\varepsilon^2$, and $A_3 = -72-12\varepsilon^2$. By [4], if

$$V = \frac{1}{\sqrt{A_1 - A_2 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{A_1 - A_2 + \varepsilon_1} & -\sqrt{\varepsilon_1} \\ -\sqrt{\varepsilon_1} & -\sqrt{A_1 - A_2 + \varepsilon_1} \end{pmatrix}, \tag{2.15}$$

where

$$2\varepsilon_1 = -A_1 + A_2 + \sqrt{(A_1 - A_2)^2 + 4A_3^2}. \tag{2.16}$$

Then

$$\begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix} = V \begin{pmatrix} \sqrt{A_1 + \varepsilon_1} & 0 \\ 0 & \sqrt{A_2 - \varepsilon_1} \end{pmatrix} V. \tag{2.17}$$

Hence

$$B_1 = \frac{(A_1 - A_2 + \varepsilon_1)\sqrt{A_1 + \varepsilon_1} + \varepsilon_1\sqrt{A_2 - \varepsilon_1}}{A_1 - A_2 + 2\varepsilon_1}. \tag{2.18}$$

When ε is very small, we have

$$\begin{gathered} 2\varepsilon_1 = -308 + 32\varepsilon^2 + \sqrt{115600 - 12800\varepsilon^2 + o(\varepsilon^2)} = 32 + \frac{224}{17}\varepsilon^2 + o(\varepsilon^2); \\ \varepsilon_1 = 16 + \frac{112}{17}\varepsilon^2 + o(\varepsilon^2); \qquad \sqrt{A_1 + \varepsilon_1} = \sqrt{340} + o(\varepsilon); \\ A_1 - A_2 + 2\varepsilon_1 = 340 + o(\varepsilon); \qquad \varepsilon_1\sqrt{A_2 - \varepsilon_1} = o(1). \end{gathered} \tag{2.19}$$

Hence by (2.18), we have $B_1 = 324/\sqrt{340} + o(1)$. Because $324/\sqrt{340} > 17$, so (2.14) is valid for some small $\varepsilon > 0$.

Therefore the proof of Theorem 1.5 is complete. □

The following corollary can be derived from Theorem 1.4.

Corollary 2.3. *Let T be an invertible operator. Then T is a log-hyponormal operator if and only if*

$$\left\| |T^*|^{2p} |T|^{-p} |T^*|^{-p} \right\| |T|^{2p} \geq |T^*|^{2p} \tag{2.20}$$

holds for any small $p > 0$.

References

[1] T. Ando, *On some operator inequalities*, Mathematische Annalen **279** (1987), no. 1, 157–159.

[2] M. Fujii, T. Furuta, and E. Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra and Its Applications **179** (1993), 161–169.

[3] T. Furuta, *Results under* $\log A \geq \log B$ *can be derived from ones under* $A \geq B \geq 0$ *by Uchiyama's method—associated with Furuta and Kantorovich type operator inequalities*, Mathematical Inequalities & Applications **3** (2000), no. 3, 423–436.

[4] K. Tanahashi, *Best possibility of the Furuta inequality*, Proceedings of the American Mathematical Society **124** (1996), no. 1, 141–146.

[5] M. Uchiyama, *Some exponential operator inequalities*, Mathematical Inequalities & Applications **2** (1999), no. 3, 469–471.

[6] T. Yamazaki, *Characterizations of* $\log A \geq \log B$ *and normaloid operators via Heinz inequality*, Integral Equations and Operator Theory **43** (2002), no. 2, 237–247.

Changsen Yang: Department of Mathematics, Henan Normal University, Xinxiang,
Henan 453007, China
E-mail address: yangchangsen0991@sina.com

Fugen Gao: Department of Mathematics, Henan Normal University, Xinxiang, Henan 453007, China
E-mail address: gaofugen@tom.com