A GRÜSS-TYPE INEQUALITY AND ITS APPLICATIONS

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We give a Grüss-type inequality which is a refinement of a result due to Dragomir and Agarwal. We also give its applications for the moments of random variables, guessing mappings, and Ozeki's inequality.

1. Introduction

Let $a = (a_1,\dots,a_n)$, $b = (b_1,\dots,b_n)$ be n-tuples (sequences) of real numbers and let $p = (p_1,\dots,p_n)$ be an n-tuple of positive numbers. Then (discrete) Grüss' inequality is an estimation of the difference

$$I(a,b;p) := P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i, \tag{1.1}$$

where $P_n = \sum_{i=1}^{n} p_i$.

If both a and b are assumed to be nondecreasing (or nonincreasing), that is,

$$a_1 \le \cdots \le a_n, b_1 \le \cdots \le b_n \quad (\text{or} \quad a_1 \ge \cdots \ge a_n, b_1 \ge \cdots \ge b_n), \tag{1.2}$$

then the above difference $I(a,b;p)$ is nonnegative, that is,

$$P_n \sum_{i=1}^{n} p_i a_i b_i \ge \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i. \tag{1.3}$$

This is well known as Čebyšev's inequality [10, page 240]. As a complement of this inequality, one of the authors proved the following theorem.

Theorem 1.1 [11, Theorem 8], [10, page 302]. *Let a, b be nondecreasing (or nonincreasing) n-tuples of real numbers and let p be an n-tuple of positive numbers. Then*

$$|I(a,b;p)| \le |a_n - a_1|\,|b_n - b_1| \max_{1\le j\le n-1} P_j(P_n - P_j), \tag{1.4}$$

where $P_j = \sum_{k=1}^{j} p_k$.

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Without any assumption of monotonicity on n-tuples a and b, the following extension of Theorem 1.1 was given by D. Andrica and C. Badea.

THEOREM 1.2 [1, Theorem 2]. *Let a, b be n-tuples of real numbers satisfying*

$$m_1 \le a_i \le M_1, \quad m_2 \le b_i \le M_2 \quad (i = 1,\dots,n), \tag{1.5}$$

and let p be an n-tuple of positive numbers. Then

$$|I(a,b;p)| \le (M_1 - m_1)(M_2 - m_2)\max_{J\subset I_n} P(J)(P_n - P(J)), \tag{1.6}$$

where $I_n = \{1,\dots,n\}$ and $P(J) = \sum_{i\in J} p_i$ for $J \subset I_n$ ($cf.$ $P_n = P(I_n)$).

There are a number of further results concerning (discrete or integral type) Grüss inequalities with or without monotonicity conditions on n-tuples a and b [2, 5, 7, 8, 9, 12], and so forth. (See [10, Chapter X].)

Recently, Dragomir and Agarwal [3] (or Dragomir and Diamond [4]) presented a general Grüss-type inequality for complex n-tuples in terms of the first differences of them.

THEOREM 1.3 [3, Theorems 22 and 23], [4, Theorems 4 and 5]. *Let a, b be n-tuples of complex numbers and let p be an n-tuple of positive numbers such that $\sum_{i=1}^n p_i = 1$, that is, p is a probability distribution. Then*

$$
\begin{gathered}
|I(a,b;p)| \\
\le \begin{cases}
\displaystyle\max_{1\le k\le n-1} |a_{k+1} - a_k| \cdot \max_{1\le i\le n} |b_i| \cdot \sum_{i,j=1}^n p_i p_j |i-j|, \\
\displaystyle n^{1/s} \max_{1\le k\le n-1} |a_{k+1}-a_k| \cdot \left(\sum_{i=1}^n |b_i|^s\right)^{1/s} \left(\sum_{i,j=1}^n p_i^q p_j^q |i-j|^q\right)^{1/q} \quad \left(s,q>1, \frac{1}{s}+\frac{1}{q}=1\right), \\
\displaystyle n \max_{1\le k\le n-1} |a_{k+1} - a_k| \cdot \sum_{i=1}^n |b_i| \cdot \max_{1\le i,j\le n} p_i p_j |i-j|,
\end{cases} \\
|I(a,b;p)| \\
\le \begin{cases}
\displaystyle\max_{1\le k\le n-1} |a_{k+1} - a_k| \cdot \max_{1\le i\le n} p_i |b_i| \cdot \sum_{i,j=1}^n p_i |i-j|, \\
\displaystyle \max_{1\le k\le n-1} |a_{k+1}-a_k| \cdot \left(\sum_{i=1}^n p_i|b_i|^s\right)^{1/s} \left(\sum_{i,j=1}^n p_i p_j |i-j|^q\right)^{1/q} \quad \left(s,q>1, \frac{1}{s}+\frac{1}{q}=1\right), \\
\displaystyle (n-1) \max_{1\le k\le n-1} |a_{k+1} - a_k| \cdot \sum_{i=1}^n p_i |b_i|.
\end{cases}
\end{gathered} \tag{1.7}
$$

In [3], the following inequality was basically used as a key fact to obtain Theorem 1.3:

$$|I(a,b;p)| \le \max_{1\le k\le n-1} |\Delta a_k| \cdot \sum_{i,j=1}^{n} p_i p_j |i-j| |b_i|, \tag{1.8}$$

where $\Delta a_i = a_{i+1} - a_i$ ($i = 1,\dots,n-1$).

In this paper, with the notations

$$P_j = \sum_{k=1}^{j} p_k, \quad \bar{P}_{j+1} = 1 - P_j \quad (j = 1,\dots,n-1) \tag{1.9}$$

for a probability distribution p, (further, conveniently putting $P_0 = \bar{P}_{n+1} = 0$ and $\Delta a_0 = \Delta a_n = 0$), we prove a fundamental identity

$$I(a,b;p) = \sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} P_j \Delta a_j - \sum_{j=i}^{n} \bar{P}_{j+1} \Delta a_j \right) p_i b_i, \tag{1.10}$$

which brings a stronger inequality (than (1.8), cf. Remark 2.4)

$$|I(a,b;p)| \le \sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} P_j |\Delta a_j| + \sum_{j=i}^{n} \bar{P}_{j+1} |\Delta a_j| \right) p_i |b_i|. \tag{1.11}$$

Using the above identity and inequality, we give our main results (Theorems 2.3 and 2.6) as refinements of Theorem 1.3. We also give some applications for the moments of discrete random variables, for guessing mappings considered in [3, 4], and for Ozeki's inequality related to [6, 7, 8].

2. Grüss-type inequalities

We prepare two useful facts before we give the main results. Recall that for n-tuples a, b and a probability distribution p, we write (cf. (1.1))

$$I(a,b;p) = \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i. \tag{2.1}$$

For this difference we have the following lemma.

Lemma 2.1.

$$I(a,b;p) = \sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} P_j \Delta a_j - \sum_{j=i}^{n} \bar{P}_{j+1} \Delta a_j \right) p_i b_i. \tag{2.2}$$

Proof. First, by definition,

$$I(a,b;p) = \sum_{i=1}^{n} p_i a_i b_i - \sum_{j=1}^{n} p_j a_j \sum_{i=1}^{n} p_i b_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_j (a_i - a_j) \right) p_i b_i. \tag{2.3}$$

Next, note the following fact which is an extension of Abel's identity for a sequence $c_1,\dots,c_n$ with $\Delta c_j = c_{j+1} - c_j$ for $j = 1,\dots,n-1$ (and conveniently putting $\Delta c_0 = \Gamma c_n = 0$):

$$\sum_{j=1}^{n} p_j c_j = c_i - \sum_{j=0}^{i-1} P_j \Delta c_j + \sum_{j=i}^{n} \bar{P}_{j+1} \Delta c_j \quad (i = 1,\dots,n). \tag{2.4}$$

Putting $c_j = a_i - a_j$ in the above identity, we have

$$\begin{aligned} \sum_{j=1}^{n} p_j (a_i - a_j) &= (a_i - a_i) - \sum_{j=0}^{i-1} P_j \Delta(a_i - a_j) + \sum_{j=i}^{n} \bar{P}_{j+1} \Delta(a_i - a_j) \\ &= \sum_{j=0}^{i-1} P_j \Delta a_j - \sum_{j=i}^{n} \bar{P}_{j+1} \Delta a_j. \end{aligned} \tag{2.5}$$

Hence from (2.3) and (2.5), we obtain the desired identity (2.2). □

Lemma 2.2 [10, page 14]. *If a and p are positive n-tuples, then the function*

$$M(r) = \begin{cases} \left(\dfrac{1}{P_n} \displaystyle\sum_{i=1}^{n} p_i a_i^r \right)^{1/r} & \text{if } r \neq 0, \\ \left(\displaystyle\prod_{i=1}^{n} a_i^{p_i} \right)^{1/P_n} & \text{if } r = 0, \\ \min\{a_1,\dots,a_n\} & \text{if } r = -\infty, \\ \max\{a_1,\dots,a_n\} & \text{if } r = \infty \end{cases} \tag{2.6}$$

is monotone increasing on $[-\infty,\infty]$.

Now we give the following result as a refinement of Theorem 1.3.

Theorem 2.3. *Let a and b be n-tuples of complex numbers and let p be a probability distribution. Then*

$$|I(a,b;p)| \leq \begin{cases} \displaystyle\sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} P_j |\Delta a_j| + \sum_{j=i}^{n} \bar{P}_{j+1} |\Delta a_j| \right) p_i |b_i|, \\ \displaystyle\sum_{i=1}^{n} \left[\left(\sum_{j=1}^{n} |i-j| p_j \right)^{1-1/q} \left(\sum_{j=0}^{i-1} P_j |\Delta a_j|^q + \sum_{j=i}^{n} \bar{P}_{j+1} |\Delta a_j|^q \right)^{1/q} \right] p_i |b_i| \quad (q > 1), \\ \displaystyle\max_{1\leq k\leq n-1} |\Delta a_k| \cdot \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |i-j| p_j \right) p_i |b_i|, \end{cases} \tag{2.7}$$

Moreover, if R_1, R_2, and R_3 denote the right-hand sides of (2.7), successively from above to below, then

$$R_1 \leq R_2 \leq R_3. \tag{2.8}$$

Proof. By Lemma 2.1, we immediately obtain

$$|I(a,b;p)| \leq \sum_{i=1}^{n}\left(\sum_{j=0}^{i-1} P_j|\Delta a_j| + \sum_{j=i}^{n}\bar{P}_{j+1}|\Delta a_j|\right)p_i|b_i|, \tag{2.9}$$

which is the first inequality in (2.7). To obtain the other inequalities, put

$$Q_i = \sum_{j=0}^{i-1} P_j + \sum_{j=i}^{n}\bar{P}_{j+1} \quad (i = 1,\dots,n). \tag{2.10}$$

Then from Lemma 2.2, we have, for $q > 1$,

$$\begin{aligned}
&\frac{1}{Q_i}\left(\sum_{j=0}^{i-1} P_j|\Delta a_j| + \sum_{j=i}^{n}\bar{P}_{j+1}|\Delta a_j|\right)\\
&\qquad \leq \left\{\frac{1}{Q_i}\left(\sum_{j=0}^{i-1} P_j|\Delta a_j|^q + \sum_{j=i}^{n}\bar{P}_{j+1}|\Delta a_j|^q\right)\right\}^{1/q}\\
&\qquad \leq \max_{1\leq j\leq n-1}|\Delta a_j|,
\end{aligned} \tag{2.11}$$

so that

$$\begin{aligned}
&\sum_{j=0}^{i-1} P_j|\Delta a_j| + \sum_{j=i}^{n}\bar{P}_{j+1}|\Delta a_j|\\
&\qquad \leq Q_i^{1-1/q}\left(\sum_{j=0}^{i-1} P_j|\Delta a_j|^q + \sum_{j=i}^{n}\bar{P}_{j+1}|\Delta a_j|^q\right)^{1/q}\\
&\qquad \leq \max_{1\leq j\leq n-1}|\Delta a_j|Q_i.
\end{aligned} \tag{2.12}$$

For Q_i, we have, by an elementary computation,

$$Q_i = \sum_{j=1}^{n}|i-j|p_j. \tag{2.13}$$

Hence from (2.9), (2.12), and (2.13), we obtain (2.8) and at the same time the second and the third inequalities in (2.7). □

Remark 2.4. The inequality

$$\sum_{j=0}^{i-1} P_j|\Delta a_j| + \sum_{j=i}^{n}\bar{P}_{j+1}|\Delta a_j| \leq \max_{1\leq j\leq n-1}|\Delta a_j|\sum_{j=1}^{n}|i-j|p_j \tag{2.14}$$

(immediately obtained from (2.12) and (2.13)) says that our basic inequality (2.9) is stronger than (1.8) which was used in [3, Theorem C].

Remark 2.5. If we define by $F(q)$ the right-hand side of the second inequality in (2.7), then we see, by Lemma 2.2, that $F(q)$ is an increasing function and $\lim_{q\to 1} F(q) \le F(q) \le \lim_{q\to\infty} F(q)$, which implies (2.8).

For positive n-tuples a and b with an assumption of monotonicity, we have the following theorem.

Theorem 2.6. *Let a and b be n-tuples of positive numbers and let p be a probability distribution. Assume that a is nondecreasing. Then*

$$I(a,b;p) \le \begin{cases} \sum_{i=1}^{n}\left(\sum_{j=0}^{i-1} P_j \Delta a_j\right) p_i b_i, \\ \sum_{i=1}^{n}\left[\left(\sum_{j=0}^{i-1}(i-j)p_j\right)^{1-1/q}\left(\sum_{j=0}^{i-1} P_j (\Delta a_j)^q\right)^{1/q}\right] p_i b_i \quad (q>1), \\ \max_{1\le k\le n-1} \Delta a_k \cdot \sum_{i=1}^{n}\left(\sum_{j=0}^{i-1}(i-j)p_j\right) p_i b_i, \end{cases} \tag{2.15}$$

(where conveniently $p_0 = 0$) and

$$I(a,b;p) \ge \begin{cases} -\sum_{i=1}^{n}\left(\sum_{j=i}^{n} \bar{P}_{j+1} \Delta a_j\right) p_i b_i, \\ -\sum_{i=1}^{n}\left[\left(\sum_{j=i}^{n}(j-i)p_j\right)^{1-1/q}\left(\sum_{j=i}^{n} \bar{P}_{j+1} (\Delta a_j)^q\right)^{1/q}\right] p_i b_i \quad (q>1), \\ -\max_{1\le k\le n-1} \Delta a_k \cdot \sum_{i=1}^{n}\left(\sum_{j=i}^{n}(j-i)p_j\right) p_i b_i. \end{cases} \tag{2.16}$$

Moreover, as in Theorem 2.3, if R_1', R_2', and R_3' denote (resp., R_1'', R_2'', and R_3'') the right-hand sides of (2.15) (resp., (2.16)), successively from above to below, then

$$R_1' \le R_2' \le R_3' \quad (\textit{resp.}, R_1'' \ge R_2'' \ge R_3''). \tag{2.17}$$

Proof. Since $\Delta a_j \ge 0$, we have

$$-\sum_{j=i}^{n} \bar{P}_{j+1}\Delta a_j \le \sum_{j=0}^{i-1} P_j \Delta a_j - \sum_{j=i}^{n} \bar{P}_{j+1}\Delta a_j \le \sum_{j=0}^{i-1} P_j \Delta a_j, \tag{2.18}$$

so that from Lemma 2.1 we obtain the first inequalities in (2.15) and (2.16). To see the remaining inequalities, note that the following identities

$$\sum_{j=0}^{i-1} P_j = \sum_{j=0}^{i-1}(i-j)p_j, \qquad \sum_{j=i}^{n} \bar{P}_{j+1} = \sum_{j=i}^{n}(j-i)p_j \tag{2.19}$$

hold. From these and the first inequalities in (2.15) and (2.16), we can obtain all other desired inequalities by the similar argument as in the proof of Theorem 2.3. □

3. Applications for the moments of random variables, guessing mappings, and Ozeki's inequality

Consider a discrete random variable

$$X : \begin{pmatrix} x_1,\dots,x_n \\ p_1,\dots,p_n \end{pmatrix}, \tag{3.1}$$

which takes positive values $x_1,\dots,x_n$ with a probability distribution $p = (p_1,\dots,p_n)$. Then the γ-moment ($\gamma > 0$) of X is defined as the expectation $E(X^\gamma)$, that is,

$$M_\gamma(X) = E(X^\gamma) = \sum_{i=1}^{n} p_i x_i^\gamma. \tag{3.2}$$

An approximation result which compares $M_{\alpha+\beta}(X)$ $(\alpha,\beta > 0)$ with the product of $M_\alpha(X)$ and $M_\beta(X)$ was shown in [3, 4]. We here give an improvement of the result by applying Theorem 2.3.

Proposition 3.1. *Under the above assumptions,*

$$\begin{aligned} &\left| M_{\alpha+\beta}(X) - M_\alpha(X)M_\beta(X) \right| \\ &\quad \le \begin{cases} \displaystyle\sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} P_j \left| \Delta x_j^\alpha \right| + \sum_{j=i}^{n} \bar{P}_{j+1} \left| \Delta x_j^\alpha \right| \right) p_i x_i^\beta, \\ \displaystyle\sum_{i=1}^{n} \left[\left(\sum_{j=1}^{n} |i-j| p_j \right)^{1-1/q} \left(\sum_{j=0}^{i-1} P_j \left| \Delta x_j^\alpha \right|^q + \sum_{j=i}^{n} \bar{P}_{j+1} \left| \Delta x_j^\alpha \right|^q \right)^{1/q} \right] p_i x_i^\beta \quad (q > 1), \\ \displaystyle\max_{1\le k\le n-1} \left| \Delta x_k^\alpha \right| \cdot \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |i-j| p_j \right) p_i x_i^\beta. \end{cases} \end{aligned} \tag{3.3}$$

Proof. Writing $x^\gamma = (x_1^\gamma,\dots,x_n^\gamma)$ for $\gamma > 0$, we see that

$$M_{\alpha+\beta}(X) - M_\alpha(X)M_\beta(X) = I(x^\alpha, x^\beta; p). \tag{3.4}$$

Hence we obtain the desired inequalities in (3.3) from Theorem 2.3. □

Now in connection with the random variable X, consider a uniformly distributed random variable

$$U : \begin{pmatrix} x_1,\dots,x_n \\ \dfrac{1}{n},\dots,\dfrac{1}{n} \end{pmatrix} \tag{3.5}$$

and its α-moment $M_\alpha(U) = (1/n)\sum_{i=1}^{n} x_i^\alpha$. Then we have the following proposition as an improvement of a result in [3, 4].

Proposition 3.2. *Under the above assumptions,*

$$|M_\alpha(X)-M_\alpha(U)| \le \begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(\sum_{j=0}^{i-1} j\,|\Delta p_j| + \sum_{j=i}^{n}(n-j)\,|\Delta p_j|\right)x_i^\alpha, \\ \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left[(S_{i-1}+S_{n-i})^{1-1/q}\left(\sum_{j=0}^{i-1} j\,|\Delta p_j|^q + \sum_{j=i}^{n}(n-j)\,|\Delta p_j|^q\right)^{1/q}\right]x_i^\alpha \quad (q>1), \\ \dfrac{1}{n}\displaystyle\max_{1\le k\le n-1}|\Delta p_k|\cdot\sum_{i=1}^{n}(S_{i-1}+S_{n-i})x_i^\alpha, \end{cases} \tag{3.6}$$

$$|M_\alpha(X)-M_\alpha(U)| \le \begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(\sum_{j=0}^{i-1} j\,|\Delta x_j^\alpha| + \sum_{j=i}^{n}(n-j)\,|\Delta x_j^\alpha|\right)p_i, \\ \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left[(S_{i-1}+S_{n-i})^{1-1/q}\left(\sum_{j=0}^{i-1} j\,|\Delta x_j^\alpha|^q + \sum_{j=i}^{n}(n-j)\,|\Delta x_j^\alpha|^q\right)^{1/q}\right]p_i \quad (q>1), \\ \dfrac{1}{n}\displaystyle\max_{1\le k\le n-1}|\Delta x_k^\alpha|\cdot\sum_{i=1}^{n}(S_{i-1}+S_{n-i})\,p_i, \end{cases} \tag{3.7}$$

where $S_k = k(k+1)/2$.

Proof. Let $\tilde{p} = (1/n,\dots,1/n)$. Then we see that

$$M_\alpha(X)-M_\alpha(U) = nI(p,x^\alpha;\tilde{p}), \qquad M_\alpha(X)-M_\alpha(U) = nI(x^\alpha,p;\tilde{p}). \tag{3.8}$$

Hence corresponding to the first or the second identity in the above (3.8), the first inequality in (3.6) or (3.7) is obtained immediately by Theorem 2.3. For the other inequalities in (3.6) and (3.7), notice that

$$nQ_i := \sum_{j=0}^{i-1} j + \sum_{j=i}^{n}(n-j) = S_{i-1}+S_{n-i}. \tag{3.9}$$

Hence again from Theorem 2.3, we can obtain all desired inequalities. □

A discrete guessing mapping G is defined [3, 4] as a special random variable

$$G : \begin{pmatrix} 1,\dots,n \\ p_1,\dots,p_n \end{pmatrix}. \tag{3.10}$$

Since $g^\gamma = (1^\gamma,\dots,n^\gamma)$ $(\gamma>0)$ is nondecreasing and

$$M_{\alpha+\beta}(G)-M_\alpha(G)M_\beta(G) = I(g^\alpha,g^\beta;p) \quad (\alpha,\beta>0), \tag{3.11}$$

we can obtain the following proposition as a refinement of a result in [3, 4] by using Theorem 2.6.

Proposition 3.3. *Under the above assumptions,*

$$M_{\alpha+\beta}(G)-M_\alpha(G)M_\beta(G)$$
$$\leq\begin{cases}\displaystyle\sum_{i=1}^{n}\left(\sum_{j=0}^{i-1}P_j\Delta j^\alpha\right)p_i i^\beta,\\ \displaystyle\sum_{i=1}^{n}\left[\left(\sum_{j=0}^{i-1}(i-j)p_j\right)^{1-1/q}\left(\sum_{j=0}^{i-1}P_j(\Delta j^\alpha)^q\right)^{1/q}\right]p_i i^\beta\quad(q>1),\\ \displaystyle\max_{1\le k\le n-1}\Delta k^\alpha\cdot\sum_{i=1}^{n}\left(\sum_{j=0}^{i-1}(i-j)p_j\right)p_i i^\beta\left(=\delta_\alpha(n)\sum_{i=1}^{n}\left(\sum_{j=0}^{i-1}(i-j)p_j\right)p_i i^\beta\right),\end{cases}$$

$$M_{\alpha+\beta}(G)-M_\alpha(G)M_\beta(G)$$
$$\geq\begin{cases}\displaystyle-\sum_{i=1}^{n}\left(\sum_{j=i}^{n}\bar{P}_{j+1}\Delta j^\alpha\right)p_i i^\beta,\\ \displaystyle-\sum_{i=1}^{n}\left[\left(\sum_{j=i}^{n}(j-i)p_j\right)^{1-1/q}\left(\sum_{j=i}^{n}\bar{P}_{j+1}(\Delta j^\alpha)^q\right)^{1/q}\right]p_i i^\beta\quad(q>1),\\ \displaystyle-\max_{1\le k\le n-1}\Delta k^\alpha\cdot\sum_{i=1}^{n}\left(\sum_{j=i}^{n}(j-i)p_j\right)p_i i^\beta\left(=-\delta_\alpha(n)\sum_{i=1}^{n}\left(\sum_{j=i}^{n}(j-i)p_j\right)p_i i^\beta\right),\end{cases}\tag{3.12}$$

where

$$\delta_\alpha(n)=\max_{1\le k\le n-1}\left|(k+1)^\alpha-k^\alpha\right|=\begin{cases}n^\alpha-(n-1)^\alpha & \text{if } \alpha\ge 1,\\ 2^\alpha-1 & \text{if } 0<\alpha<1.\end{cases}\tag{3.13}$$

Let

$$V:\begin{pmatrix}1,\dots,n\\ \dfrac{1}{n},\dots,\dfrac{1}{n}\end{pmatrix}\tag{3.14}$$

be the size-n guessing mapping with the uniform probability distribution. Then, similarly as (3.8), we see that

$$M_\alpha(G)-M_\alpha(V)=nI(p,g^\alpha;\tilde{p})\left(=nI(g^\alpha,p;\tilde{p})\right),\quad\left(M_\alpha(V)=\frac{1}{n}\sum_{i=1}^{n}i^\alpha\right).\tag{3.15}$$

Hence we can obtain the following result by the similar argument as in Proposition 3.3.

Proposition 3.4. *Under the above assumptions,*

$$M_\alpha(G)-\frac{1}{n}\sum_{i=1}^{n} i^\alpha\,(=M_a(G)-M_a(V))$$

$$\leq\begin{cases}\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(\sum_{j=0}^{i-1} j\Delta j^a\right)p_i,\\[2ex] \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left[\left(\frac{i(i-1)}{2}\right)^{1-1/q}\left(\sum_{j=0}^{i-1} j(\Delta j^\alpha)^q\right)^{1/q}\right]p_i \quad (q>1),\\[2ex] \dfrac{1}{n}\delta_\alpha(n)\displaystyle\sum_{i=1}^{n}\frac{i(i-1)}{2}p_i\left(=\frac{1}{n}\delta_\alpha(n)\left(\frac{1}{2}E(G^2)-\frac{1}{2}E(G)\right)\right),\end{cases}$$

$$M_\alpha(G)-\frac{1}{n}\sum_{i=1}^{n} i^\alpha$$

$$\geq\begin{cases}-\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(\sum_{j=i}^{n}(n-j)\Delta j^a\right)p_i,\\[2ex] -\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left[\left(\frac{(n-i)(n-i+1)}{2}\right)^{1-1/q}\left(\sum_{j=i}^{n}(n-j)(\Delta j^\alpha)^q\right)^{1/q}\right]p_i \quad (q>1),\\[2ex] -\dfrac{1}{n}\delta_\alpha(n)\displaystyle\sum_{i=1}^{n}\frac{(n-i)(n-i+1)}{2}p_i\left(=-\frac{1}{n}\delta_\alpha(n)\left(\frac{1}{2}E(G^2)-\frac{2n+1}{2}E(G)+\frac{1}{2}n(n+1)\right)\right).\end{cases} \tag{3.16}$$

Ozeki's inequality [6, 7, 8] (cf. [10, page 121]) is a complement of Cauchy-Schwarz inequality, which estimates the difference

$$I_2(a,b;p):=\sum_{i=1}^{n} p_i a_i^2\sum_{i=1}^{n} p_i b_i^2-\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2 \quad (\geq 0) \tag{3.17}$$

for positive n-tuples a and b with a probability distribution p. Note that

$$\begin{aligned} I(a^2,b^2;p)&=\sum_{i=1}^{n} p_i a_i^2 b_i^2-\sum_{i=1}^{n} p_i a_i^2\sum_{i=1}^{n} p_i b_i^2,\\ I(ab,ab;p)&=\sum_{i=1}^{n} p_i a_i^2 b_i^2-\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2,\end{aligned} \tag{3.18}$$

where $ab=(a_1b_1,\dots,a_nb_n)$. Hence, we see that

$$I_2(a,b;p)=-I(a^2,b^2;p)+I(ab,ab;p). \tag{3.19}$$

From this fact, we obtain a new estimation of the difference $I_2(a,b;p)$ in terms of the first differences of a^2 and ab.

Proposition 3.5. *Under the above assumptions,*

$$I_2(a,b;p)$$
$$\leq \begin{cases} \sum_{i=1}^{n}\left(\sum_{j=0}^{i-1}P_j\,|\,\Delta a_j^2\,|+\sum_{j=i}^{n}\bar{P}_{j+1}\,|\,\Delta a_j^2\,|\right)p_ib_i^2+\sum_{i=1}^{n}\left(\sum_{j=0}^{i-1}P_j\,|\,\Delta(ab)_j\,|+\sum_{j=i}^{n}\bar{P}_{j+1}\,|\,\Delta(ab)_j\,|\right)p_ia_ib_i, \\ \sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n}|i-j|\,p_j\right)^{1-1/q}\left(\sum_{j=0}^{i-1}P_j\,|\,\Delta a_j^2\,|^q+\sum_{j=i}^{n}\bar{P}_{j+1}\,|\,\Delta a_j^2\,|^q\right)^{1/q}\right]p_ib_i^2 \\ \quad +\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n}|i-j|\,p_j\right)^{1-1/q}\left(\sum_{j=0}^{i-1}P_j|\Delta(ab)_j|^q+\sum_{j=i}^{n}\bar{P}_{j+1}|\Delta(ab)_j|^q\right)^{1/q}\right]p_ia_ib_i \quad (q>1), \\ \max_{1\leq k\leq n-1}|\,\Delta a_k^2\,|\cdot\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|i-j|\,p_j\right)p_ia_i^2+\max_{1\leq k\leq n-1}|\,\Delta(ab)_k\,|\cdot\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|i-j|\,p_j\right)p_ia_ib_i. \end{cases} \tag{3.20}$$

Proof. By (3.19) we see that

$$I_2(a,b;p) \leq \left|I(a^2,b^2;p)\right| + \left|I(ab,ab;p)\right|. \tag{3.21}$$

Hence, we can obtain the desired inequalities in (3.20) from Theorem 2.3. □

For n-tuples a, b with an assumption of monotonicity, we have the following proposition.

Proposition 3.6. *Let a, b be positive n-tuples and let p be a probability distribition. Assume that*

$$(0<)a_1 \leq \cdots \leq a_n, \qquad (0<)b_1 \leq \cdots \leq b_n. \tag{3.22}$$

Then,

$$I_2(a,b;p) \leq \begin{cases} \sum_{i=1}^{n}\left(\sum_{j=0}^{i-1}P_j\Delta(ab)_j\right)p_ia_ib_i, \\ \sum_{i=1}^{n}\left[\left(\sum_{j=0}^{i-1}(i-j)p_j\right)^{1-1/q}\left(\sum_{j=0}^{i-1}P_j\left(\Delta(ab)_j\right)^q\right)^{1/q}\right]p_ia_ib_i \quad (q>1), \\ \max_{1\leq k\leq n-1}\Delta(ab)_k\cdot\sum_{i=1}^{n}\left(\sum_{j=0}^{i-1}(i-j)p_j\right)p_ia_ib_i. \end{cases} \tag{3.23}$$

Proof. Since all the n-tuples a^2, b^2, ab are nondecreasing, we see, by Čebyšev's inequality, that

$$I(a^2,b^2;p) \geq 0, \qquad I(ab,ab;p) \geq 0. \tag{3.24}$$

Hence from (3.19)

$$I_2(a,b;p) \le I(ab,ab;p), \tag{3.25}$$

so that from Theorem 2.6, we obtain the desired inequalities in (3.23). □

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