

CONVEX SETS AND INEQUALITIES

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We consider a natural correspondence between a family of inequalities and a closed convex set. As an application, we give new types of power mean inequalities and the Hölder-type inequalities.

1. Concept and fundamental result

Given a natural correspondence between a family of inequalities and a closed convex set in a topological linear space, one might expect that an inequality corresponding to a special point (e.g., an extreme point) would be of special interest in view of the convex analysis theory. In this paper, we realize this concept.

Let X be an arbitrary set and $\{\varphi_0, \varphi_1, \varphi\}$ a triple of nonnegative real-valued functions on X . Set

$$m = \inf_{\varphi_0(x) \neq 0} \frac{\varphi(x)}{\varphi_0(x)}, \quad M = \sup_{\varphi_1(x) \neq 0} \frac{\varphi(x)}{\varphi_1(x)}. \quad (1.1)$$

Suppose that $0 < m, M < \infty$. Then we have

$$m\varphi_0(x) \leq \varphi(x) \leq M\varphi_1(x) \quad \forall x \in X. \quad (1.2)$$

For each $x \in X$, put

$$D_\varphi(x) = \{(\alpha, \beta) \in \mathbb{R}^2 : \varphi(x) \leq \alpha\varphi_1(x) + \beta\varphi_0(x)\}. \quad (1.3)$$

We consider the intersection $D_\varphi = \bigcap_{x \in X} D_\varphi(x)$ of all such sets. Note that D_φ is a nonempty closed convex domain in \mathbb{R}^2 and that each point $(\alpha, \beta) \in D_\varphi$ corresponds to the inequality $\varphi \leq \alpha\varphi_1 + \beta\varphi_0$ on X . We want to investigate the closed convex domain D_φ . To do this, we define the constant α_φ by

$$\alpha_\varphi = \sup_{M\varphi_1(x) \neq m\varphi_0(x)} \frac{M\varphi(x) - mM\varphi_0(x)}{M\varphi_1(x) - m\varphi_0(x)}. \quad (1.4)$$

Clearly, $0 \leq \alpha_\varphi \leq M$. Also, we have the following three *fundamental facts*:

- (A) if $(\alpha, \beta) \in D_\varphi$ and $\alpha/M + \beta/m = 1$, then $\alpha \geq \alpha_\varphi$,
- (B) $\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha/M + \beta/m \geq 1, \alpha \geq \alpha_\varphi\} \subset D_\varphi$,
- (C) $D_\varphi \subset \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha/M + \beta(m\lambda) \geq 1\}$ for some $1 \leq \lambda \leq \infty$. In particular, if $\alpha_\varphi < M$, then $D_\varphi \subset \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha/M + \beta/m \geq 1\}$.

These facts will be used in the later sections to realize our concept.

Proof of (A). Suppose $(\alpha, \beta) \in D_\varphi$ and $\alpha/M + \beta/m = 1$. Then

$$\varphi(x) \leq \alpha\varphi_1(x) + m\left(1 - \frac{\alpha}{M}\right)\varphi_0(x), \quad (1.5)$$

and hence

$$\frac{M\varphi(x) - mM\varphi_0(x)}{M\varphi_1(x) - m\varphi_0(x)} \leq \alpha \quad (1.6)$$

for all $x \in X$ with $M\varphi_1(x) \neq m\varphi_0(x)$. This implies that $\alpha_\varphi \leq \alpha$. \square

Proof of (B). If $t \geq \alpha_\varphi/M$, then $\varphi(x) - m\varphi_0(x) \leq t(M\varphi_1(x) - m\varphi_0(x))$ and so $\varphi(x) \leq tM\varphi_1(x) + m(1-t)\varphi_0(x)$ for all $x \in X$. Hence, we have

$$\begin{aligned} D_\varphi &\supset \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \geq tM, \beta \geq m(1-t), t \geq \frac{\alpha_\varphi}{M} \text{ for some } t \in \mathbb{R}\right\} \\ &= \left\{(\alpha, \beta) \in \mathbb{R}^2 : \frac{\alpha}{M} + \frac{\beta}{m} \geq 1, \alpha \geq \alpha_\varphi\right\}. \end{aligned} \quad (1.7)$$

\square

Proof of (C). By the definition of M , we find a sequence $\{x_n\}$ in X such that

$$\varphi_1(x_n) \neq 0, \quad (n = 1, 2, \dots), \quad M = \lim_{n \rightarrow \infty} \frac{\varphi(x_n)}{\varphi_1(x_n)}. \quad (1.8)$$

Of course, we can assume that $\varphi(x_n) \neq 0$ for all $n = 1, 2, \dots$. Since $\{\varphi_0(x_n)/\varphi(x_n)\}$ is a bounded sequence with bound $1/m$, we can take a subsequence $\{\varphi_0(x_{n'})/\varphi(x_{n'})\}$ converging to some real number t with $0 \leq t \leq 1/m$. Set $\lambda = 1/(tm)$ so that $1 \leq \lambda \leq \infty$. We have

$$\begin{aligned} D_\varphi &\subset \bigcap_{n'} \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \frac{\varphi_1(x_{n'})}{\varphi(x_{n'})} + \beta \frac{\varphi_0(x_{n'})}{\varphi(x_{n'})} \geq 1\right\} \\ &\subset \left\{(\alpha, \beta) \in \mathbb{R}^2 : \frac{\alpha}{M} + \frac{\beta}{m\lambda} \geq 1\right\}. \end{aligned} \quad (1.9)$$

In particular, if $\alpha_\varphi < M$, then λ must be 1 by an easy geometrical consideration on the $\alpha\beta$ -plane \mathbb{R}^2 . \square

2. Application: Djokovic’s inequality

Let H be a Hlawka space, that is, a Banach space in which the Hlawka inequality holds. If n and k are natural numbers with $2 \leq k \leq n - 1$, then

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} \|x_{i_1} + \dots + x_{i_k}\| \leq \binom{n-2}{k-1} \sum_{i=1}^n \|x_i\| + \binom{n-2}{k-2} \left\| \sum_{i=1}^n x_i \right\| \tag{2.1}$$

for all $x_1, \dots, x_n \in H$. This is well known as Djokovic’s inequality (cf. [1, 2]).

Let X be the linear space $H \oplus \dots \oplus H = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in H\}$. For $1 \leq k \leq n$, set

$$\delta_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \|x_{i_1} + \dots + x_{i_k}\| \tag{2.2}$$

for all $(x_1, \dots, x_n) \in X$. Then $\{\delta_k : 1 \leq k \leq n\}$ constitutes a system of seminorms on X and satisfies

$$\binom{n-1}{k-1} \delta_n \leq \delta_k \leq \binom{n-1}{k-1} \delta_1 \quad (1 \leq k \leq n). \tag{2.3}$$

Fix k and set $\varphi_0 = \delta_n, \varphi_1 = \delta_1, \varphi = \delta_k$. Then the above Djokovic inequality can be rewritten as

$$\varphi \leq \binom{n-2}{k-1} \varphi_1 + \binom{n-2}{k-2} \varphi_0 \quad \text{on } X. \tag{2.4}$$

Also, we can see that $m = M = \binom{n-1}{k-1}$ and $\alpha_\varphi = \binom{n-2}{k-1}$ because $\binom{n-1}{k-1} = \binom{n-2}{k-1} + \binom{n-2}{k-2}$. Then we have

$$\left\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha + \beta \geq \binom{n-1}{k-1}, \alpha \geq \binom{n-2}{k-1} \right\} \subset D_\varphi \subset \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha + \beta \geq \binom{n-1}{k-1} \right\} \tag{2.5}$$

by the fundamental facts (B), (C). However, we have from [3, Theorem 1 (vi)] that

$$D_\varphi \subset \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha + \beta \geq \binom{n-1}{k-1}, \alpha \geq \binom{n-2}{k-1} \right\}. \tag{2.6}$$

It follows that D_φ coincides with the minimum domain

$$\left\{ (\alpha, \beta) \in \mathbb{R}^2 : \frac{\alpha}{M} + \frac{\beta}{m} \geq 1, \alpha \geq \alpha_\varphi \right\}. \tag{2.7}$$

Hence $(\alpha_\varphi, M - \alpha_\varphi)$ is the only extreme point of D_φ and the corresponding inequality, that is Djokovic’s inequality, is of special interest. The above argument is nearly a restatement of [3, Theorem 1].

3. Application: the power mean inequality

Let $X = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1, \dots, x_n > 0\}$ and take $t \in \mathbb{R}$. We define $\varphi_0, \varphi_1, \varphi$ by

$$\begin{aligned} \varphi_0(x_1, \dots, x_n) &= \min \{x_1, \dots, x_n\}, \\ \varphi_1(x_1, \dots, x_n) &= \max \{x_1, \dots, x_n\}, \\ \varphi(x_1, \dots, x_n) &= \begin{cases} \left(\frac{x_1^t + \dots + x_n^t}{n}\right)^{1/t} & \text{if } t \neq 0, \\ \sqrt[n]{x_1 \cdots x_n} & \text{if } t = 0, \end{cases} \end{aligned} \tag{3.1}$$

for all $(x_1, \dots, x_n) \in X$. Then $m = M = 1$. We determine the domain D_φ . For $(\alpha, \beta) \in \mathbb{R}^2$, $(\alpha, \beta) \in D_\varphi$ if and only if

$$\varphi(x_1, \dots, x_n) \leq \alpha \max \{x_1, \dots, x_n\} + \beta \min \{x_1, \dots, x_n\} \quad \forall (x_1, \dots, x_n) \in X. \tag{3.2}$$

Dividing (3.2) by $\max \{x_1, \dots, x_n\}$, we see that (3.2) is equivalent to the following condition:

$$\alpha + \beta u \geq \sup \left\{ \varphi(x_1, \dots, x_n) : \begin{array}{l} \min \{x_1, \dots, x_n\} = u \\ \max \{x_1, \dots, x_n\} = 1 \end{array} \right\} \quad \text{for } 0 < u \leq 1. \tag{3.3}$$

Denote by $f(u)$ the right side of (3.3). Then (3.3) becomes

$$\alpha + \beta u \geq f(u) \quad \text{for } 0 < u \leq 1. \tag{3.4}$$

Also, we can easily see that

$$f(u) = \begin{cases} \left(\frac{n-1}{n} + \frac{u^t}{n}\right)^{1/t} & \text{if } t \neq 0 \\ \sqrt[n]{u} & \text{if } t = 0 \end{cases} \quad (0 < u \leq 1). \tag{3.5}$$

If $t \neq 0$, then we have

$$\begin{aligned} f'(u) &= \frac{1}{n} u^{t-1} \left(\frac{n-1}{n} + \frac{u^t}{n}\right)^{1/t-1}, \\ f''(u) &= \frac{n-1}{n^2} (t-1) u^{t-2} \left(\frac{n-1}{n} + \frac{u^t}{n}\right)^{1/t-2}. \end{aligned} \tag{3.6}$$

(i) *The case of $t < 1$ and $t \neq 0$.* In this case, (3.6) implies that f is a concave function on $(0, 1]$. Hence (3.4) is equivalent to the following condition:

$$\begin{aligned} \beta &\geq f'(u(\alpha)) \quad \text{for } \lim_{u \downarrow 0} f(u) \leq \alpha \leq f(1) - f'(1), \\ \alpha + \beta &\geq f(1) \quad \text{for } \alpha > f(1) - f'(1), \end{aligned} \tag{3.7}$$

where $u(\alpha)$ is the unique solution of the equation $\alpha + f'(u)u = f(u)$. Note that $f(1) = 1$, $f(1) - f'(1) = (n - 1)/n$, and

$$\lim_{u \downarrow 0} f(u) = \begin{cases} \left(\frac{n-1}{n}\right)^{1/t} & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{3.8}$$

To investigate $f'(u(\alpha))$, set $v = u(\alpha)$ and $\gamma = f'(v)$. Then

$$\left(\frac{n-1}{n} + \frac{v^t}{n}\right)^{1/t-1} = n\gamma v^{1-t}, \quad \alpha + \gamma v = f(v). \tag{3.9}$$

Hence

$$f(v) = n\gamma v^{1-t} \left(\frac{n-1}{n} + \frac{v^t}{n}\right) = (n-1)\gamma v^{1-t} + \gamma v, \tag{3.10}$$

so that $\alpha = (n-1)\gamma v^{1-t}$. Therefore by a simple computation, we obtain the equation $(n-1)^{1/(1-t)}\alpha^{t/(t-1)} + \gamma^{t/(t-1)} = n^{1/(1-t)}$. Consequently, if $t > 0$, then

$$D_\varphi = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \begin{array}{l} \beta \geq (n^{1/(1-t)} - (n-1)^{1/(1-t)}\alpha^{t/(t-1)})^{(t-1)/t} \text{ for } \left(\frac{n-1}{n}\right)^{1/t} \alpha \leq \frac{n-1}{n} \\ \alpha + \beta \geq 1 \text{ for } \alpha > \frac{n-1}{n} \end{array} \right\}, \tag{3.11}$$

and if $t < 0$, then

$$D_\varphi = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \begin{array}{l} \beta \geq (n^{1/(1-t)} - (n-1)^{1/(1-t)}\alpha^{t/(t-1)})^{(t-1)/t} \text{ for } 0 \leq \alpha \leq \frac{n-1}{n} \\ \alpha + \beta \geq 1 \text{ for } \alpha > \frac{n-1}{n} \end{array} \right\}. \tag{3.12}$$

Also, $\alpha_\varphi = (n-1)/n$ from the fundamental facts (A), (B).

(ii) *The case of $t = 0$.* Note that f is a concave function on $(0, 1]$ and $\lim_{u \downarrow 0} f(u) = 0$, $f(1) = 1$, and $f(1) - f'(1) = (n-1)/n$. By executing the argument of (i), we obtain that

$$D_\varphi = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \begin{array}{l} \beta \geq \frac{(n-1)^{n-1}}{n^n} \alpha^{1-n} \text{ for } 0 < \alpha \leq \frac{n-1}{n} \\ \alpha + \beta \geq 1 \text{ for } \alpha > \frac{n-1}{n} \end{array} \right\}, \tag{3.13}$$

and $\alpha_\varphi = (n-1)/n$.

(iii) *The case of $t \geq 1$.* By (3.6), f is a convex function on $(0, 1]$. Therefore, (3.4) holds precisely when

$$\lim_{u \downarrow 0} f(u) \leq \alpha, \quad f(1) \leq \alpha + \beta. \tag{3.14}$$

Since $\lim_{u \rightarrow 0} f(u) = ((n-1)/n)^{1/t}$ and $f(1) = 1$, it follows that

$$D_\varphi = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha \geq \left(\frac{n-1}{n} \right)^{1/t}, \alpha + \beta \geq 1 \right\}, \quad (3.15)$$

and $\alpha_\varphi = ((n-1)/n)^{1/t}$.

We are now in a position to give the inequalities of special interest. We describe the corresponding inequality in each case.

(i) Let $0 < t < 1$. Then

$$\left(\frac{x_1^t + \cdots + x_n^t}{n} \right)^{1/t} \leq \alpha x_n + (n^{1/(1-t)} - (n-1)^{1/(1-t)} \alpha^{t/(t-1)})^{(t-1)/t} x_1 \quad (3.16)$$

for $0 < x_1 \leq \cdots \leq x_n$ and $((n-1)/n)^{1/t} < \alpha \leq (n-1)/n$. In particular,

$$\left(\frac{x_1^t + \cdots + x_n^t}{n} \right)^{1/t} \leq \frac{(n-1)x_n + x_1}{n} \quad (3.17)$$

for $0 < x_1 \leq \cdots \leq x_n$.

Let $t < 0$. Then

$$\left(\frac{x_1^t + \cdots + x_n^t}{n} \right)^{1/t} \leq \alpha x_n + (n^{1/(1-t)} - (n-1)^{1/(1-t)} \alpha^{t/(t-1)})^{(t-1)/t} x_1 \quad (3.18)$$

for $0 < x_1 \leq \cdots \leq x_n$ and $0 < \alpha \leq (n-1)/n$. In particular,

$$\left(\frac{x_1^t + \cdots + x_n^t}{n} \right)^{1/t} \leq \frac{(n-1)x_n + x_1}{n} \quad (3.19)$$

for $0 < x_1 \leq \cdots \leq x_n$.

(ii) The inequality

$$\sqrt[n]{x_1 \cdots x_n} \leq \alpha x_n + \frac{(n-1)^{n-1}}{n^n} \alpha^{1-n} x_1 \quad (3.20)$$

holds for $0 < x_1 \leq \cdots \leq x_n$ and $0 < \alpha \leq (n-1)/n$. In particular,

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_n + (n-1)^{n-1} x_1}{n} \quad (3.21)$$

holds for $0 < x_1 \leq \cdots \leq x_n$.

(iii) Let $t \geq 1$. Then

$$\left(\frac{x_1^t + \cdots + x_n^t}{n} \right)^{1/t} \leq \alpha x_n + (1 - \alpha) x_1 \quad (3.22)$$

for $0 < x_1 \leq \cdots \leq x_n$ and $((n-1)/n)^{1/t} \leq \alpha$. In particular,

$$\left(\frac{x_1^t + \cdots + x_n^t}{n} \right)^{1/t} \leq \left(\frac{n-1}{n} \right)^{1/t} x_n + \left(1 - \left(\frac{n-1}{n} \right)^{1/t} \right) x_1 \quad (3.23)$$

for $0 < x_1 \leq \cdots \leq x_n$.

4. Application: the Hölder inequality

Let (Ω, μ) be a finite measure space and $0 < p < q < r \leq \infty$. Let $X = L^r(\Omega, \mu)$ and set

$$\varphi_0(f) = \|f\|_p, \quad \varphi_1(f) = \|f\|_r, \quad \varphi(f) = \|f\|_q \tag{4.1}$$

for all $f \in X$. Then

$$m = \mu(\Omega)^{1/q-1/p}, \quad M = \mu(\Omega)^{1/q-1/r}, \tag{4.2}$$

because the map $t \mapsto \mu(\Omega)^{-1/t} \|f\|_t$ is a monotone increasing function. If $\dim X = 1$, then we have

$$\begin{aligned} D_\varphi &= \{(\alpha, \beta) \in \mathbb{R}^2 : \mu(\Omega)^{1/q} \leq \alpha\mu(\Omega)^{1/r} + \beta\mu(\Omega)^{1/p}\} \\ &= \left\{(\alpha, \beta) \in \mathbb{R}^2 : 1 \leq \frac{\alpha}{M} + \frac{\beta}{m}\right\}. \end{aligned} \tag{4.3}$$

In general, it is hard to determine the domain D_φ . We consider the following two special cases:

- (I) $\Omega = \{1, 2\}$, $\mu(\{1\}) = a > 0$, $\mu(\{2\}) = b > 0$, $p = 1$, and $r = \infty$,
- (II) $1 \leq p < q < r$ and μ is nonatomic.

(I) We first consider the case (I). In this case, $m = (a + b)^{-1+1/q}$ and $M = (a + b)^{1/q}$. Let $(\alpha, \beta) \in \mathbb{R}^2$. Then $(\alpha, \beta) \in D_\varphi$ if and only if

$$(ax^q + by^q)^{1/q} \leq \alpha \max\{x, y\} + \beta(ax + by) \quad \forall x, y \geq 0. \tag{4.4}$$

This is equivalent to the following condition:

$$\alpha + \beta t \geq \sup \left\{ (ax^q + by^q)^{1/q} : \begin{array}{l} ax + by = t, 0 \leq x, y \leq 1 \\ \max\{x, y\} = 1 \end{array} \right\} \quad \text{for } \min\{a, b\} \leq t \leq a + b, \tag{4.5}$$

namely,

$$\begin{aligned} \alpha + \beta t &\geq (a^{1-q}(t - b)^q + b)^{1/q} \quad \text{for } b \leq t \leq a + b, \\ \alpha + \beta t &\geq (b^{1-q}(t - a)^q + a)^{1/q} \quad \text{for } a \leq t \leq a + b. \end{aligned} \tag{4.6}$$

Set $f(t) = (a^{1-q}(t - b)^q + b)^{1/q}$ for $b \leq t \leq a + b$. Since $1 < q < \infty$, f is a convex function on $[b, a + b]$. Hence $\alpha + \beta t \geq f(t)$ for $b \leq t \leq a + b$ if and only if $\alpha + \beta b \geq f(b) = b^{1/q}$ and $\alpha + \beta(a + b) \geq f(a + b) = (a + b)^{1/q}$. Also, set $g(t) = (b^{1-q}(t - a)^q + a)^{1/q}$ for $a \leq t \leq a + b$. The same argument shows that $\alpha + \beta t \geq g(t)$ for $a \leq t \leq a + b$ if and only if $\alpha + \beta a \geq a^{1/q}$ and $\alpha + \beta(a + b) \geq (a + b)^{1/q}$. Therefore, in view of the condition (4.6), we have

$$D_\varphi = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \begin{array}{l} \alpha + \beta b \geq b^{1/q}, \alpha + \beta a \geq a^{1/q} \\ \alpha + \beta(a + b) \geq (a + b)^{1/q} \end{array} \right\}. \tag{4.7}$$

Moreover, since $\alpha + \beta(a + b) \geq (a + b)^{1/q}$ means $\alpha/M + \beta/m \geq 1$, it follows from the fundamental facts (A), (B) that

$$\alpha_\varphi = \frac{\max\{a, b\}^{1/q}(a + b) - (a + b)^{1/q} \max\{a, b\}}{\min\{a, b\}}. \tag{4.8}$$

Also, D_φ has two extreme points:

$$\left(\frac{\max\{a, b\}^{1/q}(a + b) - (a + b)^{1/q} \max\{a, b\}}{\min\{a, b\}}, \frac{(a + b)^{1/q} - \max\{a, b\}^{1/q}}{\min\{a, b\}} \right), \tag{4.9}$$

$$\left(\frac{a^{1/q}b - ab^{1/q}}{b - a}, \frac{b^{1/q} - a^{1/q}}{b - a} \right).$$

The first extreme point corresponds to the following inequality:

$$(ax^q + by^q)^{1/q} \leq \frac{\max\{x, y\}}{\min\{a, b\}} (\max\{a, b\}^{1/q}(a + b) - (a + b)^{1/q} \max\{a, b\}) \tag{4.10}$$

$$+ \frac{ax + by}{\min\{a, b\}} ((a + b)^{1/q} - \max\{a, b\}^{1/q})$$

for all $a, b, x, y > 0$ and $q > 1$. In particular, if $a = b$, then we have

$$(x^q + y^q)^{1/q} \leq \max\{x, y\} (2 - 2^{1/q}) + (x + y)(2^{1/q} - 1) \tag{4.11}$$

for all $x, y > 0$ and $q > 1$. Since $x + y = \max\{x, y\} + \min\{x, y\}$, it follows that

$$(x^q + y^q)^{1/q} \leq \max\{x, y\} + (2^{1/q} - 1) \min\{x, y\} \tag{4.12}$$

for all $x, y > 0$ and $q > 1$. This is just equal to (3.23) in case of $n = 2$. The second extreme point corresponds to the following inequality:

$$(ax^q + by^q)^{1/q} \leq \frac{a^{1/q}b - ab^{1/q}}{b - a} \max\{x, y\} + \frac{b^{1/q} - a^{1/q}}{b - a} (ax + by) \tag{4.13}$$

for all $a, b, x, y > 0$ and $q > 1$.

(II) We consider the case (II). Take $f \in X$. Set $t = (r - p)/(q - p)$ and $s = (r - p)/(r - q)$. Then $r/t + p/s = q$ and $1/t + 1/s = 1$. Also, we have

$$\frac{p}{sq} = \frac{rp - pq}{rq - pq} = \frac{1/q - 1/r}{1/p - 1/r}, \quad \frac{r}{tq} = 1 - \frac{p}{sq}. \tag{4.14}$$

Now put $y = (1/q - 1/r)/(1/p - 1/r)$. Then $0 < y < 1$, $p/(sq) = y$, and $r/(tq) = 1 - y$. We use the Hölder inequality to see that

$$\|f\|_q = \left(\int |f|^q dx \right)^{1/q} = \left(\int |f|^{r/t} |f|^{p/s} dx \right)^{1/q} \tag{4.15}$$

$$\leq \left(\int |f|^r dx \right)^{1/tq} \left(\int |f|^p dx \right)^{1/sq} = \|f\|_r^{r/tq} \|f\|_p^{p/sq} = \|f\|_r^{1-y} \|f\|_p^y.$$

Take $\varepsilon > 0$ arbitrarily and put $\alpha = (1 - \gamma)\varepsilon$. If $u = 1/(1 - \gamma)$ and $v = 1/\gamma$, then the Young inequality yields

$$\begin{aligned} \|f\|_r^{1-\gamma} \|f\|_p^\gamma &= (\varepsilon \|f\|_r)^{1-\gamma} (\varepsilon^{(\gamma-1)/\gamma} \|f\|_p)^\gamma \\ &\leq \frac{(\varepsilon \|f\|_r)^{(1-\gamma)u}}{u} + \frac{(\varepsilon^{(\gamma-1)/\gamma} \|f\|_p)^\gamma}{v} \\ &= (1 - \gamma)\varepsilon \|f\|_r + \gamma \varepsilon^{(\gamma-1)/\gamma} \|f\|_p \\ &= \alpha \|f\|_r + \gamma \left(\frac{\alpha}{1 - \gamma}\right)^{(\gamma-1)/\gamma} \|f\|_p. \end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16), we obtain

$$\|f\|_q \leq \alpha \|f\|_r + \gamma \left(\frac{\alpha}{1 - \gamma}\right)^{(\gamma-1)/\gamma} \|f\|_p. \tag{4.17}$$

Since $\varepsilon > 0$ is arbitrary, so is $\alpha > 0$. Hence we have

$$\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0, \beta \geq h(\alpha)\} \subset D_\varphi, \tag{4.18}$$

where $h(\alpha) = \gamma(\alpha/(1 - \gamma))^{(\gamma-1)/\gamma}$. Now, set

$$\alpha_0 = (1 - \gamma)\mu(\Omega)^{\gamma/p-\gamma/r}. \tag{4.19}$$

We observe that

$$\{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha \leq \alpha_0, \beta \geq h(\alpha)\} = D_\varphi \cap \{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha \leq \alpha_0\}. \tag{4.20}$$

Actually, the equality holds in (4.17) if and only if the equalities hold in both (4.15) and (4.16). Hence the equality condition in (4.17) is that

$$\{|f(\omega)| : \omega \in \Omega\} \subset \{0, c\} \quad \text{for some } c \in \mathbb{R}, \quad \left(\frac{\alpha}{1 - \gamma}\right)^{1/\gamma} \|f\|_r = \|f\|_p. \tag{4.21}$$

Define

$$a(\alpha) = \left(\frac{\alpha}{1 - \gamma}\right)^{pr/(r-p)\gamma} \tag{4.22}$$

for all $\alpha > 0$. Let $0 < \alpha \leq \alpha_0$. Then $0 < a(\alpha) \leq a(\alpha_0) = \mu(\Omega)$, and hence we can take a measurable set A such that $\mu(A) = a(\alpha)$, because μ is nonatomic. Since the characteristic function χ_A on A satisfies the condition (4.21), the equality in (4.17) holds for $f = \chi_A$. Consequently, we easily see that (4.20) is valid. Notice that

$$\begin{aligned} h'(\alpha) &= \frac{\gamma}{1 - \gamma} \frac{\gamma - 1}{\gamma} \left(\frac{\alpha}{1 - \gamma}\right)^{(\gamma-1)/\gamma-1} = -\left(\frac{\alpha}{1 - \gamma}\right)^{-1/\gamma} < 0 \quad (\alpha > 0), \\ h''(\alpha) &= \frac{1}{1 - \gamma} \frac{1}{\gamma} \left(\frac{\alpha}{1 - \gamma}\right)^{-1/\gamma-1} = \frac{1}{\gamma(1 - \gamma)} \left(\frac{\alpha}{1 - \gamma}\right)^{-(1+\gamma)/\gamma} > 0 \quad (\alpha > 0). \end{aligned} \tag{4.23}$$

Hence $h(\alpha)$ is a strictly monotone decreasing concave function on $(0, \infty)$. Note also that $h'(\alpha_0) = -m/M$, since $m/M = \mu(\Omega)^{-1/p+1/r}$. Next we assert that the point $(\alpha_0, h(\alpha_0))$ is on the line $\alpha/M + \beta/m = 1$. Indeed,

$$\frac{\gamma-1}{p} + \frac{1-\gamma}{r} = (\gamma-1)\left(\frac{1}{p} - \frac{1}{r}\right) = -\frac{r}{q} \frac{q-p}{r-p} \frac{r-p}{pr} = \frac{p-q}{pq} = \frac{1}{q} - \frac{1}{p}, \quad (4.24)$$

and so

$$\begin{aligned} m\left(1 - \frac{\alpha_0}{M}\right) &= \mu(\Omega)^{1/q-1/p} - \mu(\Omega)^{1/r-1/p}(1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r} \\ &= \mu(\Omega)^{1/q-1/p} - \mu(\Omega)^{(\gamma-1)/p+(1-\gamma)/r} + \gamma\mu(\Omega)^{(\gamma-1)/p+(1-\gamma)/r} \\ &= \gamma\mu(\Omega)^{(\gamma-1)/p+(1-\gamma)/r} = h(\alpha_0). \end{aligned} \quad (4.25)$$

This implies the assertion. Therefore α_φ is just equal to α_0 by the fundamental facts (A), (B). Hence the above observations imply that

$$\begin{aligned} \alpha_\varphi &= (1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r}, \\ D_\varphi &= \{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha \leq \alpha_\varphi, \beta \geq h(\alpha)\} \cup \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \geq \alpha_\varphi, \frac{\alpha}{M} + \frac{\beta}{m} \geq 1\right\}. \end{aligned} \quad (4.26)$$

Thus the corresponding inequality is

$$\|f\|_q \leq \alpha \|f\|_r + \gamma \left(\frac{\alpha}{1-\gamma}\right)^{(\gamma-1)/\gamma} \|f\|_p \quad (4.27)$$

for all $f \in L^r(\Omega, \mu)$, $1 \leq p < q < r$, $0 < \alpha \leq (1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r}$, and $\gamma = (1/q - 1/r)/(1/p - 1/r)$. In particular, we have

$$\|f\|_q \leq (1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r} \|f\|_r + \gamma\mu(\Omega)^{(\gamma-1)/p-(1-\gamma)/r} \|f\|_p \quad (4.28)$$

for all $f \in L^r(\Omega, \mu)$, $1 \leq p < q < r$, and $\gamma = (1/q - 1/r)/(1/p - 1/r)$. Moreover, as $r \rightarrow \infty$, we have

$$\mu(\Omega)^{-1/q} \|f\|_q \leq \left(1 - \frac{p}{q}\right) \|f\|_\infty + \frac{p}{q} \mu(\Omega)^{-1/p} \|f\|_p \quad (4.29)$$

for all $f \in L^\infty(\Omega, \mu)$ and $1 \leq p < q < \infty$.

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