ASYMPTOTIC BEHAVIOR OF A SYSTEM OF LINEAR FRACTIONAL DIFFERENCE EQUATIONS

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We investigate the global asymptotic behavior of solutions of the system of difference equations $x_{n+1} = (a + x_n)/(b + y_n)$, $y_{n+1} = (d + y_n)/(e + x_n)$, $n = 0, 1, \ldots$, where the parameters a, b, d, and e are positive numbers and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers. We obtain some asymptotic results for the positive equilibrium of this system.

1. Introduction and preliminaries

Consider the following system of difference equations

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \quad y_{n+1} = \frac{d + y_n}{e + x_n}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the parameters a, b, d, and e are positive numbers and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

In a modelling setting, system (1.1) of nonlinear difference equations represents the rule by which two discrete, competitive populations reproduce from one generation to the next. The phase variables x_n and y_n denote population sizes during the nth generation and the sequence or orbit $\{(x_n, y_n) : n = 0, 1, 2, \ldots\}$ depicts how the populations evolve over time. Competition between the two populations is reflected by the fact that the transition function for each population is a decreasing function of the other population size.

Hassell and Comins [6] studied 2-species competition with rational transition functions of a similar type. They discussed equilibrium stability and illustrated oscillatory and even chaotic behavior. Franke and Yakubu [4, 5] also investigated interspecific competition with rational transition functions. They established results about population exclusion where one population always goes extinct, but their assumptions included selfrepression and precluded the existence of any equilibria in the interior of the positive quadrant. A simple competition model that allows unbounded growth of a population size has been discussed in [1, 2], where it was assumed that $a = d = 0$, that is,

$$x_{n+1} = \frac{x_n}{b + y_n}, \quad y_{n+1} = \frac{y_n}{e + x_n}, \quad n = 0, 1, \ldots. \tag{1.2}$$

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Our goal in this paper is to investigate the effect of the parameters a and d on the global behavior of solutions of system (1.2). We will show that the parameters a and d can have stabilizing effect for the global behavior of solutions of system (1.2), in the sense that the unique positive equilibrium of (1.1) can become the global attractor of all positive solutions of this system for certain values of a and d.

The techniques that will be used in this paper and the results that will be obtained are essentially different from the corresponding techniques and results in [1, 2]. A more general system of the form

$$x_{n+1} = \frac{a + x_n}{b + cx_n + y_n}, \quad y_{n+1} = \frac{d + y_n}{e + x_n + f y_n}, \quad n = 0, 1, \dots, \tag{1.3}$$

has been investigated in [9]. Here we will obtain more precise results for the special case of system (1.1) by using the monotonicity properties of the map in (1.1). We will give some basic results on the equilibrium points and their stability and prove some asymptotic results for the unique equilibrium point of (1.1).

A detailed analysis for the related simpler rational difference equation of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}} \tag{1.4}$$

has been performed in [7].

An *invariant rectangle* for the system

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, 2, \dots, \tag{1.5}$$

is a rectangle $\mathcal{R} = [L_1, U_1] \times [L_2, U_2]$ with the property that if a single point (x_N, y_N) of the solution falls in $\mathcal{R}$, then all subsequent terms of the solution also belong to $\mathcal{R}$. In other words, $\mathcal{R}$ is an invariant rectangle for the system (1.5) if $(x_N, y_N) \in \mathcal{R}$ for some $N \geq 0$, implies $(x_n, y_n) \in \mathcal{R}$ for every $n > N$. Similarly, one can define the invariant set in $(-\infty, \infty) \times (-\infty, \infty)$.

The method of invariant intervals has been widely used to prove convergence and global attractivity in the case of the second-order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots, \tag{1.6}$$

see [7] and the references therein.

In this section, we present the basic notions and definitions concerning the stability of equilibrium and periodic points of system (1.5).

We will assume that the functions f and g in (1.5) are continuously differentiable. We will also assume that the considered system has solutions which exist for all $n \geq 0$.

Definition 1.1. An *equilibrium solution* of (1.5) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\bar{x} = f(\bar{x}, \bar{y}), \qquad \bar{y} = g(\bar{x}, \bar{y}). \tag{1.7}$$

Now, we define the basic notions about the stability of the equilibrium point of system (1.5).

By $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{R}^2$ given by

$$\|(x,y)\| = \sqrt{x^2+y^2}. \tag{1.8}$$

Definition 1.2. (1) An equilibrium (fixed) point $(\bar{x},\bar{y})$ of (1.5) is said to be *stable* if for any $\epsilon > 0$, there is $\delta > 0$ such that for every initial point (x_0,y_0) for which $\|(x_0,y_0)-(\bar{x},\bar{y})\| < \delta$, the iterates (x_n,y_n) of (x_0,y_0) satisfy $\|(x_n,y_n)-(\bar{x},\bar{y})\| < \epsilon$ for all $n > 0$.

(2) An equilibrium (fixed) point $(\bar{x},\bar{y})$ of (1.5) is said to be *unstable* if it is not stable. An equilibrium (fixed) point $(\bar{x},\bar{y})$ of (1.5) is said to be *asymptotically stable* if it is stable and if there exists $r > 0$ such that $(x_n,y_n) \to (\bar{x},\bar{y})$ as $n \to \infty$ for all (x_0,y_0) that satisfy $\|(x_0,y_0)-(\bar{x},\bar{y})\| < r$.

The main result in the linearized stability analysis is the following result.

Theorem 1.3 (linearized stability theorem). *Let $F = (f,g)$ be a continuously differentiable function defined on an open set W in $\mathbb{R}^2$, and let $(\bar{x},\bar{y})$ in W be a fixed point of F.*

(a) *If all the eigenvalues of the Jacobian matrix $JF(\bar{x},\bar{y})$ have modulus less than one, then the equilibrium point $(\bar{x},\bar{y})$ of (1.5) is asymptotically stable.*

(b) *If at least one of the eigenvalues of the Jacobian matrix $JF(\bar{x},\bar{y})$ has modulus greater than one, then the equilibrium point $(\bar{x},\bar{y})$ of (1.5) is unstable.*

(c) *The equilibrium point $(\bar{x},\bar{y})$ of (1.5) is locally asymptotically stable if every solution of the characteristic equation*

$$\lambda^2 - \operatorname{Tr} JF(\bar{x},\bar{y})\lambda + \operatorname{Det} JF(\bar{x},\bar{y}) = 0 \tag{1.9}$$

lies inside the unit circle, that is, if and only if

$$\left|\operatorname{Tr} JF(\bar{x},\bar{y})\right| < 1 + \operatorname{Det} JF(\bar{x},\bar{y}) < 2. \tag{1.10}$$

The next theorem gives the convergence result in the case when there exists an invariant rectangle and the functions f and g satisfy some additional conditions. The proof is similar to the proof of the related result in [9].

Theorem 1.4. *Let $[a,b]$ and $[c,d]$ be intervals of real numbers and assume that*

$$f : [a,b]\times[c,d] \longrightarrow [a,b], \qquad g : [a,b]\times[c,d] \longrightarrow [c,d] \tag{1.11}$$

are continuous functions satisfying the following properties:

(a) *$f(x,y)$ is nondecreasing in x and nonincreasing in y in $[a,b]\times[c,d]$ and $g(x,y)$ is nonincreasing in x and nondecreasing in y in $[a,b]\times[c,d]$;*

(b) *if $(m,M,m',M') \in ([a,b]\times[c,d])^2$, $m \le m'$, $M \le M'$ is a solution of the system*

$$\begin{aligned} m &= f(m,M'), \qquad & m' &= f(m',M), \\ M &= g(m',M), \qquad & M' &= g(m,M'), \end{aligned} \tag{1.12}$$

then $m = m'$ and $M = M'$.

Then every solution of (1.5) which has one point (x_n,y_n) in $[a,b]\times[c,d]$ converges to the equilibrium $(\overline{x},\overline{y})$.

2. Equilibrium points

The equilibrium points $(\overline{x},\overline{y})$ of (1.1) satisfy

$$\overline{x}=\frac{a+\overline{x}}{b+\overline{y}},\qquad \overline{y}=\frac{d+\overline{y}}{e+\overline{x}}. \tag{2.1}$$

This implies that

$$\overline{x}\,\overline{y}=a+(1-b)\overline{x}^2=d+(1-e)\overline{y}. \tag{2.2}$$

Thus, the equilibrium points $(\overline{x},\overline{y})$, whenever they exist, are the intersection points of the curves

$$y=\frac{a}{x}+1-b,\qquad y=\frac{d}{x+e-1}. \tag{2.3}$$

Notice that system (1.1), in some cases, can have equilibrium points which belong to the coordinate axes as, for instance, when $a=0$ and/or $d=0$. As we mentioned earlier, the case $a=d=0$ has been considered in [1, 2] and so we will assume that $a>0$ and $d>0$. In this case the conditions for the existence of the positive equilibrium points $E=(\overline{x},\overline{y})$ of system (1.1) are

(C1) $b>1$, $e>1$,
(C2) $b=1$, $e>1$, $a<d$,
(C3) $b>1$, $e=1$, $a>d$,
(C4) $b<1$, $e<1$,
(C5) $b=1$, $e<1$, $a>d$,
(C6) $b<1$, $e=1$, $d>a$.

3. Linearized stability analysis

System (1.1) is a special case of the general system (1.5), where

$$f(x,y)=\frac{a+x}{b+y},\qquad g(x,y)=\frac{d+y}{e+x}. \tag{3.1}$$

The Jacobian of the map $T(x,y)=((a+x)/(b+y),(d+y)/(e+x))$ that corresponds to system (1.1) evaluated at the positive equilibrium $E=(\overline{x},\overline{y})$ has the form

$$J_T(E)=\begin{bmatrix}\dfrac{\overline{x}^2(b+\overline{y})}{(a+\overline{x})^2} & -\dfrac{\overline{x}^2}{a+\overline{x}}\\[2ex] -\dfrac{\overline{y}^2}{d+\overline{y}} & \dfrac{\overline{y}^2(e+\overline{x})}{(d+\overline{y})^2}\end{bmatrix}. \tag{3.2}$$

The characteristic equation of this Jacobian is

$$\lambda^2-p\lambda-q=0, \tag{3.3}$$

where

$$\begin{aligned} p &= \frac{\bar{x}^2(b+\bar{y})}{(a+\bar{x})^2} + \frac{\bar{y}^2(e+\bar{x})}{(d+\bar{y})^2}, \\ q &= -\frac{\bar{x}^2\bar{y}^2(b+\bar{y})(e+\bar{x})}{(a+\bar{x})^2(d+\bar{y})^2} + \frac{\bar{x}^2\bar{y}^2}{(a+\bar{x})(d+\bar{y})}. \end{aligned} \tag{3.4}$$

The conditions for the local asymptotic stability of the equilibrium point $E = (\bar{x}, \bar{y})$ are (see [7, 8])

$$|\lambda_{1,2}| < 1 \Longleftrightarrow |p| < 1 - q < 2. \tag{3.5}$$

One can check that these conditions are satisfied if and only if

$$\min\{0, 1 - (b+\bar{y})(e+\bar{x})\} < \bar{x}\,\bar{y} < (b+\bar{y}-1)(e+\bar{x}-1), \tag{3.6}$$

that is, if and only if

$$\bar{x}\,\bar{y} < \sqrt{ad}. \tag{3.7}$$

Thus the following theorem holds.

Theorem 3.1. (a) *The positive equilibrium point $E = (\bar{x}, \bar{y})$ of (1.1) is locally asymptotically stable if condition (3.7) is satisfied.*

(b) *The positive equilibrium point $E = (\bar{x}, \bar{y})$ of (1.1) is a saddle if the condition*

$$\sqrt{ad} < \bar{x}\,\bar{y} < \sqrt{4\bar{x}\,\bar{y} + 2a\bar{y} + 2d\bar{x} + ad} \tag{3.8}$$

is satisfied.

(c) *The positive equilibrium point $E = (\bar{x}, \bar{y})$ of (1.1) is a repeller if the condition*

$$\bar{x}\,\bar{y} > \sqrt{4\bar{x}\,\bar{y} + 2a\bar{y} + 2d\bar{x} + ad} \tag{3.9}$$

is satisfied.

(d) *The positive equilibrium point $E = (\bar{x}, \bar{y})$ of (1.1) is nonhyperbolic if at least one of the conditions*

$$\begin{gathered} \bar{x}\,\bar{y} = \sqrt{4\bar{x}\,\bar{y} + 2a\bar{y} + 2d\bar{x} + ad}, \\ \sqrt{ad} = \bar{x}\,\bar{y} \end{gathered} \tag{3.10}$$

is satisfied.

4. Global attractivity results

System (1.1) can be represented in the form

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0,1,2,\dots, \tag{4.1}$$

where

$$f(x,y) = \frac{a+x}{b+y}, \qquad g(x,y) = \frac{d+y}{e+x}. \tag{4.2}$$

Now we find an invariant interval $[k,K]$ for system (1.1) and so, an invariant rectangle. Taking into account that $f(x,y)$ is increasing in x and decreasing in y, while $g(x,y)$ is decreasing in x and increasing in y, we have

$$k \le \frac{a+k}{b+K} = f(k,K) \le x_{n+1} = f(x_n, y_n) \le f(K,k) = \frac{a+K}{b+k} \le K, \tag{4.3}$$

$$k \le \frac{d+k}{e+K} = g(K,k) \le y_{n+1} = g(x_n, y_n) \le g(k,K) = \frac{d+K}{e+k} \le K, \tag{4.4}$$

$$a + (1-b)K \le kK \le a + (1-b)k, \tag{4.5}$$

$$d + (1-e)K \le kK \le d + (1-e)k. \tag{4.6}$$

The consistency condition for (4.5) and (4.6) implies

$$b \ge 1, \qquad e \ge 1. \tag{4.7}$$

Thus we consider several cases.

Case 4.1 ($b > 1$, $e > 1$). In this case, system (1.1) has a unique positive equilibrium $E = (\overline{x}, \overline{y})$ that is locally asymptotically stable. Indeed, we check condition (3.6):

$$\begin{gathered} \overline{x}\,\overline{y} < (b + \overline{y} - 1)(e + \overline{x} - 1) \Longleftrightarrow (b-1)(e-1) + (b-1)\overline{x} + (e-1)\overline{y} > 0, \\ 1 - (b + \overline{y})(e + \overline{x}) < \overline{x}\,\overline{y} \Longleftrightarrow 2\overline{x}\,\overline{y} + b\overline{x} + e\,\overline{y} + (be - 1) > 0, \end{gathered} \tag{4.8}$$

which is always satisfied.

Taking $k = 0$, the inequalities (4.5) and (4.6) imply

$$K \ge \frac{a}{b-1}, \qquad K \ge \frac{d}{e-1}. \tag{4.9}$$

Thus,

$$K \ge \max\left\{\frac{a}{b-1}, \frac{d}{e-1}\right\}. \tag{4.10}$$

Now it is easy to check that condition (1.12) of Theorem 1.4 is satisfied. Thus we have the following result.

Theorem 4.2. *Assume that $b > 1$ and $e > 1$. Then the rectangle $[0,K] \times [0,K]$, where K satisfies (4.10) is an invariant rectangle. The equilibrium $E = (\overline{x}, \overline{y})$ is globally asymptotically stable.*

Proof. Clearly, for any initial value (x_0, y_0) we can choose K that satisfies (4.10) such that $x_0, y_0 \in [0, K]$. Now, we apply Theorem 1.4. Condition (b) takes the form

$$\begin{aligned} m &= \frac{a+m}{b+M'}, \qquad & m' &= \frac{a+m'}{b+M}, \\ M &= \frac{d+M}{e+m'}, \qquad & M' &= \frac{d+M'}{e+m}. \end{aligned} \tag{4.11}$$

This implies

$$(m' - m)(b-1) = mM' - m'M, \tag{4.12}$$

$$(M' - M)(e-1) = m'M - mM'. \tag{4.13}$$

If $b \geq 1$, then (4.12) implies

$$mM' \geq m'M. \tag{4.14}$$

If we assume that $e \geq 1$, then (4.13) implies $M' = M$, which implies that $m = m'$. We see from (C1)–(C6) that system (1.1) has a unique positive equilibrium. □

Remark 4.3. Notice that

$$\overline{x} \leq \frac{a}{b-1}, \qquad \overline{y} \leq \frac{d}{e-1}, \tag{4.15}$$

and so

$$\overline{x}, \overline{y} \leq \max\left\{\frac{a}{b-1}, \frac{d}{e-1}\right\} \leq K, \tag{4.16}$$

which shows that the equilibrium $E = (\overline{x}, \overline{y})$ belongs to the invariant rectangle $[k, K]$.

Case 4.4 ($b = 1$, $e > 1$, $a < d$). In this case, system (1.1) has a unique positive equilibrium $E = (\overline{x}, \overline{y}) = (a(e-1)/(d-a), (d-a)/(e-1))$, that is locally asymptotically stable, as

$$\overline{x}\,\overline{y} = a = \sqrt{a}\sqrt{a} < \sqrt{a}\sqrt{d} = \sqrt{ad}. \tag{4.17}$$

Inequalities (4.5) and (4.6) imply

$$\begin{gathered} a \leq kK \leq a \Longrightarrow kK = a, \\ d + (1-e)K \leq a \leq d + (1-e)k \Longrightarrow K \geq \frac{d-a}{e-1} \geq k, \end{gathered} \tag{4.18}$$

respectively. Thus

$$0 < k = \frac{a}{K} \leq \frac{a(e-1)}{d-a}, \qquad k^2 < a < K^2 \Longrightarrow k < \sqrt{a} < K. \tag{4.19}$$

Taking into account that $E = (a(e-1)/(d-a),(d-a)/(e-1)) \in [k,K]\times[k,K]$, we obtain

$$K \geq \max\left\{\frac{a(e-1)}{d-a},\frac{d-a}{e-1}\right\}, \qquad 0 < k \leq \min\left\{\frac{a(e-1)}{d-a},\frac{d-a}{e-1}\right\}, \tag{4.20}$$
$$kK = a, \quad k < K.$$

It is easy to check that condition (1.12) of Theorem 1.4 is satisfied, and thus we have the following result.

Theorem 4.5. *Assume that $d > a$, $b = 1$, and $e > 1$. Then the rectangle $[k,K]\times[k,K]$, where k and K satisfy (4.20), is an invariant rectangle. The equilibrium $E = (\overline{x},\overline{y})$ is globally asymptotically stable.*

Proof. Clearly, for any initial value (x_0, y_0) we can choose k and K, that satisfy (4.20) such that $x_n, y_n \in [k,K]$ for $n = 1,2,\ldots$. The proof follows from (4.12) and (4.13). $\square$

Case 4.6 ($b > 1$, $e = 1$, $a > d$). In this case system (1.1) has a unique positive equilibrium $E = (\overline{x},\overline{y}) = ((a-d)/(b-1),d(b-1)/(a-d))$, that is locally asymptotically stable, as

$$\overline{x}\,\overline{y} = d = \sqrt{d}\sqrt{d} < \sqrt{a}\sqrt{d} = \sqrt{ad}. \tag{4.21}$$

Conditions (4.5) and (4.6) imply

$$\begin{gathered} d \leq kK \leq d \Longrightarrow kK = d, \\ a + (1-b)K \leq d \leq a + (1-b)k \Longrightarrow K \geq \frac{a-d}{b-1} \geq k, \end{gathered} \tag{4.22}$$

respectively. Thus

$$0 < k = \frac{d}{K} \leq \frac{d(b-1)}{a-d}, \qquad k^2 < d < K^2 \Longrightarrow k < \sqrt{d} < K. \tag{4.23}$$

Taking into account that $E = ((a-d)/(b-1),d(b-1)/(a-d)) \in [k,K]\times[k,K]$, we obtain

$$K \geq \max\left\{\frac{a-d}{b-1},\frac{d(b-1)}{a-d}\right\}, \qquad 0 < k \leq \min\left\{\frac{a-d}{b-1},\frac{d(b-1)}{a-d}\right\}, \tag{4.24}$$
$$kK = d, \quad k < K.$$

It is easy to check that condition (1.12) of Theorem 1.4 is satisfied, and thus we have the following result.

Theorem 4.7. *Assume that $a > d$, $b > 1$, and $e = 1$. Then the rectangle $[k,K]\times[k,K]$, where k and K satisfy (4.24), is an invariant rectangle. The equilibrium $E = (\overline{x},\overline{y})$ is globally asymptotically stable.*

Proof. Clearly, for any initial value (x_0, y_0) we can choose k and K, that satisfy (4.24) such that $x_n, y_n \in [k,K]$ $(n = 1,2,\ldots)$. The proof follows from (4.12) and (4.13). $\square$

5. Monotone maps technique

In cases where we cannot determine an invariant rectangle for (1.1) we will use the technique of monotone maps, as we did in [2, 10]. First, we write system (1.1) in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}_{n+1} = T\begin{pmatrix} x \\ y \end{pmatrix}_{n}, \tag{5.1}$$

where the map T is given as

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \dfrac{a+x}{b+y} \\ \dfrac{d+y}{e+x} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}. \tag{5.2}$$

The map T may be viewed as a monotone map if we define a partial order on $\mathbb{R}^2$ so that the positive cone in this new partial order is the fourth quadrant. Specifically, for $v = (v_1, v_2)$, $w = (w_1, w_2) \in \mathbb{R}^2$ we say that $v \leq w$ if $v_1 \leq w_1$ and $w_2 \leq v_2$. Two points $v, w \in \mathbb{R}^2_+$ are said to be *related* if $v \leq w$ or $w \leq v$. Also, a strict inequality between points may be defined as $v < w$ if $v \leq w$ and $v \neq w$. A stronger inequality may be defined as $v \ll w$ if $v_1 < w_1$ and $w_2 < v_2$. A map $f : \operatorname{Int}\mathbb{R}^2_+ \to \operatorname{Int}\mathbb{R}^2_+$ is *strongly monotone* if $v < w$ implies that $f(v) \ll f(w)$ for all $v, w \in \operatorname{Int}\mathbb{R}^2_+$. Clearly, being related is an invariant under iteration of a strongly monotone map. For $x, y \in \operatorname{Int}\mathbb{R}^2_+$, we observe from the Jacobian matrix $J(x,y)$ that the derivative of T has constant sign configuration

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}. \tag{5.3}$$

The mean value theorem and the convexity of $\mathbb{R}^2_+$ may be used to show that T is monotone, as in [2].

For each $v \in \mathbb{R}^2_+$, define $Q_i(v)$ for $i = 1, \dots, 4$ to be the usual four quadrants based at v and numbered in a counterclockwise direction, for example, $Q_1(v) = \{(x,y) \in \mathbb{R}^2_+ : v_1 \leq x,\ v_2 \leq y\}$.

The following result describes the monotonic character of the solutions of a system that generates a monotone map. See [2].

Proposition 5.1. (1) *Assume the map T is monotone nondecreasing and*

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \leq T\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{5.4}$$

Then

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \leq \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}, \quad n = 0, 1 \dots. \tag{5.5}$$

(2) *Assume the map T is monotone nondecreasing and*

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \geq T\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{5.6}$$

Then

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \geq \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}, \quad n = 0, 1, \dots. \tag{5.7}$$

Remark 5.2. Note that inequality (5.5) is equivalent to

$$x_n \leq x_{n+1}, \quad y_n \geq y_{n+1}, \quad n = 0, 1, \dots, \tag{5.8}$$

that is, the sequence $\{x_n\}$ is nondecreasing, and the sequence $\{y_n\}$ is nonincreasing.

Note that inequality (5.7) is equivalent to

$$x_n \geq x_{n+1}, \quad y_n \leq y_{n+1}, \quad n = 0, 1, \dots, \tag{5.9}$$

that is, the sequence $\{x_n\}$ is nonincreasing, and the sequence $\{y_n\}$ is nondecreasing.

Now, note that condition (3.8) for the equilibrium point $E = (\overline{x}, \overline{y})$ of (1.1) to be a saddle point, given in Theorem 3.1, is equivalent to the condition

$$0 < (b + \overline{y} - 1)(e + \overline{x} - 1) < \overline{x}\,\overline{y} < (e + \overline{x} + 1)(b + \overline{y} + 1). \tag{5.10}$$

Case 5.3 ($b < 1$, $e < 1$). In this case, system (1.1) has a unique positive equilibrium $E = (\overline{x}, \overline{y})$ such that

$$\begin{aligned} 1 - e < \overline{x} < \frac{d}{1-b} + 1 - e, \\ 1 - b < \overline{y} < \frac{a}{1-e} + 1 - b. \end{aligned} \tag{5.11}$$

We show that this equilibrium satisfies (5.10) and so is a saddle point. Indeed, the first inequality in (5.10) gives

$$\begin{aligned} &\overline{x}\,\overline{y} > (b + \overline{y} - 1)(e + \overline{x} - 1) \\ &\Longleftrightarrow (1-b)\overline{x} + (1-e)\overline{y} - (1-b)(1-e) > 0 \\ &\Longleftrightarrow (1-b)(\overline{x} - 1 + e) + (1-e)\overline{y} > 0, \end{aligned} \tag{5.12}$$

which, in view of $b < 1$ and $e < 1$, is satisfied. Likewise, the second inequality in (5.10) is satisfied as

$$\overline{x}\,\overline{y} < (e + \overline{x} + 1)(b + \overline{y} + 1) \Longleftrightarrow (b+1)(e + \overline{x} + 1) + (e+1)\overline{y} > 0. \tag{5.13}$$

We apply Proposition 5.1. Condition (5.4) implies

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \leq \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \dfrac{a + x_0}{b + y_0} \\ \dfrac{d + y_0}{e + x_0} \end{pmatrix} \Longleftrightarrow \begin{cases} x_0 \leq \dfrac{a + x_0}{b + y_0} \\ \dfrac{d + y_0}{e + x_0} \leq y_0 \end{cases} \Longleftrightarrow \begin{cases} y_0 \leq \dfrac{a}{x_0} + 1 - b \\ x_0 \geq \dfrac{d}{y_0} + 1 - e. \end{cases} \tag{5.14}$$

Thus, if $x_0 \geq d/y_0 + (1-e)$ and $y_0 \leq a/x_0 + (1-b)$, the sequence $\{x_n\}$ is nondecreasing, and the sequence $\{y_n\}$ is nonincreasing.

If $x_0 > d/y_0 + (1-e)$ and $y_0 < a/x_0 + (1-b)$, the sequence $\{x_n\}$ is increasing, and the sequence $\{y_n\}$ is decreasing, which implies

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} y_n = 0. \tag{5.15}$$

Otherwise, $\lim_{n\to\infty} x_n = a < \infty$ and $\lim_{n\to\infty} y_n = b > 0$, which means that system (1.1) has an equilibrium point in the first quadrant different from $E = (\overline{x}, \overline{y})$.

We apply Proposition 5.1. Condition (5.6) gives

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \geq \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \dfrac{a+x_0}{b+y_0} \\ \dfrac{d+y_0}{e+x_0} \end{pmatrix} \Longleftrightarrow \begin{cases} x_0 \geq \dfrac{a+x_0}{b+y_0} \\ \dfrac{d+y_0}{e+x_0} \geq y_0 \end{cases} \Longleftrightarrow \begin{cases} y_0 \geq \dfrac{a}{x_0} + 1 - b \\ x_0 \leq \dfrac{d}{y_0} + 1 - e. \end{cases} \tag{5.16}$$

Thus, the sequence $\{x_n\}$ is nonincreasing, and the sequence $\{y_n\}$ is nondecreasing. The case $x_0 = d/y_0 + 1 - e$, $y_0 = a/x_0 + 1 - b$ is trivial. If $x_0 < d/y_0 + 1 - e$ and $y_0 > a/x_0 + 1 - b$, the sequence $\{x_n\}$ is decreasing, and the sequence $\{y_n\}$ is increasing. Now we have

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = \infty, \tag{5.17}$$

since otherwise system (1.1) would have an additional equilibrium point in the first quadrant.

Define the sets S_1 and S_2 as follows:

$$S_1 = \left\{ (x,y) \in \mathbb{R}_+^2 : \frac{d}{x+e-1} \leq y \leq \frac{a}{x} + 1 - b \right\}, \tag{5.18}$$

$$S_2 = \left\{ (x,y) \in \mathbb{R}_+^2 : \frac{a}{y+b-1} \leq x \leq \frac{d}{y} + 1 - e \right\}. \tag{5.19}$$

Set

$$\begin{aligned} \phi_1(x) &= \frac{d}{x+e-1}, \qquad & \phi_2(x) &= \frac{a}{x} + 1 - b, \\ \psi_1(y) &= \frac{a}{y+b-1}, \qquad & \psi_2(y) &= \frac{d}{y} + 1 - e. \end{aligned} \tag{5.20}$$

Note that for $x > \overline{x}$, $y > \overline{y} : \phi_i(x) \in Q_4(E)$, $\psi_i(y) \in Q_2(E)$ $(i = 1,2)$, and that for $(x,y) \in S_1$, $x > \overline{x} : \phi_1(x) < y < \phi_2(x) < \overline{y}$ while for $(x,y) \in S_2$, $y > \overline{y} : \psi_1(y) < x < \psi_2(y) < \overline{x}$. Consequently, $S_1 \subset Q_4(E)$ and $S_2 \subset Q_2(E)$.

Next, we will prove that S_1 and S_2 are invariant sets.

Lemma 5.4. *S_1 and S_2 are invariant sets.*

Proof. To prove that S_1 is an invariant set, we have to show that

$$(x_0, y_0) \in S_1 \Longrightarrow (x_n, y_n) \in S_1, \quad n = 1,2,\ldots, \tag{5.21}$$

and then apply the induction.

By definition of S_1 it follows that

$$(x_0, y_0) \in S_1 \Longleftrightarrow d + (1-e)y_0 \le x_0 y_0 \le a + (1-b)x_0. \tag{5.22}$$

We want to prove that

$$d + (1-e)y_1 \le x_1 y_1 \le a + (1-b)x_1. \tag{5.23}$$

First inequality is equivalent to

$$\begin{aligned}
& d + (1-e)y_1 \le x_1 y_1 \\
& \Longleftrightarrow d + (1-e)\frac{d+y_0}{e+x_0} \le \frac{a+x_0}{b+y_0} \cdot \frac{d+y_0}{e+x_0} \\
& \Longleftrightarrow (b+y_0)(d+(1-e)y_0 - x_0 y_0) + x_0(b+y_0)(d+y_0) \\
& \qquad \le (a+x_0)(d+y_0) \\
& \Longleftrightarrow (b+y_0)(d+(1-e)y_0 - x_0 y_0) - (d+y_0)(a+(1-b)x_0 - x_0 y_0) \le 0.
\end{aligned} \tag{5.24}$$

By (5.22), the last inequality is satisfied.

Next, we have

$$\begin{aligned}
& a + (1-b)x_1 \ge x_1 y_1 \\
& \Longleftrightarrow a + (1-b)\frac{a+x_0}{b+y_0} \ge \frac{a+x_0}{b+y_0} \cdot \frac{d+y_0}{e+x_0} \\
& \Longleftrightarrow (e+x_0)(a+(1-b)x_0 - x_0 y_0) + y_0(a+x_0)(e+x_0) \\
& \qquad \ge (a+x_0)(d+y_0) \\
& \Longleftrightarrow (e+x_0)(a+(1-b)x_0 - x_0 y_0) - (a+x_0)(d+(1-e)y_0 - x_0 y_0) \ge 0.
\end{aligned} \tag{5.25}$$

By (5.22), the last inequality is satisfied.

The proof that S_2 is an invariant set is similar. □

So far, we have proved that S_1 and S_2 are invariant under the map T and that every orbit in $S_1 \setminus E$ is attracted to $(\infty, 0)$, while every orbit in $S_2 \setminus E$ is attracted to $(0, \infty)$.

In other words, the *basin of attraction* of the point $(\infty, 0)$, see [3, page 61] and [11], which is denoted by $\mathcal{B}((\infty,0))$, contains $S_1 \setminus E$, $\mathcal{B}((\infty,0)) \supseteq S_1 \setminus E$, while the basin of attraction of $(0,\infty)$, $\mathcal{B}((0,\infty))$, contains $S_2 \setminus E$, $\mathcal{B}((0,\infty)) \supseteq S_2 \setminus E$.

Thus we have the following result.

Theorem 5.5. *Assume that $b < 1$ and $e < 1$.*

(1) *The set S_1, defined by (5.18), is an invariant set of (1.1) and every solution $\{(x_n, y_n)\}$ of (1.1), with initial conditions $(x_0, y_0) \in S_1 \setminus E$, satisfies*

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} y_n = 0, \qquad \mathcal{B}((\infty,0)) \supseteq S_1 \setminus E. \tag{5.26}$$

(2) *The set S_2, defined by (5.19), is an invariant set of (1.1) and every solution $\{(x_n, y_n)\}$ of (1.1), with initial conditions $(x_0, y_0) \in S_2 \setminus E$, satisfies*

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = \infty, \qquad \mathcal{B}((0,\infty)) \supseteq S_2 \setminus E. \tag{5.27}$$

Case 5.6 ($b = 1$, $e < 1$, $a > d$). In this case, system (1.1) has a unique positive equilibrium $E = (a(1-e)/(a-d), (a-d)/(1-e))$. We show that this equilibrium satisfies (3.8) and so is a saddle point. Indeed, the first inequality in (3.8) gives

$$\overline{x}\,\overline{y} = a = \sqrt{a}\sqrt{a} > \sqrt{a}\sqrt{d} = \sqrt{ad}. \tag{5.28}$$

The second inequality in (3.8) yields

$$\begin{aligned}
&\overline{x}^2\overline{y}^2 < 4\overline{x}\,\overline{y} + 2a\overline{y} + 2d\overline{x} + ad \\
&\iff a < 4 + 2\frac{a-d}{1-e} + 2d\frac{1-e}{a-d} + d \\
&\iff a(1-e)(a-d) < 4(1-e)(a-d) + 2(a-d)^2 + 2d(1-e)^2 \\
&\qquad\qquad + d(1-e)(a-d) \\
&\iff (1-e)(a-d)^2 - 4(1-e)(a-d) < 2(a-d)^2 + 2d(1-e)^2 \\
&\iff (a-d)^2(1+e) + 2d(1-e)^2 + 4(1-e)(a-d) > 0,
\end{aligned} \tag{5.29}$$

which is always satisfied.

Now, we apply Proposition 5.1. Condition (5.4) implies

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \le \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \dfrac{a+x_0}{1+y_0} \\ \dfrac{d+y_0}{e+x_0} \end{pmatrix} \iff \begin{cases} x_0 \le \dfrac{a+x_0}{1+y_0} \\ \dfrac{d+y_0}{e+x_0} \le y_0 \end{cases} \iff \begin{cases} y_0 \le \dfrac{a}{x_0} \\ x_0 \ge \dfrac{d}{y_0} + 1 - e. \end{cases} \tag{5.30}$$

In the case where $x_0 \ge d/y_0 + 1 - e$ and $y_0 \le a/x_0$, the sequence $\{x_n\}$ is nondecreasing and the sequence $\{y_n\}$ is nonincreasing.

In the case where $x_0 > d/y_0 + 1 - e$ and $y_0 < a/x_0$, the sequence $\{x_n\}$ is increasing, and the sequence $\{y_n\}$ is decreasing, and so

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} y_n = 0. \tag{5.31}$$

Otherwise, that is, if $\lim_{n\to\infty} x_n = a < \infty$ and $\lim_{n\to\infty} y_n = b > 0$, system (1.1) would possess another equilibrium point, as the limiting point of the sequence $\{(x_n, y_n)\}$.

Condition (5.6) of Proposition 5.1 implies

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \ge \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \dfrac{a+x_0}{1+y_0} \\ \dfrac{d+y_0}{e+x_0} \end{pmatrix} \iff \begin{cases} x_0 \ge \dfrac{a+x_0}{1+y_0} \\ \dfrac{d+y_0}{e+x_0} \ge y_0 \end{cases} \iff \begin{cases} y_0 \ge \dfrac{a}{x_0} \\ x_0 \le \dfrac{d}{y_0} + 1 - e. \end{cases} \tag{5.32}$$

Thus, the sequence $\{x_n\}$ is nonincreasing and the sequence $\{y_n\}$ is nondecreasing. Case $x_0 = d/y_0 + 1 - e$, $y_0 = a/x_0$ is trivial. When $x_0 < d/y_0 + 1 - e$ and $y_0 > a/x_0$, the sequence $\{x_n\}$ is decreasing, and the sequence $\{y_n\}$ is increasing. Consequently,

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = \infty, \tag{5.33}$$

since otherwise system (1.1) would possess another equilibrium point different from $E = (a(1-e)/(a-d), (a-d)/(1-e))$.

We introduce the sets O_1 and O_2 as follows:

$$O_1 = \left\{(x,y) \in \mathbb{R}_+^2 : \frac{d}{x+1-e} \le y \le \frac{a}{x}\right\}, \tag{5.34}$$

$$O_2 = \left\{(x,y) \in \mathbb{R}_+^2 : \frac{a}{y} \le x \le \frac{d}{y} + 1 - e\right\}. \tag{5.35}$$

Set

$$\phi_3(x) = \frac{d}{x+e-1}, \qquad \phi_4(x) = \frac{a}{x}, \qquad \psi_3(y) = \frac{a}{y}, \qquad \psi_4(y) = \frac{d}{y} + 1 - e. \tag{5.36}$$

Notice that for $x > \overline{x}$, $y > \overline{y} : \phi_i(x) \in Q_4(E)$, $\psi_i(y) \in Q_2(E)$ $(i = 3,4)$, and that for $(x,y) \in O_1$, $x > \overline{x} : \phi_3(x) < y < \phi_4(x) < \overline{y}$ while for $(x,y) \in O_2$, $y > \overline{y} : \psi_3(y) < x < \psi_4(y) < \overline{x}$. Consequently, $O_1 \subset Q_4(E)$ and $O_2 \subset Q_2(E)$.

As in Lemma 5.4, we can show that the sets O_1 and O_2 are invariant under the map T. In addition, every orbit in $O_1 \setminus E$ is attracted to $(\infty, 0)$, while every orbit in $O_2 \setminus E$ is attracted to $(0,\infty)$.

In other words, the basin of attraction of the point $(\infty,0)$, contains $O_1 \setminus E$, $\mathcal{B}((\infty,0)) \supseteq O_1 \setminus E$, while the basin of attraction of $(0,\infty)$, $\mathcal{B}((0,\infty))$, contains $O_2 \setminus E$, $\mathcal{B}((0,\infty)) \supseteq O_2 \setminus E$.

Thus we have the following result.

Theorem 5.7. *Assume that $b = 1$, $e < 1$, and $a > d$.*

(1) *The set O_1, defined by (5.34), is an invariant set of (1.1) and every solution $\{(x_n, y_n)\}$ of (1.1), with initial conditions $(x_0, y_0) \in O_1 \setminus E$, satisfies*

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} y_n = 0, \qquad \mathcal{B}((\infty,0)) \supseteq O_1 \setminus E. \tag{5.37}$$

(2) *The set O_2, defined by (5.35), is an invariant set of (1.1) and every solution $\{(x_n, y_n)\}$ of (1.1), with initial conditions $(x_0, y_0) \in O_2 \setminus E$, satisfies*

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = \infty, \qquad \mathcal{B}((0,\infty)) \supseteq O_2 \setminus E. \tag{5.38}$$

Case 5.8 ($b < 1$, $e = 1$, $a < d$). In this case, system (1.1) has a unique positive equilibrium $E = ((d-a)/(1-b), d(1-b)/(d-a))$. We show that this equilibrium satisfies (3.8) and so is a saddle point. Indeed, the first inequality in (3.8) gives

$$\overline{x}\,\overline{y} = d = \sqrt{d}\sqrt{d} > \sqrt{a}\sqrt{d} = \sqrt{ad}. \tag{5.39}$$

The second inequality in (3.8) yields

$$\begin{aligned}
&\overline{x}^2\overline{y}^2 < 4\overline{x}\,\overline{y} + 2a\overline{y} + 2d\overline{x} + ad \\
&\Longleftrightarrow d < 4 + 2\frac{d-a}{1-b} + 2a\frac{1-b}{d-a} + a \\
&\Longleftrightarrow d(1-b)(d-a) < 4(1-b)(d-a) + 2(d-a)^2 + 2a(1-b)^2 \\
&\qquad\qquad + a(1-b)(d-a) \\
&\Longleftrightarrow (1-b)(d-a)^2 - 4(1-b)(d-a) < 2(d-a)^2 + 2a(1-b)^2 \\
&\Longleftrightarrow (d-a)^2(1+b) + 2a(1-b)^2 + 4(1-b)(d-a) > 0.
\end{aligned} \tag{5.40}$$

Using Proposition 5.1 we obtain a similar result as in Cases 5.1 and 5.2.

We introduce the sets P_1 and P_2 as follows:

$$P_1 = \left\{(x,y) \in \mathbb{R}_+^2 : \frac{d}{x} \le y \le \frac{a}{x} + 1 - b\right\}, \tag{5.41}$$

$$P_2 = \left\{(x,y) \in \mathbb{R}_+^2 : \frac{a}{y+b-1} \le x \le \frac{d}{y}\right\}. \tag{5.42}$$

Set

$$\phi_5(x) = \frac{d}{x}, \qquad \phi_6(x) = \frac{a}{x} + 1 - b, \qquad \psi_5(y) = \frac{a}{y+b-1}, \qquad \psi_6(y) = \frac{d}{y}. \tag{5.43}$$

Note that for $x > \overline{x}$, $y > \overline{y} : \phi_i(x) \in Q_4(E)$, $\psi_i(y) \in Q_2(E)$ $(i = 5,6)$, and that for $(x,y) \in P_1$, $x > \overline{x} : \phi_5(x) < y < \phi_6(x) < \overline{y}$ while for $(x,y) \in P_2$, $y > \overline{y} : \psi_5(y) < x < \psi_6(y) < \overline{x}$. Consequently, $P_1 \subset Q_4(E)$ and $P_2 \subset Q_2(E)$.

As in Lemma 5.4, we can show that the sets P_1 and P_2 are invariant under the map T. In addition, every orbit in $P_1 \setminus E$ is attracted to $(\infty, 0)$, while every orbit in $P_2 \setminus E$ is attracted to $(0,\infty)$.

In other words, the basin of attraction of the point $(\infty,0)$, contains $P_1 \setminus E$, $\mathcal{B}((\infty,0)) \supseteq P_1 \setminus E$, while the basin of attraction of $(0,\infty)$, $\mathcal{B}((0,\infty))$, contains $P_2 \setminus E$, $\mathcal{B}((0,\infty)) \supseteq P_2 \setminus E$.

Thus we have the following result.

Theorem 5.9. *Assume that $b < 1$, $e = 1$, and $a < d$.*

(1) *The set P_1, defined by (5.41), is an invariant set of (1.1) and every solution $\{(x_n, y_n)\}$ of (1.1), with initial conditions $(x_0, y_0) \in P_1 \setminus E$, satisfies*

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} y_n = 0, \qquad \mathcal{B}((\infty,0)) \supseteq P_1 \setminus E. \tag{5.44}$$

(2) *The set P_2, defined by (5.42), is an invariant set of (1.1) and every solution* $\{(x_n, y_n)\}$ *of (1.1), with initial conditions* $(x_0, y_0) \in P_2 \setminus E$, *satisfies*

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = \infty, \qquad \mathcal{B}((0,\infty)) \supseteq P_2 \setminus E. \tag{5.45}$$

We believe that, in Cases 5.1, 5.2, and 5.3, a stronger result holds. In fact, based on the results in [2], for system (1.2), we believe that there exists the global stable manifold $W^s(E) \in Q_1(E) \cup Q_3(E)$ of E that separates the positive quadrant and serves as a threshold for mutual exclusion, that is, for all orbits below this manifold, the y sequence converges to zero and the x sequence becomes unbounded and for all orbits above this manifold, the x sequence converges to zero and the y sequence becomes unbounded. Precisely, we have the following conjecture.

Conjecture 5.10. Each orbit in $\operatorname{Int}\mathbb{R}^2_+$ starting above $W^s(E)$ remains above $W^s(E)$ and is asymptotic to $(0,\infty)$. Each orbit in $\operatorname{Int}\mathbb{R}^2_+$ starting below W^s remains below W^s and is asymptotic to $(\infty,0)$.

6. Conclusion

The results obtained in Cases 4.1, 4.2, and 4.3 explain the effect of the positive parameters a and d on the global behavior of system (1.1). Namely, if $a = d = 0$, that is, in case of system (1.2), the zero equilibrium is globally asymptotically stable provided that $b > 1$, $e > 1$, see [1]. Introducing the positive parameters a and d, we have created the unique positive equilibrium and we have shifted this property to hold for this unique positive equilibrium.

In Case 4.2, we have shown that the equilibrium points of system (1.2) consist of all points on the y-axis and that every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of (1.2) converges to some of these equilibrium points which depend continuously on the initial point (x_0, y_0), see [1]. Introducing the positive parameters a and d, we have created the unique positive equilibrium and we have forced all solutions of (1.1) to converge to this unique positive equilibrium. A similar conclusion holds in Case 4.3 with the roles of b and e interchanged.

In Case 5.1, we have shown that the equilibrium points of system (1.2) are $(0,0)$ and $(1-e, 1-b)$, the former being a global repeller and latter being a saddle point, see [2]. We have shown that the global stable manifold of this system separates the positive quadrant into basins of attraction of two types of asymptotic behavior, one being asymptotic to $(\infty,0)$ and the other being asymptotic to $(0,\infty)$. In the case where $b = e$, we found an explicit equation for the stable manifold. In this case, the introduction of the positive parameters a and d does not seem to affect the global qualitative behavior at least in the region $S_1 \cup S_2$.

In Case 5.3, we have shown that the equilibrium points of system (1.2) are all the points on the y-axis and they are all nonhyperbolic except $(0,0)$. We have shown that the solutions of (1.2) may have two different global asymptotic behaviors with solutions approaching $(0,\overline{y})$ and $(\infty,0)$ as $n \to \infty$. An introduction of the positive parameters a and d changed the global behavior by generating the positive equilibrium point if $a < d$, and it seems that all solutions, with the exception of the stable manifold, are asymptotic to either $(\infty, 0)$ or to $(0, \infty)$. A similar conclusion holds in Case 5.2 with the roles of b and e interchanged.

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