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BALLS LEFT EMPTY BY A CRITICAL BRANCHING WIENER PROCESS

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ABSTRACT

At time $t=0$ we have a Poisson random field on $\mathbb{R}^d$. Each particle executes a critical branching Wiener process starting from its position at time $t=0$. Let R_T be the radius of the largest ball around the origin of $\mathbb{R}^d$ which does not contain any particle at time T. Our goal is to characterize the properties of the stochastic process $\{R_T, T \geq 0\}$.

This article is dedicated to the memory of Professor Roland L. Dobrushin.

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1. Introduction

Consider the following

Model 1:
(*i*) a particle starts from the position $0 \in \mathbb{R}^d$ and executes a Wiener process $W(t) \in \mathbb{R}^d$;
(*ii*) arriving at time $t=1$ to the new location $W(1)$, it dies;
(*iii*) at death, it is replaced by Y offspring, where

$$\mathbf{P}\{Y=0\} = \mathbf{P}\{Y=2\} = 1/2;$$

(*iv*) each offspring, starting from where its ancestor dies, executes a Wiener process (from its starting point) and repeats the above given steps and so on. All Wiener processes and offspring numbers are assumed independent of each other.

A more formal definition is given in Chapter 6 of [1], p. 91.

Let $A \subset \mathbb{R}^d$ be a Borel set and let $\lambda(A,t)$ $(t = 0,1,2,\ldots)$ be the number of particles located in A at time t. Then

$$B(t) = \lambda(\mathbb{R}^d, t)$$

is the number of particles living at t and $\{B(t), t = 0,1,2,\ldots\}$ is a branching process.

We also consider the following

Model 2: At time $t=0$ we have a Poisson random field of parameter μ, i.e., in a Borel set $A \subset \mathbb{R}^d$, we have k particles with probability

$$\pi(A,k)=\frac{(\mu\,|\,A\,|\,)^k}{k!}e^{-\mu\,|\,A\,|},$$

where $|\,A\,|$ is the Lebesgue measure of A. It is also assumed that the numbers of particles in disjoint Borel sets are independent r.v.'s. Each particle executes a critical branching Wiener process (starting from its position at time $t=0$) according to Model 1.

A more formal definition is given in Chapter 8 of [1], p. 129. Let $\Lambda(A,t)$ be the number of particles located in A at time t. Then clearly

$$\mathbf{P}\{\Lambda(A,0)=k\}=\pi(A,k).$$

Let

$$\mathfrak{C}(x,r)=\{y\colon \|\,y-x\,\|\ \leq r\}\subset\mathbb{R}^d$$

and

$$R_T=\sup\{R\colon \Lambda(\mathfrak{C}(0,R),T)=0\}\quad (T=0,1,2,\ldots),$$

i.e., R_T is the radius of the largest ball around the origin of $\mathbb{R}^d$ which does not contain any particle at time T.

We are interested in the limit behavior of R_T as $T\to\infty$.

In the case $d=1$, this problem is very simple. In fact we have,

Theorem A: (Theorem 8.2 p. 129 in [1]). *Let $d=1$. Then for any $\epsilon>0$ we have*

$$\Lambda(\mathfrak{C}(0,T(\log T)^{-1-\epsilon}),T)=0\ \textit{a.s.}$$

for all but finitely many T,

$$\Lambda(\mathfrak{C}(0,\epsilon T),T)\geq 1\ \textit{i.o. a.s.},$$

$$\Lambda(\mathfrak{C}(0,\epsilon^{-1}T),T)=0\ \ \textit{i.o. a.s.},$$

and

$$\Lambda(\mathfrak{C}(0,T(\log T)^{1+\epsilon}),T)\geq 1\ \textit{a.s.}$$

for all but finitely many T.

We note that Theorem 8.2 of [1] is formulated in a slightly different way, but the above Theorem A can be obtained directly by the method presented there.

Now we formulate our main result.

Theorem 1: *We have*

$$\Lambda(\mathfrak{C}(0,R_1(T,d)),T)\geq 1\ \textit{a.s.}$$

for all but finitely many T,

$$\Lambda(\mathfrak{C}(0,R_2(T,d)),T)=0\ \textit{i.o. a.s.},$$

$$\Lambda(\mathfrak{C}(0,R_3(T,d)),T)\geq 1\quad \textit{i.o.}\quad \textit{a.s.}$$

and

$$\Lambda(\mathfrak{C}(0,R_4(T,d)),T)=0\ \ \textit{a.s.}$$

for all but finitely many T, where

$$R_1(T,d)=\begin{cases} T(\log T)^{1+\epsilon} & \textit{if } d=1,\\ K(T\log T)^{1/2} & \textit{if } d=2,\\ K(\log T)^{1/(d-2)} & \textit{if } d\geq 3,\end{cases}$$

$$R_2(T,d)=\begin{cases}\epsilon^{-1}T & \text{if } d=1,\\ T^{1/2}(g(T))^{-1} & \text{if } d=2,\\ K^{-1}(\log\log\log T)^{1/(d-2)} & \text{if } d\geq 3,\end{cases}$$

$$R_3(T,d)=\begin{cases}\epsilon T & \text{if } d=1,\\ (\log T)^{-1/2+\epsilon} & \text{if } d=2,\\ K^{-1}(\log T)^{-1/d} & \text{if } d\geq 3,\end{cases}$$

$$R_4(T,d)=\begin{cases}T(\log T)^{-1-\epsilon} & \text{if } d=1,\\ T^{-1/2}(\log T)^{-1/2-\epsilon} & \text{if } d=2,\\ T^{-1/d}(\log T)^{-1/d-\epsilon} & \text{if } d\geq 3,\end{cases}$$

K is large enough, $g(T)$ is an arbitrary function with $g(T)\uparrow\infty$ and ϵ is an arbitrary positive number.

Remark: Intuitively it is clear that if $R_1(T)$ $(T=1,2,\ldots)$ is a function going to infinity fast enough, then the ball around the origin, of radius $R_1(T)$ will contain at least one living particle at time T for any T large enough. Theorem 1 claims that in $\mathbb{R}^2$ we might choose $R_1(T)=K(T\log T)^{1/2}$, while in $\mathbb{R}^3$ it is enough to choose $R_1(T)=K\log T$. We are also interested to characterize those functions $R_3(T)$ for which it is still true that the ball, around the origin, of radius $R_3(T)$ contains particles at time T for infinitely many T. Theorem 1 claims that in $\mathbb{R}^2$ we might choose $R_3(T)=(\log T)^{-1/2+\epsilon}$ while in $\mathbb{R}^3$ we might have $R_3(T)=K^{-1}(\log T)^{-1/3}$. The results on R_2 and R_4 tell us how exact are the results on R_1 and R_3. Unfortunately, it turns out that we have a very big gap.

We also prove two theorems describing some properties of $\lambda(\cdot,\cdot)$ of Model 1, which will be used in the proof of Theorem 1 and which seem to be interesting in themselves.

Let $f(t)$ $(t=1,2,\ldots)$ be a positive, real valued function with $f(t)\to\infty$ (as $t\to\infty$), let $\alpha\in\mathbb{R}^d$ and let

$$\mathfrak{C}=\mathfrak{C}(\alpha T^{1/2},T^{1/2}(f(T))^{-1}).$$

Then we have,

Theorem 2: *In case $d=1$ we have*

$$1+(1-\epsilon)\Big(1-\frac{1}{K}\Big)\frac{c(\alpha)}{2}\frac{T}{f(T)}\leq \mathbf{E}(\lambda(\mathfrak{C},T)\mid\lambda(\mathfrak{C},T)>0)$$
$$\leq 1+(1+\epsilon)c(\alpha)\frac{T}{f(T)}+K\frac{T}{f^2(T)}$$

for any $K>0$, $\epsilon>0$ if T is large enough, where

$$c(\alpha)=\Big(\frac{2}{\pi}\Big)^{1/2}\int_0^{\pi/2}\exp\Big(-\frac{\alpha^2}{2}\frac{1-\sin x}{1+\sin x}\Big)\,dx.$$

If we also assume that $f(T)\leq T^{1/2}$ and $K\geq 2$ then

$$1+(1-\epsilon)\ \Big(1-\frac{1}{K}\Big)\frac{c(\alpha)}{2}\frac{T}{f(T)}$$

$$\leq \mathbf{E}(\lambda(\mathfrak{C},T)\mid\lambda(\mathfrak{C},T)>0)\leq 1+(1+\epsilon)\frac{c(\alpha)}{2}\ \Big(1+\frac{2}{K}\Big)\frac{T}{f(T)}+K\frac{T}{f^2(T)}.$$

In case $d=2$ we have

$$1+(1-\epsilon)\ \left(1-\frac{1}{K}\right)\frac{1}{4}\ \frac{T}{f^2(T)}\log f(T)$$

$$\leq \mathbf{E}(\lambda(\mathbb{C},T)\mid\lambda(\mathbb{C},T)>0)\leq 1+\tfrac{1}{2}(1+\epsilon)\ \frac{T}{f^2(T)}\log f(T)+K\frac{T}{f^2(T)}$$

for any $K>0$ if T is large enough.

If we also assume that $f(T)\leq T^{1/2}$ and $K\geq 2$, then

$$1+(1-\epsilon)\ \left(1-\frac{1}{K}\right)\frac{1}{4}\ \frac{T}{f^2(T)}\ \log f(T)$$

$$\leq \mathbf{E}(\lambda(\mathbb{C},T)\mid\lambda(\mathbb{C},T)>0)$$

$$\leq 1+\frac{1+\epsilon}{4}\ \left(1+\frac{2}{K}\right)\frac{T}{f^2(T)}\log f(T)+K\frac{T}{f^2(T)}.$$

In case $d\geq 3$ for any $K>0$ if T is large enough, we have

$$1+(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{1}{d-2}\ \frac{\omega_d}{(4\pi)^{d/2}}\ \frac{T}{f^2(T)}\ K^{-(d-2)/2}$$

$$\leq E(\lambda(\mathbb{C},T)\mid\lambda(\mathbb{C},T)>0)$$

$$\leq 1+(1+\epsilon)\frac{2}{d-2}\ \frac{\omega_d}{(4\pi)^{d/2}}\ \frac{T}{f^2(T)}\ K^{-(d-2)/2}+K\frac{T}{f^2(T)},$$

where

$$\omega_d=\begin{cases}2 & \text{if } d=1,\\ \pi & \text{if } d=2,\\ \dfrac{\pi^{d/2}}{\Gamma(d/2+1)} & \text{if } d\geq 3\end{cases}$$

is the volume of a ball in $\mathbb{R}^d$ of radius 1.

Consequences: In case $d=1$,

$$\frac{c(\alpha)}{2}\leq \liminf_{T\to\infty}\frac{f(T)}{T}\mathbf{E}(\lambda(\mathbb{C},T)\mid\lambda(\mathbb{C},T)>0)$$

$$\leq \limsup_{T\to\infty}\frac{f(T)}{T}\mathbf{E}(\lambda(\mathbb{C},T)\mid\lambda(\mathbb{C},T)>0)<c(\alpha),$$

provided that

$$\lim_{T\to\infty}\frac{T}{f(T)}=\infty.$$

If $d=1$ and

$$\lim_{T\to\infty}\frac{T}{f(T)}=\beta,\quad 0\leq\beta<\infty,$$

then

$$1+\frac{c(\alpha)}{2}\beta\leq \liminf_{T\to\infty}\mathbf{E}(\lambda(\mathbb{C},T)\mid\lambda(\mathbb{C},T)>0)$$

$$\leq \limsup_{T\to\infty}\mathbf{E}(\lambda(\mathbb{C},T)\mid\lambda(\mathbb{C},T)>0)\leq 1+c(\alpha)\beta.$$

If $d=1$ and $f(T)\leq T^{1/2}$, then

$$\lim_{T\to\infty}\frac{f(T)}{T}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)=\frac{c(\alpha)}{2}.$$

Note that

$$c(0)=\left(\frac{\pi}{2}\right)^{1/2}.$$

In case $d=2$ we have

$$\frac{1}{4}\le\liminf_{T\to\infty}\frac{f^2(T)}{T\log f(T)}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)$$

$$\le\limsup_{T\to\infty}\frac{f^2(T)}{T\log f(T)}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)\le\frac{1}{2},$$

provided that

$$\lim_{T\to\infty}\frac{T}{f^2(T)}\log f(T)=\infty.$$

If $d=2$ and

$$\lim_{T\to\infty}\frac{T}{f^2(T)}\log f(T)=\beta,$$

then

$$1+\frac{\beta}{4}\le\liminf_{T\to\infty}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)$$

$$\le\limsup_{T\to\infty}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)\le 1+\frac{\beta}{2}.$$

If $d=2$ and $f(T)\le T^{1/2}$, then

$$\lim_{T\to\infty}\frac{f^2(T)}{T\log f(T)}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)=\frac{1}{4}.$$

In case $d\ge 3$,

$$\frac{2}{d-2}\frac{\omega_d}{(8\pi)^{d/2}}\le\liminf_{T\to\infty}\frac{f^2(T)}{T}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)$$

$$\le\limsup_{T\to\infty}\frac{f^2(T)}{T}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)\le\frac{4}{d-2}\frac{\omega_d}{(8\pi)^{d/2}}+2,$$

provided that

$$\lim_{T\to\infty}\frac{T}{f^2(T)}=\infty.$$

If $d\ge 3$ and

$$\lim_{T\to\infty}\frac{T}{f^2(T)}=\beta,$$

then

$$1+\frac{2}{d-2}\frac{\omega_d}{(8\pi)^{d/2}}\beta\le\liminf_{T\to\infty}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)$$

$$\le\limsup_{T\to\infty}\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)$$

$$\le 1+2\beta+\frac{4}{d-2}\frac{\omega_d}{(8\pi)^{d/2}}\beta.$$

Theorem 3: *Consider the case where*

$$d=1,\ \lim_{T\to\infty}\frac{T}{f(T)}=\infty.$$

Then,

$$\frac{1}{(2\pi)^{1/2}}\frac{\exp(-\alpha^2/2)}{c(\alpha)}\le\liminf_{T\to\infty}\mathbf{P}\{\lambda(\mathcal{C},T)>0\mid B(T)>0\}$$

$$\leq \limsup_{T\to\infty} \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$

$$\leq \left(\frac{2}{\pi}\right)^{1/2} \frac{\exp(-\alpha^2/2)}{c(\alpha)}. \tag{1.1}$$

If $d=1$ *and*

$$\lim_{T\to\infty} \frac{T}{f(T)} = \beta, \quad 0 \leq \beta < \infty,$$

then

$$\frac{\beta}{(2\pi)^{1/2}} \frac{\exp(-\alpha^2/2)}{1 \leq c(\alpha)\beta} \leq \liminf_{T\to\infty} \frac{f(T)}{T} \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$

$$\leq \limsup_{T\to\infty} \frac{f(T)}{T} \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$

$$\leq \frac{\beta}{(2\pi)^{1/2}} \frac{2\exp(-\alpha^2/2)}{2+c(\alpha)\beta}. \tag{1.2}$$

If

$$d=1 \text{ and } f(T) \leq T^{1/2},$$

then

$$\lim_{T\to\infty} \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\} = \left(\frac{2}{\pi}\right)^{1/2} \frac{e^{-\alpha^2/2}}{c(\alpha)}.$$

Now consider the case

$$d=2, \lim_{T\to\infty} \frac{T}{f^2(T)} \log f(T) = \infty.$$

Then,

$$\frac{1}{2} e^{-\alpha^2/2} \leq \liminf_{T\to\infty} (\log f(T)) \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$

$$\leq \limsup_{T\to\infty} (\log f(T)) \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$

$$\leq e^{-\alpha^2/2}. \tag{1.3}$$

If

$$d=2 \text{ and } \lim_{T\to\infty} \frac{T}{f^2(T)} \log f(T) = \beta,$$

then

$$\frac{\exp(-\alpha^2/2)}{4+2\beta} \leq \liminf_{T\to\infty} \frac{f^2(T)}{T} \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$

$$\leq \limsup_{T\to\infty} \frac{f^2(T)}{T} \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$

$$\leq \frac{\exp(-\alpha^2/2)}{4+\beta}. \tag{1.4}$$

If

$$d=2, \lim_{T\to\infty} \frac{T}{f^2(T)} \log f(T) = \infty \text{ and } f(T) \leq T^{1/2},$$

then

$$\lim_{T\to\infty} (\log f(T)) \mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0 \mid = \exp\left(-\frac{\alpha^2}{2}\right). \tag{1.5}$$

Now consider the case

$$d \geq 3, \lim_{T\to\infty} \frac{T}{f^2(T)} = \infty.$$

Then,

$$\frac{2^d \omega_d (d-2) e^{-\alpha^2/2}}{8\omega_d + 4(8\pi)^{d/2}(d-2)}$$

$$\leq \liminf_{T\to\infty}(f(T))^{d-2}\mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$
$$\leq \limsup_{T\to\infty}(f(T))^{d-2}\mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$
$$\leq (d-2)2^{d-2}\exp\left(-\frac{\alpha^2}{2}\right). \tag{1.6}$$

If

$$d \geq 3 \quad and \quad \lim_{T\to\infty}\frac{T}{f^2(T)} = \beta,$$

then

$$\frac{2^d\omega_d(d-2)e^{-\alpha^2/2}}{8\beta\omega_d + 2(d-2)(8\pi)^{d/2}(1+\beta)}$$
$$\leq \liminf_{T\to\infty}\frac{(f(T))^d}{T}\mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$
$$\leq \limsup_{T\to\infty}\frac{(f(T))^d}{T}\mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\}$$
$$\leq \frac{(d-2)2^d\omega_d e^{-\alpha^2/2}}{2(d-2)(8\pi)^{d/2}+4\omega_d\beta}. \tag{1.7}$$

2. Lemmas

Let

$$W_1(t) = \{W_{11}(t), W_{12}(t), \ldots, W_{1d}(t)\},$$
$$W_2(t) = \{W_{21}(t), W_{22}(t), \ldots, W_{2d}(t)\},$$
$$W_3(t) = \{W_{31}(t), W_{32}(t), \ldots, W_{3d}(t)\}$$

be independent Wiener processes and let

$$\Gamma_1(t,s,T) = \begin{cases} W_1(t) & \text{if } 0 \leq t \leq s, \\ W_1(s) + W_2(t-s) & \text{if } s \leq t \leq T, \end{cases}$$

$$\Gamma_2(t,s,T) = \begin{cases} W_1(t) & \text{if } 0 \leq t \leq s, \\ W_1(s) + W_3(t-s) & \text{if } s \leq t \leq T. \end{cases}$$

Let

$$\gamma(x) = \gamma_{T,s}(x) = \mathbf{P}\{\Gamma_2(T,s,T) = x \mid \Gamma_1(T,s,T) = z\}$$

be the conditional density function of Γ_2 given $\Gamma_1 = z$.

Lemma 1:

$$\gamma(x) = (2\pi\sigma^2)^{-d/2}\exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right),$$

where

$$\nu = \mathbf{E}(\Gamma_2(T,s,T) \mid \Gamma_1(T,s,T) = z) = \frac{sz}{T}$$

and

$$\sigma^2 = \mathbf{E}((\Gamma_2(T,s,T)-\nu)^2 \mid \Gamma_1(T,s,T) = z) = T\left(1-\frac{s^2}{T^2}\right).$$

Proof is trivial.

Lemma 2: *Let $A \subseteq \mathbb{R}^d$ be a Borel set. Then*

$$\mathbf{P}\{\Gamma_2(T,s,T)\in A \mid \Gamma_1(T,s,T)\in A\} = \frac{\int\limits_A\Big(\int\limits_A \gamma(x)dx\Big)\psi(z)dz}{\int\limits_A \psi(z)dz},$$

where

$$\psi(z)=\psi_T(z)=(2\pi T)^{-d/2}\exp\left(-\frac{z^2}{2T}\right)$$

is the density function of $\Gamma_1(T,s,T)$.

Proof: Since

$$\mathbf{P}\{\Gamma_2\in A \mid \Gamma_1\in A\} = \frac{\int\limits_A \mathbf{P}\{\Gamma_2\in A\mid \Gamma_1 = z\}\psi(z)dz}{\int\limits_A \psi(z)dz},$$

Lemma 2 follows.

Lemma 3: *Let*

$$\kappa_T = T - K\frac{T}{f^2(T)} \quad (K>0)$$

and

$$P(T)=\sum_{s=1}^{\kappa_T}\mathbf{P}\{\Gamma_2\in\mathfrak{C}\mid\Gamma_1\in\mathfrak{C}\}.$$

Then in the case $d=1$, for any $\epsilon>0$, there exists a $T_0=T_0(\epsilon)>0$ such that

$$(1-\epsilon)\Big(1-\frac{1}{K}\Big)c(\alpha)\le\frac{f(T)}{T}P(T)\le(1+\epsilon)c(\alpha)$$

if $T\ge T_0$, where

$$c(\alpha)=\Big(\frac{2}{\pi}\Big)^{1/2}\int_0^{\pi/2}\exp\left(-\frac{\alpha^2}{2}\,\frac{1-\sin x}{1+\sin x}\right)dx.$$

In the case $d=2$, for any $\epsilon>0$, there exists a $T_0=T_0(\epsilon)>0$ such that

$$(1-\epsilon)\ \Big(1-\frac{1}{K}\Big)\frac{1}{2}\,\frac{T}{f^2(T)}\log f(T)\le P(T)\le(1+\epsilon)\frac{1}{2}\,\frac{T}{f^2(T)}\log f(T)$$

if $T\ge T_0$.

In the case $d\ge 3$, for any $\epsilon>0$, there exists a $T_0=T_0(\epsilon)>0$ such that

$$(1-\epsilon)\Big(1-\frac{1}{K}\Big)\frac{\omega_d}{(4\pi)^{d/2}}\,\frac{2}{d-2}\,\frac{T}{f^2(T)}K^{-(d-2)/2}$$

$$\le P(T)\le(1+\epsilon)\frac{\omega_d}{(4\pi)^{d/2}}\,\frac{2}{d-2}\,\frac{T}{f^2(T)}K^{-(d-2)/2}$$

if $T\ge T_0$.

Proof: By Lemma 2,

$$P(T)=\frac{\sum\limits_{s=1}^{\kappa_T}\int\limits_{\mathfrak{C}}\int\limits_{\mathfrak{C}}\gamma(x)dx\psi(z)dz}{\int\limits_{\mathfrak{C}}\psi(z)dz}.$$

Let

$$x=\alpha T^{1/2}+u\frac{T^{1/2}}{f(T)},$$

$$z=\alpha T^{1/2}+v\frac{T^{1/2}}{f(T)}.$$

Then,

$$u\in\mathfrak{C}(0,1)\text{ and }v\in\mathfrak{C}(0,1)$$

if

$$x \in \mathfrak{C} \text{ and } z \in \mathfrak{C}.$$

Hence,

$$\exp\left(-\frac{z^2}{2T}\right) = \exp\left(-\frac{\alpha^2}{2}\right)\exp\left(-\frac{1}{2}\left(\frac{2(\alpha, v)}{f(T)} + \frac{v^2}{f^2(T)}\right)\right) =$$

$$= \exp\left(-\frac{\alpha^2}{2}\right)\left(1 + O\left(\frac{1}{f(T)}\right)\right),$$

$$\int_{\mathfrak{C}} \psi(z)dz = (2\pi T)^{-d/2} |\mathfrak{C}| \exp\left(-\frac{\alpha^2}{2}\right)\left(1 + O\left(\frac{1}{f(T)}\right)\right)$$

and

$$P(T) = \left(1 + O\left(\frac{1}{f(T)}\right)\right)\frac{1}{|\mathfrak{C}|}(2\pi)^{-d/2}\sum_{s=1}^{\kappa_T}\sigma^{-d}\int_{\mathfrak{C}}\int_{\mathfrak{C}}\exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right)dxdz.$$

Observe that

$$\frac{x-\nu}{\sigma} = \alpha\left(\frac{T-s}{T+s}\right)^{1/2} + (u-v)\frac{T}{f(T)((T-s)(T+s))^{1/2}} + v\left(\frac{T-s}{T+s}\right)^{1/2}\frac{1}{f(T)},$$

$$\left(\frac{x-\nu}{\sigma^2}\right)^2 = \alpha^2\frac{T-s}{T+s} + (u-v)^2\frac{T^2}{f^2(T)(T^2-s^2)} + O\left(\frac{1}{f(T)}\right)$$

and

$$\frac{T}{\sigma^2 f^2(T)} = \frac{T^2}{(T^2-s^2)f^2(T)} \le \frac{1}{K},$$

provided that

$$1 \le s \le \kappa_T.$$

Since

$$dxdz = \frac{T^d}{(f(T))^{2d}}dudv,$$

we have

$$\int_{\mathfrak{C}}\int_{\mathfrak{C}}\exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right)dxdz$$

$$= \left(1 + O\left(\frac{1}{f(T)}\right)\right)\frac{T^d}{(f(T))^{2d}}\exp\left(-\frac{\alpha^2}{2}\frac{T-s}{T+s}\right)\int_{\mathfrak{C}(0,1)}\int_{\mathfrak{C}(0,1)}\exp\left(-\frac{(u-v)^2}{2}\frac{T}{\sigma^2 f^2(T)}\right)dudv$$

$$\le \left(1 + O\left(\frac{1}{f(T)}\right)\right)\frac{T^d}{(f(T))^{2d}}\omega_d^2\exp\left(-\frac{\alpha^2}{2}\frac{T-s}{T+s}\right)$$

and

$$\int_{\mathfrak{C}}\int_{\mathfrak{C}}\exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right)dxdz > \left(1 + O\left(\frac{1}{f(T)}\right)\right)\left(1 - \frac{1}{K}\right)\frac{T^d}{(f(T))^{2d}}\omega_d^2\exp\left(-\frac{\alpha^2}{2}\frac{T-s}{T+s}\right).$$

Hence

$$\left(1 + O\left(\frac{1}{f(T)}\right)\right)\left(1 - \frac{1}{K}\right) I \le P(T) \le \left(1 + O\left(\frac{1}{f(T)}\right)\right)I,$$

where

$$I = \omega_d(2\pi)^{-d/2}\frac{T^{d/2}}{(f(T))^d}\sum_{s=1}^{\kappa_T}\sigma^{-d}\exp\left(-\frac{\alpha^2}{2}\frac{T-s}{T+s}\right)$$

$$= \omega_d (2\pi)^{-d/2} \frac{T}{(f(T))^d} \sum_{s=1}^{\kappa_T} \frac{1}{T} \left(1 - \frac{s^2}{T^2}\right)^{-d/2} \exp\left(-\frac{\alpha^2}{2} \frac{1 - s/T}{1 + s/T}\right)$$

$$\sim \omega_d (2\pi)^{-d/2} \frac{T}{(f(T))^d} \int_0^{1 - K(f(T))^{-2}} (1 - u^2)^{-d/2} \exp\left(-\frac{\alpha^2}{2} \frac{1-u}{1+u}\right) du.$$

In the case $d = 1$,

$$I \sim \left(\frac{2}{\pi}\right)^{1/2} \frac{T}{f(T)} \int_0^1 (1 - u^2)^{-1/2} \exp\left(-\frac{\alpha^2}{2} \frac{1-u}{1+u}\right) du$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{T}{f(T)} \int_0^{\pi/2} \exp\left(-\frac{\alpha^2}{2} \frac{1 - \sin x}{1 + \sin x}\right) dx.$$

Hence, for $d = 1$ and for any $K > 0$, we have

$$\left(1 + O\left(\frac{1}{f(T)}\right)\right)\left(1 - \frac{1}{K}\right) c(\alpha) \frac{T}{f(T)} \le P(T) \le \left(1 + O\left(\frac{1}{f(T)}\right)\right) c(\alpha) \frac{T}{f(T)}.$$

Hence, Lemma 3 is proved for $d = 1$.

In the case $d \ge 2$,

$$I \sim \omega_d (2\pi)^{-d/2} \frac{T}{(f(T))^d} \int_{K(f(T))^{-2}}^1 (v(2 - v))^{-d/2} \exp\left(-\frac{\alpha^2}{2} \frac{v}{2 - v}\right) dv$$

$$\sim \omega_d (4\pi)^{-d/2} \frac{T}{(f(T))^d} \int_{K(f(T))^{-2}}^1 v^{-d/2} dv.$$

Hence, in the case $d = 2$,

$$I \sim \frac{1}{2} \frac{T}{f^2(T)} \log f(T)$$

and we have Lemma 3 for $d = 2$.

In the case $d \ge 3$,

$$I \sim \frac{2\omega_d}{d - 2} (4\pi)^{-d/2} \frac{T}{(f(T))^2} \frac{1}{K^{(d-2)/2}}.$$

Lemma 4: *Let X, Y be i.i.d.r.v.'s with*

$$\mathbf{P}\{X \ge 0\} = \mathbf{P}\{Y \ge 0\} = 1,$$

$$\mathbf{P}\{X > 0\} = \mathbf{P}\{Y > 0\} = p \quad (0 \le p \le 1).$$

Then

$$\mathbf{E}(X + Y \mid X + Y > 0) = \frac{1}{2 - p} \mathbf{E}Y + \mathbf{E}(X \mid X > 0).$$

Proof:

$$\mathbf{E}(X + Y \mid X + Y > 0) = \frac{2}{\mathbf{P}\{X + Y > 0\}} \int_{\{X + Y > 0\}} X d\mathbf{P} = \frac{2}{2p - p^2} \mathbf{E}X = \frac{2p}{2p - p^2} \mathbf{E}(X \mid X > 0)$$

$$= \left(1 + \frac{p}{2 - p}\right) \mathbf{E}(X \mid X > 0) = \mathbf{E}(X \mid X > 0) + \frac{1}{2 - p} \mathbf{E}Y.$$

Lemma 4 is proved.

Lemma 5:

$$\mathbf{E}(\lambda(\mathfrak{C}, T) \mid B(T) > 0) \sim \frac{T}{2} \mid \mathfrak{C} \mid (2\pi T)^{-d/2} \exp\left(-\frac{\alpha^2}{2}\right)$$

$$= \frac{T}{2}(2\pi)^{-d/2}(f(T))^{-d}\omega_d \exp\left(-\frac{\alpha^2}{2}\right).$$

Proof: Clearly,

$$\mathbf{E}(\lambda(\mathfrak{C},T) \mid B(T)) = B(T)(2\pi T)^{-d/2}\int_{\mathfrak{C}} \exp\left(-\frac{x^2}{2T}\right)dx \sim B(T) \mid \mathfrak{C} \mid (2\pi T)^{-d/2}\exp\left(-\frac{\alpha^2}{2}\right).$$

Since

$$\mathbf{E}B(T) = 1 \text{ and } \mathbf{P}\{B(T) > 0\} \sim \frac{2}{T},$$

we have

$$\mathbf{E}\lambda(\mathfrak{C},T) = \mathbf{E}\mathbf{E}(\lambda(\mathfrak{C},T) \mid B(T)) \sim (2\pi T)^{-d/2} \mid \mathfrak{C} \mid \exp\left(-\frac{\alpha^2}{2}\right)$$

and

$$\mathbf{E}(\lambda(\mathfrak{C},T) \mid B(T) > 0) = \frac{1}{\mathbf{P}\{B(T) > 0\}}\int_{\{B(T) > 0\}} \lambda(\mathfrak{C},T)d\mathbf{P} = \frac{1}{\mathbf{P}\{B(T) > 0\}}\mathbf{E}\lambda(\mathfrak{C},T)$$

and consequently we have Lemma 5.

Lemma 6: $$\mathbf{P}\{\lambda(\mathfrak{C},T) > 0\} \le \mathbf{P}\{B(T) > 0\} \sim \frac{2}{T}.$$

Proof is trivial.

3. Proofs of Theorems 2 and 3

Having the condition $\{\lambda(\mathfrak{C},T) > 0\}$ we have two particles at time $t = 1$ and we know that at least one of them has at least one living offspring located in $\mathfrak{C}$ at time T. Let $\lambda_{11}(\mathfrak{C},T-1)$, respectively $\lambda_{12}(\mathfrak{C},T-1)$ be the number of those offspring of the first respectively, second particle which are located in $\mathfrak{C}$ at time T. Clearly,

$$\lambda(\mathfrak{C},T) = \lambda_{11}(\mathfrak{C},T-1) + \lambda_{12}(\mathfrak{C},T-1).$$

Then by Lemma 4,

$$\mathbf{E}(\lambda(\mathfrak{C},T) \mid \Lambda(\mathfrak{C},T) > 0)$$
$$= \mathbf{E}(\lambda_{11}(\mathfrak{C},T-1) \mid \lambda_{11}(\mathfrak{C},T-1) > 0) + \frac{1}{2-p_1}\mathbf{E}\lambda_{12}(\mathfrak{C},T-1), \tag{3.1}$$

where

$$p_1 = \mathbf{P}\{\lambda_{11}(\mathfrak{C},T-1) > 0\} = \mathbf{P}\{\lambda_{12}(\mathfrak{C},T-1) > 0\}.$$

Consider that particle at time $t = 1$ which has at least one offspring living at time T and located in $\mathfrak{C}$. (In the case both particles have such an offspring, consider one of them.) This particle has at time $t = 2$ two offspring and we know that at least one of them has at least one offspring located in $\mathfrak{C}$ at time T. Let $\lambda_{21}(\mathfrak{C},T-2)$ respectively, $\lambda_{22}(\mathfrak{C},T-2)$ be the number of those offspring of the first respectively, second particle which are located in $\mathfrak{C}$ at time T. Clearly,

$$\lambda_{11}(\mathfrak{C},T-1) = \lambda_{21}(\mathfrak{C},T-2) + \lambda_{22}(\mathfrak{C},T-2).$$

Then by Lemma 4,

$$\mathbf{E}(\lambda_{11}(\mathfrak{C},T-1) \mid \lambda_{11}(\mathfrak{C},T-1) > 0)$$
$$\mathbf{E}(\lambda_{21}(\mathfrak{C},T-2) \mid \lambda_{21}(\mathfrak{C},T-2) > 0) + \frac{1}{2-p_2}\mathbf{E}\lambda_{22}(\mathfrak{C},T-2), \tag{3.2}$$

where

$$p_2 = \mathbf{P}\{\lambda_{21}(\mathfrak{C},T-2) > 0\} = \mathbf{P}\{\lambda_{22}(\mathfrak{C},T-2) > 0\}.$$

(3.1) and (3.2), combined, imply

$$\mathbf{E}(\lambda(\mathfrak{C},T) \mid \lambda(\mathfrak{C},T) > 0) = \mathbf{E}(\lambda_{21}(\mathfrak{C},T-2) \mid \lambda_{21}(\mathfrak{C},T-2) > 0)$$
$$+ \frac{1}{2-p_1}\mathbf{E}\lambda_{12}(\mathfrak{C},T-1) + \frac{1}{2-p_2}\mathbf{E}\lambda_{22}(\mathfrak{C},T-2).$$

Continuing this procedure we obtain

$$\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)=\sum_{s=1}^{T}\frac{1}{2-p_s}\mathbf{E}\lambda_{s2}(\mathcal{C},T-s)$$

$$=\sum_{s=1}^{\kappa_T}\frac{1}{2-p_s}\mathbf{E}\lambda_{s2}(\mathcal{C},T-s)+\sum_{s=\kappa_T+1}^{T-1}\frac{1}{2-p_s}\mathbf{E}\lambda_{s2}(\mathcal{C},T-s)+1=I+II+1, \tag{3.3}$$

where

$$p_s=\mathbf{P}\{\lambda_{s1}(\mathcal{C},T-s)>0\}=\mathbf{P}\{\lambda_{s2}(\mathcal{C},T-s)>0\}$$

and

$$\kappa_T=T-K\frac{T}{f^2(T)}.$$

Clearly,

$$\mathbf{E}\lambda_{s2}(\mathcal{C},T-s)=\mathbf{P}\{\Gamma_2(T,s,T)\in\mathcal{C}\mid\Gamma_1(T,s,T)\in\mathcal{C}\}, \tag{3.4}$$

$$0\le II\le\sum_{s=\kappa_T+1}^{T-1}\mathbf{E}\lambda_{s2}(\mathcal{C},T-s)=\sum_{s=\kappa_T+1}^{T-1}\mathbf{P}\{\Gamma_2\in\mathcal{C}\mid\Gamma_1\in\mathcal{C}\}$$

$$\le T-\kappa_T=K\frac{T}{f^2(T)} \tag{3.5}$$

and

$$\frac{1}{2}\sum_{s=1}^{\kappa_T}\mathbf{E}\lambda_{s2}(\mathcal{C},T-s)\le I\le\sum_{s=1}^{\kappa_T}\mathbf{E}\lambda_{s2}\mathcal{C},T-s). \tag{3.6}$$

Then by Lemma 3, (3.4) and (3.6) if $d=1$, we have

$$(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{c(\alpha)}{2}\frac{T}{f(T)}\le I\le(1+\epsilon)c(\alpha)\frac{T}{f(T)}. \tag{3.7}$$

(3.3), (3.5) and (3.7) imply

$$1+(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{c(\alpha)}{2}\frac{T}{f(T)}\le\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)$$

$$\le 1+(1+\epsilon)c(\alpha)\frac{T}{f(T)}+K\frac{T}{f^2(T)} \tag{3.8}$$

for any $K>0$.

Note that if

$$f(T)\le T^{1/2}\text{ and }s\le\kappa_T,$$

then by Lemma 6,

$$p_s\le\frac{2}{T-s}\le\frac{2f^2(T)}{KT}\le\frac{2}{K}$$

and

$$I\le\frac{1}{2}\left(1+\frac{2}{K}\right)\sum_{s=1}^{\kappa_T}\mathbf{E}\lambda_{s2}(\mathcal{C},T-s) \tag{3.9}$$

if $K\ge 2$.

If we assume that $d=1$, $f(T)\le T^{1/2}$ and $K\ge 2$, then by (3.3), (3.4), (3.5), (3.6) and (3.9), we have

$$1+(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{c(\alpha)}{2}\frac{T}{f(T)}\le\mathbf{E}(\lambda(\mathcal{C},T)\mid\lambda(\mathcal{C},T)>0)$$

$$\le 1+(1+\epsilon)\frac{1}{2}\left(1+\frac{2}{K}\right)c(\alpha)\frac{T}{f(T)}+K\frac{T}{f^2(T)}. \tag{3.10}$$

Hence, we have Theorem 2 in the case $d=1$.

In the case $d=2$, Lemma 3, (3.4) and (3.6) imply

$$(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{1}{4}\frac{T}{f^2(T)}\log f(T) \le I \le \frac{1}{2}(1+\epsilon)\frac{T}{f^2(T)}\log f(T). \tag{3.11}$$

(3.3), (3.5) and (3.11) imply

$$1+(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{1}{4}\frac{T}{f^2(T)}\log f(T)$$
$$\le \mathbf{E}(\lambda(\mathfrak{C},T) \mid \lambda(\mathfrak{C},T)>0) \le 1+\frac{1}{2}(1+\epsilon)\frac{T}{f^2(T)}\log f(T)+K\frac{T}{f^2(T)}$$

for any $K>0$.

If we assume that $d=2$, $f(T)\le T^{1/2}$ and $K\ge 2$, then by (3.3), (3.4), (3.5), (3.6) and (3.9) we have

$$1+(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{1}{4}\frac{T}{f^2(T)}\log f(T) \le \mathbf{E}(\lambda(\mathfrak{C},T)\mid\lambda(\mathfrak{C},T)>0)$$
$$\le 1+\frac{1+\epsilon}{4}\left(1+\frac{2}{K}\right)\frac{T}{f^2(T)}\log f(T)+K\frac{T}{f^2(T)}. \tag{3.12}$$

Hence, we have Theorem 2 in the case $d=2$.

In the case $d\ge 3$, Lemma 3, (3.4) and (3.6) imply

$$(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{1}{2}\frac{\omega_d}{(4\pi)^{d/2}}\frac{2}{d-2}\frac{T}{f^2(T)}K^{-(d-2)/2}$$
$$\le I \le (1+\epsilon)\frac{\omega_d}{(4\pi)^{d/2}}\frac{2}{d-2}\frac{T}{f^2(T)}K^{-(d-2)/2}. \tag{3.13}$$

(3.3), (3.5) and (3.13) imply

$$1+(1-\epsilon)\left(1-\frac{1}{K}\right)\frac{1}{d-2}\frac{\omega_d}{(4\pi)^{d/2}}\frac{T}{f^2(T)}K^{-(d-2)/2} \le \mathbf{E}(\lambda(\mathfrak{C},T)\mid\lambda(\mathfrak{C},T)>0)$$
$$\le 1+(1+\epsilon)\frac{\omega_d}{(4\pi)^{d/2}}\frac{2}{d-2}\frac{T}{f^2(T)}K^{-(d-2)/2}+K\frac{T}{f^2(T)}.$$

Hence, we have Theorem 2 in the case $d\ge 3$.

Theorem 3 is a simple outcome of the consequences of Theorem 2, Lemma 5 and the following:

Lemma 7:

$$\mathbf{P}\{\lambda(\mathfrak{C},T)>0 \mid B(T)>0\} = \frac{\mathbf{E}(\lambda(\mathfrak{C},T)\mid B(T)>0)}{\mathbf{E}(\lambda(\mathfrak{C},T)\mid\lambda(\mathfrak{C},T)>0)},$$

whose proof is trivial.

4. Proof of Theorem 1

Let $B(A,T)$ be the number of those particles which are located in $A\subset\mathbb{R}^d$ at time $t=0$ and which have at least one offspring living at time T. The following lemma is trivial.

Lemma 8:

$$\mathbf{P}\{B(A,T)=k\}=\frac{\nu^k}{k!}e^{-\nu} \quad (k=0,1,2,\ldots)$$

and

$$\mathbf{E}\exp(-zB(A,T))=\exp(\nu(e^{-z}-1)),$$

where

$$\nu=\nu(A,T)\sim\frac{2\mu\,|\,A\,|}{T},$$

as $T\to\infty$.

Introduce the following notations:

$$\mathfrak{C}(R) = \mathfrak{C}(0,R),$$
$$\mathfrak{C}_i = \mathfrak{C}_i(\Delta,T) = \mathfrak{C}(0,(i+1)\Delta T^{1/2}) - \mathfrak{C}(0,i\Delta T^{1/2}),$$
$$B(R) = B(\mathfrak{C}(R),T),$$
$$B_i = B_i(\Delta,T) = B(\mathfrak{C}_i,T).$$

Then we have

$$|\,\mathfrak{C}(R)\,| = R^d\omega_d,$$
$$= d\omega_d\Delta^dT^{d/2}i^{d-1} \leq |\,\mathfrak{C}_i\,| = \Delta^dT^{d/2}((i+1)^d - i^d)\omega_d \leq 2^d\omega_d\Delta^dT^{d/2}i^{d-1},$$
$$\mathbf{E}B(R) \sim \frac{2\mu\omega_dR^d}{T}, \tag{4.1}$$
$$\exp\left(-(1+\epsilon)\frac{2\mu\omega_dR^d}{T}\right) \leq \mathbf{P}\{B(R)=0\} \leq \exp\left(-(1-\epsilon)\frac{2\mu\omega_dR^d}{T}\right), \tag{4.2}$$
$$\exp\left(-(1+\epsilon)\frac{2\mu\omega_dR^d}{T}(e^{-z}-1)\right) \leq \mathbf{E}\exp(-zB(R)) \leq \exp\left(-(1-\epsilon)\frac{2\mu\omega_dR^d}{T}(e^{-z}-1)\right), \tag{4.3}$$
$$(1-\epsilon)2\mu d\omega_d\Delta^di^{d-1}T^{(d-2)/2} \leq \mathbf{E}B_i \sim \frac{2\mu\,|\,\mathfrak{C}_i\,|}{T} \leq (1+\epsilon)2\mu 2^d\omega_d\Delta^di^{d-1}T^{(d-2)/2}, \tag{4.4}$$
$$\exp(-(1+\epsilon)2\mu 2^d\omega e_d\Delta^di^{d-1}T^{(d-2)/2}) \leq \mathbf{P}\{B_i = 0\}$$
$$\sim \exp\left(-\frac{2\mu\,|\,\mathfrak{C}_i\,|}{T}\right) \leq \exp(-(1-\epsilon)2\mu d\omega_d\Delta^di^{d-1}T^{(d-2)/2}) \tag{4.5}$$
$$\exp((1+\epsilon)2\mu 2^d\omega_d\Delta^di^{d-1}T^{(d-2)/2}(e^{-z}-1)) \leq \mathbf{E}\exp(-zB_i) \sim \exp\left(\frac{2\mu\,|\,\mathfrak{C}_i\,|}{T}(e^{-z}-1)\right)$$
$$\leq \exp((1-\epsilon)2\mu d\omega_d\Delta^di^{d-1}T^{(d-2)/2}(e^{-z}-1)). \tag{4.6}$$

Now we present the proof of Theorem 1 in eight steps.

Step 1: Let $d=2$ and

$$R_1 = R_1(T) = K(T\log T)^{1/2}.$$

Then by (4.2),

$$\mathbf{P}\{B(R_1)=0\} \leq \exp(-(1-\epsilon)2\mu\pi K^2\log T). \tag{4.7}$$

Consider a particle which is located in $\mathfrak{C}(R_1)$ at time $t=0$ and which has a living offspring at time T. Let $V(0)$ be the location of the considered particle at $t=0$ and let $V(T)$ be the location of an arbitrary, fixed offspring of the considered particle at time T. Then,

$$\mathbf{P}\{\,|\,V(T)-V(0)\,| \geq R_1\} \leq \exp\left(-\frac{K^2}{2}\log T\right).$$

Consequently,

$$\mathbf{P}\{\Lambda(\mathfrak{C}(2R_1),T)=0\} \leq \exp(-(1-\epsilon)2\mu\pi K^2\log T) + \exp\left(-\frac{K^2}{2}\log T\right).$$

Hence, by Borel-Cantelli lemma,

$$\Lambda(\mathfrak{C}(2R_1),T) > 0, \text{ a.s.}$$

for all but finitely many T provided that

$$K > \max\{2^{1/2},(2\mu\pi)^{-1/2}\},$$

Step 2: Let $d=2$ and

$$R_2 = R_2(T) = \frac{T^{1/2}}{f(T)},$$

where

$$f(T)\uparrow\infty, \quad R_2(T)\uparrow\infty.$$

Consider B_i particles located in the ring $\mathcal{C}_i$ at time $t=0$ having offspring living at time T. Let $\lambda_j^{(i)}(A,T)$ $(j=1,2,\ldots,B_i)$ be the number of those offspring of the jth particle which are located in A at time T. Then, by (1.3),

$$\mathbf{P}\{\lambda_j^{(i)}(\mathcal{C}(R_2),T)>0\} \le \frac{\exp\left(-\frac{i^2\Delta^2}{2}\right)}{\log f(T)} \tag{4.8}$$

and by (4.6)

$$\mathbf{P}\left\{\prod_{j=1}^{B_i}\{\lambda_j^{(i)}(\mathcal{C}(R_2),T)=0\}\right\} = \mathbf{EP}\left\{\prod_{j=1}^{B_i}\{\lambda_j^{(i)}(\mathcal{C}(R_2),T)=0\}\mid B_i\right\}$$

$$\ge \mathbf{E}\left(-\frac{\exp\left(-\frac{i^2\Delta^2}{2}\right)}{\log f(T)}\right)^{B_i} = \exp(\nu(e^{-z}-1))$$

where

$$\nu=\nu_i=\frac{2\mu\mid\mathcal{C}_i\mid}{T}\le 8\mu\pi\Delta^2 i \tag{4.9}$$

and

$$e^{-z}=e^{-z_i}=1-\frac{\exp\left(-\frac{i^2\Delta^2}{2}\right)}{\log f(T)}. \tag{4.10}$$

Hence,

$$\mathbf{P}\{\Lambda(\mathcal{C}(R_2),T)=0\}\ge\prod_{i=0}^{\infty}\exp(\nu_i(e^{-z_i}-1))\ge\exp\left(-\sum_{i=0}^{\infty}8\mu\pi i\Delta^2\frac{\exp\left(-\frac{i^2\Delta^2}{2}\right)}{\log f(T)}\right)$$

$$\ge\exp\left(-\frac{8\mu\pi}{\log f(T)}\sum_{i=0}^{\infty}i\Delta^2\exp\left(-\frac{i^2\Delta^2}{2}\right)\right). \tag{4.11}$$

Since, as $\Delta\to\infty$,

$$\sum_{i=0}^{\infty}i\Delta^2\exp\left(-\frac{i^2\Delta^2}{2}\right)\sim\int_0^{\infty}x\ \exp\left(-\frac{x^2}{2}\right)dx=1,$$

we have

$$\mathbf{P}\{\Lambda(\mathcal{C}(R_2),T)=0\}\ge 1-\frac{8\mu\pi}{\log f(T)}\to 1.$$

Which, in turn, implies

$$\Lambda(\mathcal{C}(R_2),T)=0 \quad \text{i.o. a.s.}$$

Step 3: Let $d=2$,

$$R_3=R_3(T)=\log T)^{-1/2+\epsilon}, \quad (\epsilon>0)$$
$$T_k=e^k, \quad \rho_k=T_k^{1/2},$$
$$R_3(k)=R_3(T_k).$$

Then by Lemma 8,

$$\mathbf{P}\{B(\mathcal{C}(\rho_{k+1})-\mathcal{C}(\rho_k),T_{k+1})=0\}\le\exp\left(-(1-\epsilon)\frac{2\mu\pi}{T_{k+1}}(\rho_{k+1}^2-\rho_k^2)\right)$$

$$=\exp\left(-(1-\epsilon)\frac{2\mu\pi(e-1)}{e}\right)<1. \tag{4.12}$$

Consider a particle which is located in $\mathcal{C}(\rho_{k+1})-\mathcal{C}(\rho_k)$ at time $t=0$ and which has a living offspring at time T_{k+1}. Note that by (4.12) with positive probability there exists such a particle. Let $\lambda_3(k)$ be the number of those offspring of the considered particle which are located in $\mathcal{C}(R_3(k+1))$ at time T_{k+1}. Then by (1.3),

$$\mathbf{P}\{\lambda_3(k)>0\}\geq\frac{1-\epsilon}{k+1}e^{-1/2}. \tag{4.13}$$

Since the events $\{\lambda_3(k)>0\}$ are independent we have

$$\Lambda(\mathcal{C}(R_3),T)>0 \quad \text{i.o} \quad \text{a.s.}$$

Step 4: Let $d=2$,

$$R_4=R_4(T)=T^{-1/2}(\log T)^{-1/2-\epsilon}.$$

Then by (4.4),

$$\mathbf{E}B_i\leq(1+\epsilon)8\mu\pi\Delta^2 i. \tag{4.14}$$

Consider the particles located in the ring $\mathcal{C}_i$ at time $t=0$ having offspring living at time T. Let $\lambda_j^{(i)}(A,T)$ $(j=1,2,\ldots,B_i)$ be the number of those offspring of the jth particle which are located in A at time T. Then by (1.4),

$$\mathbf{P}\{\lambda_j^{(i)}(\mathcal{C}(R_4),T)>0\}\leq\frac{R_4^2}{4}\exp\left(-\frac{i^2\Delta^2}{2}\right) \tag{4.15}$$

and

$$\mathbf{P}\left\{\sum_{j=1}^{B_i}\lambda_j^{(i)}(\mathcal{C}(R_4),T)>0\right\}\leq\mathbf{EP}\left\{\sum_{j=1}^{B_i}\lambda_j^{(i)}(\mathcal{C}(R_4),R)>0\mid B_i\right\}$$

$$\leq\mathbf{E}B_i\frac{R_4^2}{4}\exp\left(-\frac{i^2\Delta^2}{2}\right)\leq 2\mu\pi R_4^2\Delta^2 i\exp\left(-\frac{i^2\Delta^2}{2}\right). \tag{4.16}$$

Hence,

$$\mathbf{P}\{\Lambda(\mathcal{C}(R_4),T)>0\}\leq\sum_{i=0}^{\infty}\mathbf{P}\left\{\sum_{j=1}^{B_i}\lambda_j^{(i)}(\mathcal{C}(R_4),T)>0\right\}$$

$$\leq 2\mu\pi R_4^2\sum_{i=0}^{\infty}\Delta^2 i\exp\left(-\frac{i^2\Delta^2}{2}\right). \tag{4.17}$$

Consequently,

$$\Lambda(\mathcal{C}(R_4),T)=0 \quad \text{a.s.}$$

for all but finitely many T.

Step 5: Let $d\geq 3$ and

$$R_1=R_1(T)=K(\log T)^{1/(d-2)}.$$

Define $\lambda_j^{(i)}(\cdot,\cdot)$ as in Step 2. Then by (1.6),

$$\mathbf{P}\{\lambda_j^{(i)}(\mathcal{C}(R_1),T)>0\}\geq\frac{2^d\omega_d(d-2)}{8\omega_d+4(8\pi)^{d/2}(d-2)}\exp\left(-\frac{(i+1)^2\Delta^2}{2}\right)\left(\frac{R_1}{T^{1/2}}\right)^{d-2}$$

and

$$\mathbf{P}\left\{\prod_{j=1}^{B_i}\{\lambda_j^{(i)}(\mathcal{C}(R_1),T)=0\}\right\}=\mathbf{EP}\left\{\prod_{j=1}^{B_i}\{\lambda_j^{(i)}(\mathcal{C}(R_1),T)=0\}\mid B_i\right\}$$

$$\leq\mathbf{E}\left(1-M\exp\left(-\frac{(i+1)^2\Delta^2}{2}\right)\left(\frac{R_1}{T^{1/2}}\right)^{d-2}\right)^{B_i}=\exp(\nu(e^{-z}-1)),$$

where

$$M=\frac{2^d\omega_d(d-2)}{8\omega_d+4(8\pi)^{d/2}(d-2)},$$

$$\nu = \nu_i = \frac{2\mu \mid \mathfrak{C}_i \mid}{T} = \frac{2\mu\omega_d}{T}((i+1)^d - i^d)T^{d/2}\Delta^d \geq \frac{2\mu d\omega_d}{T} i^{d-1} T^{d/2}\Delta^d,$$

$$e^{-z} = e^{-z_i} = 1 - M\exp\left(-\frac{(i+1)^2\Delta^2}{2}\right)\left(\frac{R_1}{T^{1/2}}\right)^{d-2}.$$

Hence,

$$\mathbf{P}\{\Lambda(\mathfrak{C}(R_1), T) = 0\} \leq \prod_{i=0}^{\infty} \exp(\nu_i(e^{-z_i} - 1))$$

$$\leq \exp\left(-\sum_{i=0}^{\infty} \frac{2\mu d\omega_d}{T} T^{d/2} i^{d-1} M\left(\frac{R_L}{T^{1/2}}\right)^{d-2}\Delta^d \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right)\right)$$

$$\leq \exp\left(-2\mu d\omega_d M K^{d-2}\log T\sum_{i=0}^{\infty} i^{d-1}\Delta^d \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right)\right).$$

Choose K such that

$$2\mu d\omega_d M K^{d-2}\sum_{i=0}^{\infty} i^{d-1}\Delta^d \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right) > 1.$$

Then, we have

$$\Lambda(\mathfrak{C}(R_1), T) > 0 \quad \text{a.s.}$$

for all but finitely many T.

Step 6: Let $d \geq 3$ and

$$R_2 = R_2(T) = K(\log\log\log T)^{1/(d-2)}.$$

Now follow the proof of Step 2, with the following modifications: instead of (4.8) by (1.6), we have

$$\mathbf{P}\{\lambda_j^{(i)}(\mathfrak{C}(R_2), T) > 0\} \leq 2^{d-2}(d-2)\exp\left(-\frac{i^2\Delta^2}{2}\right)R_2^{d-2}T^{-(d-2)/2};$$

instead of (4.9), we have

$$\nu = \nu_i = \frac{2\mu \mid \mathfrak{C}_i \mid}{T} \leq 2^{d+1}\mu\omega_d\Delta^d T^{(d-2)/2} i^{d-1};$$

instead of (4.10), we have

$$e^{-z} = e^{-z_i} = 1 - 2^{d-2}(d-2)\exp\left(-\frac{i^2\Delta^2}{2}\right)R_2^{d-2}T^{-(d-2)/2};$$

instead of (4.11), we have

$$\mathbf{P}\{\Lambda(\mathfrak{C}(R_2), T) = 0\} \geq \exp\left(-2^{2d-1}\mu\omega_d(d-2)R_2^{d-2}\sum_{i=0}^{\infty} i^{d-1}\Delta^d \exp\left(-\frac{i^2\Delta^2}{2}\right)\right).$$

Hence, if K is small enough, then

$$\mathbf{P}\{\Lambda(\mathfrak{C}(R_2(T_k)), T_k) = 0\} \geq \frac{1}{k},$$

where

$$T_k = \exp(\exp k^2).$$

Observe that the probability that at least one particle among the ones who are located in $\mathfrak{C}(T_k)$ at time $t = 0$ would live at time T_{k+1}, is equal to

$$1 - \exp\left(-2\omega_d\mu\frac{T_k^d}{T_{k+1}}\right) \sim 2\omega_d\mu\frac{T_k^d}{T_{k+1}} = 2\omega_d\mu\exp(-e^{k^2}(e^{2k+1} - d)).$$

Hence, there is no particle in $\mathfrak{C}(T_k)$ living up to time T_{k+1}.

Consequently, by Borel-Cantelli lemma we have

$$\Lambda(\mathfrak{C}(R_2), T) = 0 \text{ i.o. a.s.}$$

Step 7: Let $d \geq 3$,

$$R_3 = R_3(T) = M(\log T)^{-1/d}.$$

Now follow the proof of Step 3 with the following modifications: instead of (4.12), we have

$$\mathbf{P}\{B(\mathcal{C}(\rho_{k+1}) - \mathcal{C}(\rho_k), T_{k+1}) = 0\} \le \left(-(1-\epsilon)\frac{2\mu\omega_d}{T_{k+1}}(T_{k+1}^{d/2} - T_k^{d/2})\right) \to 0.$$

Consider the $B_k = B(\mathcal{C}(\rho_{k+1}) - \mathcal{C}(\rho_k), T_{k+1})$ particles located in $\mathcal{C}(\rho_{k+1}) - \mathcal{C}(\rho_k)$ at time $t = 0$ having offspring living at time T_{k+1}. Let $\lambda_j^{(k)}(A,T)(j = 1, 2, \ldots, B_k)$ be the number of those offspring of the jth particle which are located in A at time T. Then by (1.7) (with $\beta = 0$), we have

$$\mathbf{P}\{\lambda_j^{(k)}(\mathcal{C}(R_3), T_{k+1}) > 0\} \ge (1-\epsilon)\frac{\omega_d e^{-1/2}}{2(2\pi)^{d/2}M^d}\frac{T_{k+1}}{(T_{k+1})^{d/2}\log T_{k+1}}$$

$$= (1-\epsilon)\frac{\omega_d e^{-1/2}}{2(2\pi)^{d/2}M^d}\frac{R_3^d(T_{k+1})}{T_{k+1}^{(d-2)/2}}$$

and

$$\mathbf{P}\{\Lambda(\mathcal{C}(R_3(T_{k+1})), T_{k+1}) = 0\} = \mathbf{E}\mathbf{P}\{\Lambda(\mathcal{C}(R_3(T_{k+1})), T_{k+1}) = 0 \mid B_k\}$$

$$\le \mathbf{E} \exp(-zB_k) = \exp(\nu(e^{-z} - 1)),$$

where

$$\nu = \nu_k = \frac{2\mu}{T_{k+1}}\omega_d(\rho_{k+1}^d - \rho_k^d)$$

and

$$e^{-z} = e^{-z_k} = 1 - (1-\epsilon)\frac{\omega_d e^{-1/2}}{2(2\pi)^{d/2}M^d}\frac{R_3^d(T_{k+1})}{T_{k+1}^{(d-2)/2}}.$$

Hence,

$$\mathbf{P}\{\Lambda(\mathcal{C}(R_3(T_{k+1})), T_{k+1}) > 0\}$$

$$\ge 1 - \exp\left(-\frac{2\mu}{T_{k+1}}\omega_d(\rho_{k+1}^d - \rho_k^d)(1-\epsilon)\frac{\omega_d e^{-1/2}}{2(2\pi)^{d/2}M^d}\frac{R_3^d(T_{k+1})}{T_{k+1}^{(d-2)/2}}\right) \ge \frac{1}{k}$$

if M is small enough. Consequently,

$$\Lambda(\mathcal{C}(R_3(T)), T) > 0 \quad \text{i.o.} \quad \text{a.s.}$$

Step 8: Let $d \ge 3$ and

$$R_4 = R_4(T) = T^{-1/d}(\log T)^{-1/d-\epsilon}.$$

Now follow the proof of Step 4 with the following modifications: instead of (4.14) we have

$$\mathbf{E}B_i \le 2^{d+1}\mu\omega_d\Delta^d i^{d-1}T^{(d-2)/2},$$

instead of (4.15), by (1.7) we have

$$\mathbf{P}\{\lambda_j^{(i)}(\mathcal{C}(R_4), T) > 0\} \le \frac{\omega_d \exp\left(-\frac{i^2\Delta^2}{2}\right)}{2(2\pi)^{d/2}}R_4^d T^{-(d-2)/2};$$

instead of (4.16), we have

$$\mathbf{P}\left\{\sum_{j=1}^{B_i}\lambda_j^{(i)}(\mathcal{C}(R_4), T) > 0\right\} \le \mu\omega_d^2\left(\frac{2}{\pi}\right)^{d/2} i^{d-1}\Delta^d \exp\left(-\frac{i^2\Delta^2}{2}\right)R_4^d;$$

instead of (4.17), we have

$$\mathbf{P}\{\Lambda(\mathcal{C}(R_4),T)>0\}\leq \mu\omega_d^2\Big(\frac{2}{\pi}\Big)^{d/2}R_4^d\sum_{i=0}^{\infty}i^{d-1}\Delta^d\exp\left(-\frac{i^2\Delta^2}{2}\right).$$

Consequently,

$$\Lambda(\mathcal{C}(R_4),T)=0 \quad \text{a.s.}$$

for all but finitely many T.

References

[1] Révész, P., *Random Walks of Infinitely Many Particles*, World Scientific, Singapore 1994.