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A CHARACTERIZATION AND MOVING AVERAGE REPRESENTATION FOR STABLE HARMONIZABLE PROCESSES

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ABSTRACT

In this paper we provide a characterization for symmetric α-stable harmonizable processes for $1 < \alpha \leq 2$. We also deal with the problem of obtaining a moving average representation for stable harmonizable processes discussed by Cambanis and Soltani [3], Makegan and Mandrekar [9], and Cambanis and Houdre [2]. More precisely, we prove that if Z is an independently scattered countable additive set function on the Borel field with values in a Banach space of jointly symmetric α-stable random variables, $1 < \alpha \leq 2$, then there is a function $k \in L^2(\lambda)$ (λ is the Lebesgue measure) and a certain symmetric-α-stable random measure Y for which

$$\int_{-\infty}^{\infty} e^{itx} dZ(x) = \int_{-\infty}^{\infty} k(t-s) dY(s), \quad t \in \mathbf{R},$$

if and only if $Z(A) = 0$ whenever $\lambda(A) = 0$. Our method is to view $S\alpha S$ processes with parameter space R as $S\alpha S$ processes whose parameter spaces are certain L^{β} spaces.

Key words: Stable Processes, Harmonizable Processes, Spectral Representations, Moving Average Representation, Grothendieck Measure.

AMS (MOS) subject classifications: 60G20, 60G25, 60G57, 60H05, 28B05.

1. Introduction

It is well known that certain stationary Gaussian processes can be represented as the Fourier transform of independently scattered Gaussian random measures and as moving averages of Gaussian motions as well. The work of Schilder [16] enables one to define the Fourier transform of certain stable random measures and the moving average of a stable motion separately. A natural question that arises is to investigate the connection between these two types of stable processes as the moving average representation has its special importance in time domain analysis. It was demonstrated in [3] that the situation in the stable case is rather complicated. It is not possible to obtain a result, in the stable case, similar to the one that is available in the Gaussian case. It is proved in [9] and [2] that the moving averages of the stable motion are not Fourier transforms of stable random measures. In the summability sense, discussed in [2], Cambanis and Houdre provided a connection between the Fourier transforms and moving averages of different stable random measures.

In the present work, we deal with the same problem. We prove in Theorem 3.2 that every strongly harmonizable symmetric α-stable ($S\alpha S$) process, $1 < \alpha \leq 2$,

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda), \quad t \in \mathbf{R},$$

has a moving average representation in the strong sense (not summability sense), in the case that the random measure Z is absolutely continuous with respect to the Lebesgue measure. Our method is to associate to every harmonizable (or moving average) process a unique continuous linear mapping on a certain L^{β} space with values in a Banach space of jointly stable random variables. The notion of (α, β) boundedness and some of the results given by Houdre in [6] will be used in our work.

Early results on moving average representation, in the Gaussian case, are due to [7]. For more recent research in prediction theory of stable or Gaussian processes, see [10-13] and [15].

The paper is organized as follows. In Section 2, we present notations and preliminaries. In Section 3, we characterize harmonizable $S\alpha S$ processes and present the main results of the article which are Theorems 3.1, 3.2 and Remark 3.1. Theorem 3.2 brings into sight, an important class of $S\alpha S$ processes in the time domain that, to the best of our knowledge, has not been treated before.

2. Notations and Preliminaries

In this section we adopt some of the notions and results in [6]. Except for Theorem 2.2 which is new, we do not give proofs for the rest of the material in this section. Readers can easily derive the proofs by applying the corresponding techniques presented in [6]. In this work we only consider the case for which $1 < \alpha \leq 2$.

The Schilder's norm of a $S\alpha S$ random variable X is denoted by $\| X \|$. The space of all jointly $S\alpha S$ random variables equipped with $\| \cdot \|$ is a weakly complete Banach space which is denoted by $(\mathcal{S}, \| \cdot \|)$. The convergence in $(\mathcal{S}, \| \cdot \|)$ is equivalent to the convergence in probability. A process $\{\Phi(i): i \in I\}$ is $S\alpha S$ if $\Phi(i) \in \mathcal{S}$ for each i in the index set I. If an index set I is equipped with a topology τ, then we call the process Φ "continuous" if $\Phi: (I, \tau) \rightarrow (\mathcal{S}, \| \cdot \|)$ is continuous. The Lebesgue measure is denoted by λ and for $f \in L^p(\lambda)$, $p \geq 1$, $\hat{f}$ and $\check{f}$ stand for the Fourier transform and the inverse Fourier transform of f respectively, whenever they are well defined.

Let $\mathcal{B}_0(\mathbf{R})$ denote those elements of the Borel field $\mathcal{B}(\mathbb{R})$ that are bounded. Let μ be a regular measure on $\mathcal{B}(\mathbf{R})$. By $\mathbf{M}^{\beta}(\mu)$, $1 \leq \beta < \infty$, we mean the space of all random measures Z: $\mathcal{B}_0(\mathbf{R}) \rightarrow \mathcal{S}$ such that for each Z, there is some constant C for which

$$\| \sum_{i=1}^{n} a_i Z(A_i) \| \leq C \{ \sum_{i=1}^{n} | a_i |^{\beta} \mu(A_i) \}^{1/\beta} \tag{2.1}$$

for any pairwise disjoint sets $A_1, A_2, \ldots$ in $\mathcal{B}_0(\mathbf{R})$ and for any real numbers $a_1, a_2, \ldots$.

Note that every element of $\mathbf{M}^{\beta}(\mu)$ is a $S\alpha S$-valued random measure on $\mathcal{B}_0(\mathbf{R})$ with continuity property (2.1).

Let μ be a regular measure and let $\Phi: L^{\beta}(\mu) \rightarrow \mathcal{S}$ be a continuous linear mapping. Clearly $\Phi = \{\Phi(f), f \in L^{\beta}(\mu)\}$ is a continuous $S\alpha S$ process for which

$$\| \Phi(f) \| \leq C \| f \|_{L^\beta(\mu)}$$

for any $f \in L^\beta(\mu)$, where C is a constant not depending on f. The space of such processes is denoted by $\mathcal{M}^\beta(\mu)$.

For $Z \in \mathbf{M}^\beta(\mu)$ and $f \in L^\beta(\mu)$, $\int f dZ$ is well defined. It is a $S\alpha S$ random variable for which

$$\| \int f dZ \| \leq C \| f \|_{L^\beta(\mu)}, \quad f \in L^\beta(\mu).$$

If we let

$$\psi(f) = \int f dZ, \quad f \in L^\beta(\mu),$$

then, $\psi \in \mathcal{M}^\beta(\mu)$ and in this case we denote ψ by ψ_Z and Z by Z_ψ respectively.

An element $Z \in \mathbf{M}^\beta(\mu)$ is said to be independently scattered, I.S. in short, if $Z(A_1), \ldots, Z(A_n)$ are independent random variables whenever $A_1, \ldots, A_n$ are pairwise disjoint elements of $\mathcal{B}_0(\mathbf{R})$. In this case, we have

$$\| \int g dZ \| = \| g \|_{L^\alpha(\nu)}, \quad g \in L^\alpha(\nu),$$

where ν is the control measure of Z; i.e.,

$$\nu(A) = \| Z(A) \|^\alpha, \quad A \in \mathcal{B}_0(\mathbf{R}).$$

An element $\psi \in \mathcal{M}^\beta(\mu)$ has independent values at each point of $L^\beta(\mu)$ if $\psi(f_1)$, $\psi(f_2)$ are independent random variables whenever $f_1, f_2 \in L^\beta(\mu)$ and $f_1 f_2 = 0$ a.e. μ; see [5] part 4.

Theorem 2.1: *Let ψ be a $S\alpha S$ process in $\mathcal{M}^\beta(\mu)$, $1 \leq \beta < \infty$. Then there is a unique Z in $\mathbf{M}^\beta(\mu)$ for which*

$$\psi(f) = \int f dZ, \quad f \in L^\beta(\mu).$$

In the following, we present an elementary but very useful lemma.

Lemma 2.1: *(i) Every element of $\mathbf{M}^\beta(\mu)$ is a countable additive $\mathcal{G}$-valued measure on the δ-ring $\mathcal{B}_\mu(\mathbf{R}) = \{A \in \mathcal{B}(\mathbf{R}): \mu(A) < \infty\}$.*

(ii) Let μ be a finite measure on $\mathcal{B}(\mathbf{R})$. Then $\mathbf{M}^\beta(\mu) \subset \mathbf{M}^\gamma(\mu)$ and $\mathcal{M}^\beta(\mu) \subset \mathcal{M}^\gamma(\mu)$ whenever $1 \leq \beta \leq \gamma < \infty$.

(iii) Suppose that Z is I.S. and $Z \in \mathbf{M}^\beta(\mu)$. Then for some constant $c > 0$

$$\nu(A) \leq c^\alpha(\mu(A))^{\alpha/\beta}, \quad A \in \mathcal{B}_\mu(\mathbf{R}), \tag{2.2}$$

where ν is the control measure of Z. Moreover, if the function $f \in L^\beta(\mu)$, then $f \in L^\alpha(\nu)$.

(iv) Let $1 \leq \beta < \alpha$ and $Z \in \mathcal{M}^\beta(\mu)$. If Z is I.S., then $Z \equiv 0$.

The following theorem is the new result of this section. It has significant applications in the subsequent section, where we show that certain harmonizable processes are moving averages.

Theorem 2.2: *Suppose μ is a regular measure for which $\mu \ll \lambda$. If $\psi \in \mathcal{M}^\beta(\mu)$, then there is some $M \in \mathbf{M}^\beta(\lambda)$ such that*

$$\psi(f) = \int fg dM, \quad f \in L^\beta(\mu), \tag{2.3}$$

where $1 \leq \beta < \infty$ and $\frac{d\mu}{d\lambda} = |g|^\beta$. In addition, M is I.S. whenever ψ has independent values at each point of $L^\beta(\mu)$.

Proof: From Theorem 2.1, $\psi(f) = \int f dZ_\psi$, $f \in L^\beta(\mu)$. We define $M(A) = \int_A g^{-1} I_{[g \neq 0]} dZ_\psi$, $A \in \mathcal{B}_0(\mathbf{R})$. Clearly, M is a well defined random measure on $\mathcal{B}_0(\mathbf{R})$ which belongs to $\mathbf{M}^\beta(\lambda)$. It follows that

$$\int k dM = \int k g^{-1} I_{[g \neq 0]} dZ_\psi, \quad k \in L^\beta(\lambda). \tag{2.4}$$

Now $fg \in L^{\beta}(\lambda)$ whenever $f \in L^{\beta}(\mu)$ and it follows from (2.4) that

$$\int fgdM = \int fI_{[g \neq 0]}dZ_{\psi} = \int fdZ_{\psi} = \psi(f)$$

for all $f \in L^{\beta}(\mu)$, since $Z_{\psi}([g=0]) = 0$. The proof is complete.

The following spectral type theorem also has useful applications.

Theorem 2.3: *Let $\psi \in \mathcal{M}^{\beta}(\mu)$. Then ψ has independent values at each point if and only if Z_{ψ} is I.S. In this case, there is a unique $h \geq 0$ and an I.S. random measure $W \in \mathbf{M}^{\alpha}(\mu)$ with control measure $m = \mu I_{[h \neq 0]}$ for which*

$$\psi(f) = \int fh^{1/\alpha}dW, \quad f \in L^{\beta}(\mu). \tag{2.4}$$

Moreover, $h \in L^{\beta/(\beta-\alpha)}(\mu)$ if $\alpha < \beta < \infty$, $h \in L^{\infty}(\mu)$ if $\alpha = \beta$, and $h = 0$ if $1 \leq \beta < \alpha$.

We now introduce the following subspaces of $(\mathfrak{I}, \|\ \|)$. For $\psi \in \mathcal{M}^{\beta}(\mu)$, let

$$\mathcal{A}_{\psi} = \text{closure } \{\psi(f), f \in L^{\beta}(\mu)\},$$

and for a classical $S\alpha S$ process $X = \{X(t), t \in \mathbf{R}\}$, let

$$\mathcal{A}_X = \text{span closure } \{X(t), t \in \mathbf{R}\}.$$

For a process Φ on $L^2(\lambda)$, i.e., $\Phi \in \mathcal{M}^2(\lambda)$, the Fourier transform of Φ is defined by

$$\widehat{\Phi}(f) = \Phi(\widehat{f}), \quad f \in L^2(\lambda).$$

Clearly, $\widehat{\Phi} \in \mathcal{M}^2(\lambda)$, because $\|\widehat{f}\|_{L^2(\lambda)} = (2\pi)^{1/2} \|f\|_{L^2(\lambda)}$.

3. Harmonizables are Moving Averages

In this section, we show that a rather wide class of $S\alpha S$-harmonizable processes are contained in a certain class of moving average processes. Let us first introduce the notions of harmonizability and moving averageness in detail. By a classical $S\alpha S$ process $X = \{X(t): t \in \mathbf{R}\}$, we mean a continuous function $X: R \rightarrow \mathfrak{I}$.

Definition 3.1: A classical $S\alpha S$ process $X = \{X(t): t \in R\}$ is called *harmonizable* (*strongly harmonizable*) if

$$X(t) = \int_{-\infty}^{\infty} e^{itx}dZ(x), \quad t \in \mathbf{R}, \tag{3.1}$$

where Z is a countable additive (an I.S. countable additive) set function of $\mathfrak{B}(R)$ into $\mathfrak{I}$.

Definition 3.2: A classical $S\alpha S$ process $\{X(t): t \in \mathbf{R}\}$ is called a *moving average of a countable additive set function Y*, of $\mathfrak{B}_0(\mathbf{R})$ into $\mathfrak{I}$, if

$$X(t) = \int_{-\infty}^{\infty} k(t-s)dY(s), \quad t \in \mathbf{R}, \tag{3.2}$$

with a function k for which the integral in (3.2) makes sense in $(\mathfrak{I}, \|\cdot\|)$. If Y in (3.2) is the stable motion, then we say that $X(t)$ is a strongly moving average process.

The question whether a stable harmonizable process is a moving average of a certain stable measure has inspired some deep results in the literature. In Theorem 3.1 in [3] the following three assertions are considered.

1. Two classes of strongly harmonizable processes and strongly moving average processes are disjoint.

2. Strongly moving average processes are harmonizable (not necessarily strongly).

3. Strongly harmonizable processes are moving average.

Assertion 1 and its proof as presented in [3] are correct. Assertion 2 is not correct, see [9] and [2]. Also the proof presented in [3] for assertion 3 is not correct and this brought some doubt on the validity of assertion 3, see [2]. In Theorem 3.2 we prove that assertion 3 is correct.

The following theorem gives a characterization for harmonizable processes.

Theorem 3.1: *Let $\{X(t): t \in \mathbf{R}\}$ be a harmonizable process. Then there is a finite Borel measure μ and a generalized process $\psi \in \mathcal{M}^2(\mu)$ for which $X(t) = \psi(e^{it\cdot})$, $t \in \mathbf{R}$.*

Proof: Note that if Z is a countable additive set function of $\mathfrak{B}(\mathbf{R})$ into $\mathfrak{S}$, i.e., $\| Z(\bigcup_{i=0}^{\infty} A_i) - \sum_{i=1}^{n} Z(A_i) \| \to 0$ as $n \to \infty$, for every sequence of disjoint sets in $\mathfrak{B}(\mathbf{R})$, then the integral $\int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda)$ is well defined in $(\mathfrak{S}, \| \cdot \|)$; see [4] part I, IV.10. Moreover,

$$\| \int f dZ \| \leq C \| f \|_{\infty}, \qquad f \in L^{\infty}(\mu), \tag{3.3}$$

where C is a constant number and μ is a finite measure on $\mathfrak{B}(\mathbf{R})$ (see [4] part I, I.V.10.5, lemma 5) satisfying $\mu(A) = 0$ if and only if $Z(A)$ is degenerate at zero. Now, by Theorem 5.1 in [6] there is a finite Borel measure h such that

$$\| \int f dZ \| \leq C_1 \{ \int | f |^2 dh \}^{\frac{1}{2}}, \; f \in C_0(\mathfrak{R}).$$

Clearly, $Z \ll h$ (we can replace h by $\mu + h$). Thus by Lusin's Theorem [14] and Theorem 10, page 328 [4] we infer that

$$\| \int f dZ \| \leq C_1 \{ \int | f |^2 dh \}^{\frac{1}{2}}, \quad f \in L^{\infty}(h).$$

Now let $f = \sum_{i=1}^{n} a_i A_{A_i}$ be a simple function. Then,

$$\| \sum_{i=1}^{n} a_i Z(A_i) \| \leq C' (\sum_{i=1}^{n} | a_i |^2 h(A_i))^{\frac{1}{2}},$$

i.e, $Z \in \mathbf{M}^2(h)$. Thus, $X(t) = \psi(e^{it\cdot})$, $t \in \mathbf{R}$, for some $\psi \in \mathcal{M}^2(h)$ (precisely: $\psi(f) = \int f dZ$, $f \in L^2(h)$). The proof is complete.

We call the measure μ given in Theorem 3.1 a *Grothendieck measure* of the process $X(t)$.

The following theorem is the main result of this article.

Theorem 3.2: *Let $\{X(t): t \in \mathbf{R}\}$ be a classical $S\alpha S$ process. Then*

(*i*) *if $\{X(t): t \in \mathbf{R}\}$ is a strongly harmonizable process for which its control measure $\nu \ll \lambda$, then*

$$X(t) = \int_{-\infty}^{\infty} k(t-s) dY(s), \quad t \in \mathbf{R},$$

where $Y \in M^2(\lambda)$ with $\widehat{Y}$ being independently scattered and $k \in L^2(\lambda)$ with $| \widehat{k} |^2 = d\nu / d\lambda$. Moreover, $\mathcal{A}_X = \mathcal{A}_Y$ if $\widehat{k} \neq 0$ a.e., λ;

(*ii*) *if $\{X(t): t \in \mathbf{R}\}$ is a harmonizable process given by* (3.1) *and has a Grothendieck measure $\mu \ll \lambda$, then the conclusion of part* (*i*) *holds except that $\widehat{Y}$ may not be independently scattered and $| \widehat{k} |^2 = d\mu / d\lambda$.*

Parts (*i*) and (*ii*) of Theorem 3.2 follow immediately from the propositions 3.1 and 3.2, given below, respectively. Only define ψ by $\psi(e^{it\cdot}) = X(t)$.

Proposition 3.1: *Let a generalized $S\alpha S$ process ψ belong to $\mathcal{M}^{\beta}(\mu)$, where μ is a finite Borel* *measure $\mu \ll \lambda$ and $1 \le \beta < \infty$. Suppose that ψ has independent values at each point of $L^{\beta}(\mu)$. Then there is a process Φ in $\mathcal{M}^2(\lambda)$ and a function $k \in L^2(\lambda)$ for which:*

(*i*) $\widehat{\Phi}$ *has independent values at each point of $L^2(\lambda)$,*

(*ii*) $\psi(e^{it\,\cdot}) = \Phi(k(t-\cdot))$, $t \in \mathbf{R}$,

(*iii*) *if $\widehat{k} \neq 0$, a.e., λ, then $\mathcal{A}_{\psi} = \mathcal{A}_X = \mathcal{A}_{\Phi}$, where $X(t) = \psi(e^{it\,\cdot})$, $t \in \mathbf{R}$.*

Proof: Let $\psi \in \mathcal{M}^{\beta}(\mu)$, where μ is a finite Borel measure. By Theorem 2.3, there is an I.S. $Z_{\psi} \in \mathbf{M}^{\beta}(\mu)$ for which

$$\psi(f) = \int f dZ_{\psi}, \quad f \in L^{\beta}(\mu). \tag{3.4}$$

Since μ is a finite measure it follows from Lemma 2.1(*iii*) that ν is a finite measure and $\nu \ll \mu$. Also note that $Z_{\psi} \in \mathbf{M}^{\alpha}(\nu)$. Now let

$$G(g) = \int g dZ_{\psi}, \quad g \in L^{\alpha}(\nu). \tag{3.5}$$

Thus $G \in \mathcal{M}^{\alpha}(\nu)$, and by Lemma 2.1(*ii*), $G \in \mathcal{M}^2(\nu)$. Hence by Theorem 2.2, there is some I.S. random measure $M \in \mathbf{M}^2(\lambda)$ for which

$$G(f) = \int f h^* dM, \quad f \in L^2(\nu), \tag{3.6}$$

where $|h^*|^2 = \frac{d\nu}{d\lambda} \in L^1(\lambda)$ and $h^*(x) = h(-x)$. It follows from (3.4), (3.5) and (3.6) that

$$\psi(e^{it\,\cdot}) = \int e^{its} h^*(s) dM(s)$$

$$= \int \check{h}(t-x) d\widehat{M}(x),$$

where the last equality is the Parseval's type formula which is given by Theorem 4.1 in [6]; (see also the paragraph after Theorem 4.3 in [6]).

Now let

$$\Phi(g) = \int g d\widehat{M}, \quad g \in L^2(\lambda),$$

then $\widehat{\Phi}$ has independent values at each point and

$$\psi(e^{it\,\cdot}) = \Phi(\check{h}(t-\cdot)), \quad t \in \mathbf{R},$$

given (*i*) and (*ii*). For (*iii*), note that it follows from (*ii*) that

$$\mathcal{A}_X = \overline{sp}\{\Phi(k(t-\cdot)), \ t \in \mathbf{R}\}.$$

But since $k \in L^2(\lambda)$ under the assumption that $\widehat{k} \neq 0$, a.e., λ we obtain

$$\overline{sp}\{k(t-\cdot), t \in \mathbf{R}\} = L^2(\lambda),$$

see [8]. It also follows from the continuity of Φ that

$$\overline{sp}\{\Phi(k(t-\cdot)), t \in \mathbf{R}\} = \overline{sp}\{\Phi(g), g \in L^2(\lambda)\}.$$

Thus, $\mathcal{A}_X = \mathcal{A}_{\psi} = \mathcal{A}_{\Phi}$. The proof is complete.

Proposition 3.2: *Let μ be a finite measure and let $\psi \in \mathcal{M}^{\beta}(\mu)$. Suppose μ is absolutely continuous with respect to the Lebesgue measure λ. Then there is a function $k \in L^2(\lambda)$ and a stable process $\Phi \in \mathcal{M}^2(\lambda)$ for which*

(*i*) $\psi(e^{it\,\cdot}) = \Phi(k(t-\cdot))$, $t \in \mathbf{R}$, (*ii*) *if* $\widehat{k} \neq 0$ *a.e.* λ, *then* $\mathcal{A}_\psi = \mathcal{A}_X = \mathcal{A}_\Phi$, *where* $X(t) = \psi(e^{it\,\cdot})$, $t \in \mathbf{R}$.

Proof: Let $\psi \in \mathcal{M}^\beta(\mu)$, then $\psi(f) = \int f dZ$, for $f \in L^\beta(\mu)$, where $Z \in \mathbf{M}^\beta(\mu)$.

For $1 \leq \beta \leq 2$, it follows that $\psi \in \mathcal{M}^\beta(\mu)$ by Lemma 2.1(*ii*), and therefore, as shown in the proof of Proposition 3.1, the conclusions (*i*) and (*ii*) are satisfied.

For $\beta > 2$ we use a version of Theorem 5.2 in [6] (see the discussion on page 183 in [6]) and write

$$\| \psi(f) \| = \| \int f dZ \| \leq \Big(\int | f |^2 h d\mu \Big)^{\frac{1}{2}}, \quad f \in L^\beta(\mu),$$

where $h \in L^{\beta/(\beta-2)}(\mu)$ (put γ in place of α, $1 < \gamma < \alpha$, in Theorem 5.1 [6]). Thus $\psi \in \mathcal{M}^2(\nu)$, where $d\nu = h d\mu$. Note that Hölder's inequality provides that $\nu(A) \leq K(\mu(A))^{2/\beta}$, $A \in \mathfrak{B}(\mathbf{R})$, where $K = \| h \|_{L^{\beta/(\beta-2)}(\mu)}$. Therefore, ν is a finite measure and $\nu \ll \lambda$. Now again, we are in a position to apply Theorem 2.2 and conclude the result. The proof is complete.

The following theorem provides sufficient conditions under which a moving average process is harmonizable.

Theorem 3.3: *Let* $\Phi \in \mathcal{M}^\beta(\lambda)$, $2 \leq \beta < \infty$. *Suppose there is a function* $k \in L^\beta(\lambda)$ *for which* $k = \check{g}$ *for some* $g \in L^{\beta'}(\lambda)$, *where* $\frac{1}{\beta} + \frac{1}{\beta'} = 1$. *Then,*

(*i*) *there is a finite Borel measure* μ, $\mu \ll \lambda$, *and a process* $\psi \in \mathcal{M}^{\beta'}(\mu)$ *for which*

$$\Phi(k(t - \cdot)) = \psi(e^{it\,\cdot}), \quad t \in \mathbf{R},$$

(*ii*) *there is a process* $\Phi_1 \in \mathcal{M}^2(\lambda)$ *and a* $k_1 \in L^2(\lambda)$ *with*

$$\Phi(k(t - \cdot)) = \Phi_1(k_1(t - \cdot)), \quad t \in \mathbf{R}.$$

Proof: For (*i*), let $M = Z_\Phi$. Then

$$\Phi(k(t - \cdot)) = \int k(t-s) dM(s)$$

$$= \int \check{g}(t-s) dM(s) = \int e^{-itx} g(x) d\widehat{M}(x)$$

$$= \int e^{itx} g^*(x) d\widehat{M}^*(x) = \int e^{itx} dZ(x),$$

where the third equality is by Theorem 4.2 in [6] and $Z(A) = \int_A g^* d\widehat{M}^*$, $\widehat{M}^*(B) = \widehat{M}(-B)$, for all $B \in \mathfrak{B}_0(\mathbf{R})$, $A \in \mathfrak{B}(\mathbf{R})$. Therefore, $Z \in \mathbf{M}^{\beta'}(\mu)$, where $\mu(A) = \int_A | g |^{\beta'} d\lambda$, $A \in \mathfrak{B}(\mathbf{R})$.

For (*ii*), apply Proposition 3.2. The proof is complete.

Remark 3.1: It is interesting to note that if

$$\int e^{its} dZ(s) = \int k(t-x) dY(x), \quad t \in \mathbf{R}, \tag{3.7}$$

for a $\mathfrak{H}$-valued Borel measure Z, $k \in L^2(\lambda)$ and $Y \in M^2(\lambda)$, then $Z \ll \lambda$. To see this, let $k = \check{g}$, $g = \widehat{k} \in L^2(\lambda)$. Then, by Theorem 4.2 in [6],

$$\int k(t-x) dY(x) = \int e^{its} g^*(s) d\widehat{Y}^*(s)$$

for all $t \in \mathbf{R}$. Now by letting $Z_1(A) = \int_A g^* d\widehat{Y}^*$ for $A \in \mathfrak{B}(\mathbf{R})$, and $\mu(B) = \int_B | g^* |^2 d\lambda$ for $B \in \mathfrak{B}(\mathbf{R})$, we obtain that

$$\int e^{its} dZ(s) = \int e^{its} dZ_1(s), \quad t \in \mathbf{R}.$$

Thus $Z = Z_1$. But $Z_1 \ll \lambda$, because $Z_1 \in \mathbf{M}^2(\mu)$ and $\mu \ll \lambda$. Note that μ is, indeed, a Grothendieck measure of $\mathbf{Z}$.

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