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INVARIANT PROBABILITIES FOR FELLER-MARKOV CHAINS[1]

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ABSTRACT

We give necessary and sufficient conditions for the existence of invariant probability measures for Markov chains that satisfy the Feller property.

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1. Introduction

The existence of invariant probabilities for Markov chains is an important issue for studying the long-term behavior of the chains and also for analyzing Markov control processes under the long-run expected average cost criterion. Inspired by the latter control problems, we present in this paper, two **necessary and sufficient** conditions for the existence of invariant probabilities for Markov chains that satisfy the Feller property. Our study extends previous results using stronger assumptions, such as the **strong** Feller property in Beneš [1], nondegeneracy assumptions (see condition (2) in Beneš [2]), and a uniform countable-additivity hypothesis in Liu and Susko [8]. As can be seen in the references, it is also worth noting that there are many reported results providing (only) **sufficient** conditions for the existence of invariant measures; in contrast however, our conditions are also necessary.

The setting for this paper is specified in Section 2, and our main result is presented in Section 3.

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2. Notation and Definitions

Let X be a σ-compact metric space, and let $\{x_t,\ t=0,1,\ldots\}$ be an X-valued Markov chain with time-homogeneous kernel P, i.e.,

$$P(B \mid x) = \text{Prob}(x_{t+1} \in B \mid x_t = x) \forall t = 0,1,\ldots, x \in X,\ B \in \mathfrak{B}(X),$$

where $\mathfrak{B}(X)$ denotes the Borel σ-algebra of X. A probability measure (p.m.) μ on $\mathfrak{B}(X)$ is said to be *invariant for P* if

$$\mu(B) = \int_X P(B \mid x)\mu(dx) \qquad \forall B \in \mathfrak{B}(X).$$

Here, we give necessary and sufficient conditions for the existence of invariant p.m.'s when P satisfies the *Feller property*:

$$x \rightarrow \int u(y)P(dy \mid x) \text{ is in } C(X) \text{ whenever } u \in C(X), \tag{1}$$

where $C(X)$ denotes the space of all bounded and continuous functions on X. Our conditions use a *moment* function, defined as follows.

Definition: A nonnegative Borel-measurable function v on X is said to be a *moment* if, as $n \rightarrow \infty$, $\inf\{v(x) \mid x \notin K_n\} \uparrow \infty$ for some sequence of compact sets $K_n \uparrow X$.

Moment functions have been used by several authors to study the existence of invariant measures for Markov processes (e.g., see Beneš [1, 2], Hernández-Lerma [6], Liu and Susko [8], and Meyn and Tweedie [9]). The key feature used in these studies is the following (easily proved) fact.

Lemma: *Let M be a family of p.m.'s on X. If there exists a moment v on X such that $\sup_{\mu \in M} \int v d\mu < \infty$, then M is tight, i.e., for every positive ϵ there exists a compact set K such that $\mu(K) > 1 - \epsilon$ for all $\mu \in M$.*

Therefore by Prohorov's Theorem [3, 9], the family M in the lemma is relatively compact, i.e., every sequence in M contains a weakly convergent subsequence.

Our theorem below (see Section 3) extends a result by Beneš [2] where our conditions (a) and (b) are new and, most importantly, we do *not* require Beneš' "nondegeneracy" condition, according to which

$$\lim_{x \rightarrow \infty} P^t(K \mid x) = 0 \text{ for } t = 1, 2, \ldots, K \text{ compact},$$

with $P^t(\cdot \mid x)$ being the t-step transition probability given the initial state $x_0 = x$. This condition excludes important classes of ergodic Markov chains, such as those that have a "minorant"; see e.g., Dynkin and Yushkevich [5], or condition R1 in Hernández-Lerma *et al.* [7]. See also Remarks 2 and 3 (Section 3) for additional comments on related results.

3. The Theorem

If ν is a p.m. on X, $E_\nu(\cdot)$ stands for the expectation given the "initial distribution" ν.

Theorem: *If P satisfies the Feller property, then the following conditions (a), (b), and (c) are equivalent:*

(*a*) *There exists a p.m.* ν *and a moment* v *such that*

$$\limsup_{n\to\infty} J_n(\nu) < \infty,$$

where $J_n(\nu) := n^{-1}E_\nu\left[\sum_{t=0}^{n-1} v(x_t)\right]$;

(*b*) *There exists a p.m.* ν *and a moment* v *such that*

$$\limsup_{\alpha\uparrow 1} V_\alpha(\nu) < \infty,$$

where $V_\alpha(\nu) := (1-\alpha)E_\nu\left[\sum_{t=0}^{\infty} \alpha^t v(x_t)\right]$;

(*c*) *There exists an invariant probability for* P.

Proof: We will show that (*a*)⇒(*b*)⇒(*c*)⇒(*a*).

(*a*) *implies* (*b*): This follows from a well-known Abelian theorem (e.g., see Sznajder and Filar [11], Theorem 2.2), which states that

$$\limsup_{\alpha\uparrow 1} V_\alpha(\nu) \le \limsup_{n\to\infty} J_n(\nu).$$

(Since a direct proof that (*a*) implies (*c*) is surprisingly simple, it will also be included here; see Remark 1 below.)

(*b*) *implies* (*c*): Suppose that (*b*) holds and for each $\alpha \in (0,1)$, let μ_α be the probability measure on X defined as

$$\mu_\alpha(B) := (1-\alpha)\sum_{t=0}^{\infty} \alpha^t \int_X P^t(B \mid z)\nu(dz), \quad B \in \mathfrak{B}(X).$$

Then we may write $V_\alpha(\nu)$ as $V_\alpha(\nu) = \int v d\mu_\alpha$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\alpha_n \uparrow 1$ and, by (*b*),

$$\limsup_{\alpha\uparrow 1} V_\alpha(\nu) = \lim_{n\to\infty} V_{\alpha_n}(\nu) = \lim_{n\to\infty} \int v d\mu_{\alpha_n} < \infty.$$

By the lemma in Section 2, $\{\mu_{\alpha_n}\}$ is tight and therefore, by Prohorov's Theorem, $\{\mu_{\alpha_n}\}$ contains a weakly convergent subsequence, which we denote by $\{\mu_{\alpha_n}\}$ again; that is, there exists a probability measure μ on X such that

$$\lim_n \int u d\mu_{\alpha_n} = \int u d\mu \ \forall u \in C(X).$$

We claim that μ is invariant for P.

To see this, first note that by the Markov property, we may write μ_α as

$$\mu_\alpha(B) = (1-\alpha)\nu(B) + \alpha \int P(B \mid x)\mu_\alpha(dx) \quad \forall \alpha \in (0,1), \quad B \in \mathfrak{B}(X).$$

Hence, for any $u \in C(X)$,

$$\int u d\mu_\alpha = (1-\alpha)\int u(x)\nu(dx) + \alpha \int \int u(y)P(dy \mid x)\mu_\alpha(dx),$$

and furthermore, note that by the Feller property (1), $\int u(y)P(dy \mid \cdot)$ is in $C(X)$. Thus, replacing α by α_n and letting $n \to \infty$, we obtain

$$\int u d\mu = \int \int u(y)P(dy \mid x)\mu(dx). \tag{2}$$

Finally, since $u \in C(X)$ was arbitrary, we conclude from (2) that μ is invariant for P.

(*c*) *implies* (*a*): Let ν be an invariant probability for P, and let $\{K_n\}$ be an increasing sequence of compact sets such that $K_n \uparrow X$ and $\nu(K_{n+1} - K_n) < 1/n^3$, $n = 1, 2, \ldots$. (Here we have used the fact that every p.m. on a σ-compact metric space is tight; see [3], p. 9.) Define a function $v(\cdot) := 0$ on K_1 and $v(x) := n$ for $x \in K_{n+1} - K_n$, $n \geq 1$. Then v is a moment and

$$\limsup_{n \to \infty} J_n(\nu) = \int v(x)\nu(dx) \leq \sum_{n=1}^{\infty} n^{-2} < \infty. \qquad \square$$

Remark 1: We will prove directly that (*a*) implies (*c*). Suppose that (*a*) holds and for every $n = 1, 2, \ldots$, let μ_n be the probability measure on X defined as

$$\mu_n(B) := n^{-1} \sum_{t=0}^{n-1} \int P^t(B \mid z)\nu(dz), \quad B \in \mathcal{B}(X),$$

so that we may rewrite the condition in (*a*) as

$$\limsup_{n \to \infty} \int v d\mu_n < \infty.$$

Hence, by the lemma in Section 2, $\{\mu_n\}$ has a subsequence $\{\mu_{n_i}\}$ which converges weakly to some probability measure μ. We will show that (cf. (2))

$$\int Lu(x)\mu(dx) = 0 \qquad \forall u \in C(X), \tag{3}$$

where $Lu(x) := \int u(y)P(dy \mid x) - u(x)$, thus showing that μ is invariant for P.

Indeed, for any bounded measurable function u on X, the sequence

$$M_n(u) := u(x_n) - \sum_{t=0}^{n-1} Lu(x_t), \quad n = 1, 2, \ldots,$$

with $M_0(u) := u(x_0)$, is a martingale, which implies

$$E_\nu[M_n(u)] = E_\nu[M_0(u)] \forall n,$$

i.e.,

$$E_\nu[u(x_n)] - n \int Lu(x)\mu_n(dx) = \int u(x)\nu(dx).$$

Finally, let u be in $C(X)$; replace n by n_i; multiply by $1/n_i$; and then let $i \to \infty$ to get (3).

Remark 2: In [8], it is shown that

$$\sup_{t \geq 1} \int \int g(y)P^t(dy \mid x)\nu(dx) < \infty \tag{4}$$

for some moment g and initial p.m. ν, is also a necessary and sufficient condition for existence of invariant probabilities *provided that the Markov chain satisfies the uniform countable-additivity property*

$$\lim_{A \downarrow \emptyset} \sup_{x \in K} P(A \mid x) = 0 \tag{5}$$

for every compact set K in x.

Note that (4) is stronger than our condition (a) and that (5) implies: *For every compact set* $K \subset X$, *the family of p.m.'s* $\{P(\cdot \mid x)\}_{x \in K}$ *is tight.*

Remark 3: It is worth noting that the theorem still holds if we replace "lim sup" by "lim inf" in both conditions (a) and (b). Now, (b)$\Rightarrow$(a) by a well-known Abelian theorem [11]. With similar arguments as in Remark 1, (a)$\Rightarrow$(c). We finally prove (c)$\Rightarrow$(b) by exhibiting the same moment function v and show that

$$\liminf_{\alpha \uparrow 1} V_\alpha(\nu) = \liminf_{\alpha \uparrow 1} (1-\alpha) E_\nu \sum_{t=0}^{\infty} \alpha^t v(x_t) = \int v(x)\nu(dx) \leq \sum_{t=0}^{\infty} n^{-2} < \infty.$$

In conclusion, we mention that the theorem can be extended in the obvious way to continuous-time Markov processes, as in [2]. Conditions for *uniqueness* and *ergodicity* of invariant measures can be found, for instance, in [4, 7, 10] and references therein.

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