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EXTENSION OF THE METHOD OF QUASILINEARIZATION FOR STOCHASTIC INITIAL VALUE PROBLEMS

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ABSTRACT

In this paper we extend the method of quasilinearization to stochastic initial value problems. Further we prove that the iterates converge uniformly almost surely to the unique solution and the convergence is quadratic.

Key words: Quasilinearization, quadratic convergence, monotone sequence, stochastic initial value problem.

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1. Introduction

Quasilinearization is a well known technique for obtaining approximate solutions of nonlinear differential equations [1, 2]. It provides a monotone sequence of approximate solutions that converges quadratically to the unique solution of the IVP (initial value problem)

$$u' = f(t,u), u(0) = u_0 \text{ on } J = [0,T], \tag{1.1}$$

if f is convex. Recently, this method has been generalized and extended using less restrictive conditions on f so as to be applicable to a large class of problems [4-10, 12]. In particular, in [4, 8], this technique has been extended to obtain monotone sequences that converge quadratically to the unique solution of (1.1) when f can be decomposed into a difference of two convex functions. In this paper we extend the technique used in [8] to stochastic initial value problems.

2. Main Result

Let $(\Omega, \mathcal{A}, P)$ be a probability measure space and $u_0: \Omega \longrightarrow \mathbb{R}$ be a given measurable function. Consider the stochastic initial value problem (SIVP)

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$$u'(t,\omega) = f(t,u(t,\omega),\omega) + g(t,u(t,\omega),\omega), \quad a.e. \text{ on } J = [0,T], \tag{2.1}$$

$$u(0,\omega) = u_0(\omega),$$

where $f\colon J \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ and $g\colon J \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ satisfy:

(*i*) $f(t,u,\cdot)$ and $g(t,u,\cdot)$ are measurable for all (t,u);

(*ii*) $f(\cdot,u,\cdot)$ and $g(\cdot,u,\cdot)$ are measurable for every u;

(*iii*) $f(t,\cdot,\omega)$ and $g(t,\cdot,\omega)$ are continuous for all (t,ω).

Suppose that

1) $|f(t,x,\omega)| \leq K(t,\omega)$ on $J \times \mathbb{R} \times \Omega$, where $K\colon J \times \Omega \longrightarrow \mathbb{R}_+$ is measurable in t and $\int_0^T K(s,\omega)ds < \infty$ on Ω.

A stochastic process $u\colon J \times \Omega \longrightarrow \mathbb{R}$ is called a *sample solution* of (2.1) if $u(0,\cdot) = u_0$ and is absolutely continuous (*a.c.*) on J and satisfies $u'(t,\omega) = f(t,u(t,\omega),\omega) + g(t,u(t,\omega),\omega)$, *a.e.* on J.

A stochastic process $\alpha\colon J \times \Omega \longrightarrow \mathbb{R}$ is said to be a *sample lower solution* of (2.1) if for almost all $\omega \in \Omega$, $\alpha(\cdot,\omega)$ is *a.c.* and $\alpha'(t,\omega) \leq f(t,\alpha(t,\omega),\omega) + g(t,\alpha(t,\omega),\omega)$, *a.e.* on J. The definition of sample upper solution is obtained by reversing the inequality above. For further details we refer to [3].

Theorem 2.1: *Assume that*

A_1) *α_0 and β_0 are lower and upper sample solutions of* (2.1) *such that $\alpha_0 \leq \beta_0$ on $J \times \Omega$;*

A_2) *$f_u(t,u,\omega), g_u(t,u,\omega), f_{uu}(t,u,\omega), g_{uu}(t,u,\omega)$ exist, are continuous in u, measurable in*

ω, measurable in (t,ω) and satisfy $f_{uu}(t,u,\omega) \geq 0$, $g_{uu}(t,u,\omega) \leq 0$;

A_3) *f_u, g_u, f_{uu} and g_{uu} satisfy* (2.1) *with different bounds.*

Then there exist monotone sequences $\{\alpha_n(t,\omega)\}$, $\{\beta_n(t,\omega)\}$ which converge uniformly, for almost all $\omega \in \Omega$, to the unique sample solution of (2.1) *and the convergence is quadratic.*

Proof: Let us first observe that (A_2) implies, for any $u \geq v$,

$$\begin{cases} f(t,u,\omega) \geq f(t,v,\omega) + f_u(t,v,\omega)(u-v), \\ g(t,u,\omega) \geq g(t,v,\omega) + g_u(t,u,\omega)(u-v). \end{cases} \tag{2.2}$$

Moreover, for any u_1, u_2 such that $\alpha_0(t,\omega) \leq u_2 \leq u_1 \leq \beta_0(t,\omega)$, it follows that

$$\begin{cases} f(t,u_1,\omega) - f(t,u_2,\omega) \leq L_1(t,\omega)(u_1 - u_2), \\ g(t,u_1,\omega) - g(t,u_2,\omega) \leq L_2(t,\omega)(u_1 - u_2), \end{cases} \tag{2.3}$$

a.e. on J, where $L_i(t,\omega) > 0$, is measurable for every t and $\int_0^T L_i(t,\omega)ds < \infty$ on Ω, for $i = 1, 2$.

Let $\alpha_1(t,\omega)$, $\beta_1(t,\omega)$ be sample solutions of the linear SIVPs

$$\begin{cases} \alpha_1' = f(t,\alpha_0,\omega) + f_u(t,\alpha_0,\omega)(\alpha_1 - \alpha_0) + g(t,\alpha_0,\omega) + g_u(t,\beta_0,\omega)(\alpha_1 - \alpha_0), \ \alpha_1(0,\omega) = u_0(\omega), \\ \beta_1' = f(t,\beta_0,\omega) + f_u(t,\alpha_0,\omega)(\beta_1 - \beta_0) + g(t,\beta_0,\omega) + g_u(t,\beta_0,\omega)(\beta_1 - \beta_0), \ \beta_1(0,\omega) = u_0(\omega), \end{cases} \tag{2.4}$$

a.e. on J, where $\alpha_0(0,\omega) \leq u_0(\omega) \leq \beta_0(0,\omega)$.

We shall prove that $\alpha_0 \leq \alpha_1$ on $J \times \Omega$. To do this, let $p = \alpha_0 - \alpha_1$ so that $p(0,\omega) \leq 0$. Then, using (2.4), we get

$$\begin{aligned} p' &= \alpha_0' - \alpha_1' \\ &\leq f(t,\alpha_0,\omega) + g(t,\alpha_0,\omega) - [f(t,\alpha_0,\omega) + f_u(t,\alpha_0,\omega)(\alpha_1 - \alpha_0) \\ &\quad + g(t,\alpha_0,\omega) + g_u(t,\beta_0,\omega)(\alpha_1 - \alpha_0)] \\ &= [f_u(t,\alpha_0,\omega) + g_u(t,\beta_0,\omega)]p, \text{ a.e. on } J. \end{aligned}$$

This implies, by Theorem 1.1 [11], that $p(t,\omega) \leq 0$ on $J \times \Omega$. Now set $p = \alpha_1 - \beta_0$ and note that $p(0,\omega) \leq 0$. Using (2.2) and (2.4), we obtain

$$\begin{aligned} p' &= \alpha_1' - \beta_0' \\ &\leq [f(t,\alpha_0,\omega) + f_u(t,\alpha_0,\omega)(\alpha_1 - \alpha_0) + g(t,\alpha_0,\omega) + g_u(t,\beta_0,\omega)(\alpha_1 - \alpha_0)] \\ &\quad - f(t,\beta_0,\omega) - g(t,\beta_0,\omega) \\ &\leq [f(t,\alpha_0,\omega) + f_u(t,\alpha_0,\omega)(\alpha_1 - \alpha_0) + g(t,\beta_0,\omega) - g_u(t,\beta_0,\omega)(\beta_0 - \alpha_0) \\ &\quad + g_u(t,\beta_0,\omega)(\alpha_1 - \alpha_0)] - f(t,\alpha_0,\omega) - f_u(t,\alpha_0,\omega)(\beta_0 - \alpha_0) - g(t,\beta_0,\omega) \\ &= [f_u(t,\alpha_0,\omega) + g_u(t,\beta_0,\omega)]p, \quad \text{a.e. on } J, \end{aligned}$$

which again implies $p(t,\omega) \leq 0$ on $J \times \Omega$. As a result we have $\alpha_0(t,\omega) \leq \alpha_1(t,\omega) \leq \beta_0(t,\omega)$ on $J \times \Omega$. Similarly, we can find that $\alpha_0(t,\omega) \leq \beta_1(t,\omega) \leq \beta_0(t,\omega)$ on $J \times \Omega$. We need to show that $\alpha_1(t,\omega) \leq \beta_{11}(t,\omega)$ on $J \times \Omega$ so that it yields

$$\alpha_0(t,\omega) \leq \alpha_1(t,\omega) \leq \beta_1(t,\omega) \leq \beta_0(t,\omega) \text{ on } J \times \Omega. \tag{2.5}$$

Using (2.2) and (2.4), we see that

$$\begin{aligned} \alpha_1' &= f(t,\alpha_0,\omega) + f_u(t,\alpha_0,\omega)(\alpha_1 - \alpha_0) + g(t,\alpha_0,\omega) + g_u(t,\beta_0,\omega)(\alpha_1 - \alpha_0) \\ &\leq f(t,\alpha_1,\omega) + g(t,\alpha_1,\omega) - g_u(t,\alpha_1,\omega)(\alpha_1 - \alpha_0) + g_u(t,\beta_0,\omega)(\alpha_1 - \alpha_0) \\ &= f(t,\alpha_1,\omega) + g(t,\alpha_1,\omega) + [g_u(t,\beta_0,\omega) - g_u(t,\alpha_1,\omega)](\alpha_1 - \alpha_0) \\ &\leq f(t,\alpha_1,\omega) + g(t,\alpha_1,\omega), \text{ a.e. on } J, \end{aligned}$$

because of the fact that $g_u(t,u,\omega)$ is nonincreasing in u and $\alpha_1 \leq \beta_0$ on $J \times \Omega$. Similarly, using (2.2) again, we obtain

$$\begin{aligned} \beta_1' &= f(t,\beta_0,\omega) + f_u(t,\alpha_0,\omega)(\beta_1 - \beta_0) + g(t,\beta_0,\omega) + g_u(t,\beta_0,\omega)(\beta_1 - \beta_0) \\ &\geq f(t,\beta_1,\omega) + f_u(t,\beta_1,\omega)(\beta_1 - \beta_0) + f_u(t,\alpha_0,\omega)(\beta_1 - \beta_0) + g(t,\beta_1,\omega) \\ &= f(t,\beta_1,\omega) + [-f_u(t,\beta_1,\omega) + f_u(t,\alpha_0,\omega)](\beta_1 - \beta_0) + g(t,\beta_1,\omega) \\ &\geq f(t,\beta_1,\omega) + g(t,\beta_1,\omega), \text{ a.e. on } J, \end{aligned}$$

because of the fact that $f_u(t,u,\omega)$ is nondecreasing in u and $\alpha_0 \leq \beta_1$ on $J \times \Omega$. It then follows from Theorem 1.1 [11] and (2.3), that $\alpha_1(t,\omega) \leq \beta_1(t,\omega)$ on $J \times \Omega$ which shows that (2.5) is valid.

Assume that for some $k > 1$, $\alpha_k' \leq f(t,\alpha_k,\omega) + g(t,\alpha_k,\omega)$, $\beta_k' \geq f(t,\beta_k,\omega) + g(t,\beta_k,\omega)$, *a.e.* on J and $\alpha_k(t,\omega) \leq \beta_k(t,\omega)$ on $J \times \Omega$. We shall prove that

$$\alpha_k(t,\omega) \leq \alpha_{k+1}(t,\omega) \leq \beta_{k+1}(t,\omega) \leq \beta_k(t,\omega) \text{ on } J \times \Omega, \tag{2.6}$$

where $\alpha_{k+1}(t,\omega)$ and $\beta_{k+1}(t,\omega)$ are sample solutions of the linear stochastic SIVPs

$$\begin{aligned} \alpha_{k+1}' &= f(t,\alpha_k,\omega) + f_u(t,\alpha_k,\omega)(\alpha_{k+1} - \alpha_k) + g(t,\alpha_k,\omega) \\ &\quad + g_u(t,\beta_k,\omega)(\alpha_{k+1} - \alpha_k), \\ \alpha_{k+1}(0,\omega) &= u_0(\omega) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \beta_{k+1}' &= f(t,\beta_k,\omega) + f_u(t,\alpha_k,\omega)(\beta_{k+1} - \beta_k) + g(t,\beta_k,\omega) \\ &\quad + g_u(t,\beta_k,\omega)(\beta_{k+1} - \beta_k), \\ \beta_{k+1}(0,\omega) &= u_0(\omega), \end{aligned} \tag{2.8}$$

a.e. on J.

Setting $p = \alpha_k - \alpha_{k+1}$, we have, as before, that $p' \leq [f_u(t,\alpha_k,\omega) + g_u(t,\beta_k,\omega)]p$, *a.e.* on J and $p(0,\omega) = 0$. This proves that $p(t,\omega) \leq 0$ on $J \times \Omega$. On the other hand, letting $p = \alpha_{k+1} - \beta_k$ yields

$$\begin{aligned} p' &= \alpha_{k+1}' - \beta_k' \\ &\leq f(t,\alpha_k,\omega) + f_u(t,\alpha_k,\omega)(\alpha_{k+1} - \alpha_k) + g(t,\alpha_k,\omega) + g_u(t,\beta_k,\omega)(\alpha_{k+1} - \alpha_k) \\ &\quad - f(t,\beta_k,\omega) - g(t,\beta_k,\omega). \end{aligned}$$

Since $\alpha_k \leq \beta_k$, (2.2) gives, after some computation,

$$p' \leq [f_u(t,\alpha_k,\omega) + g_u(t,\beta_k,\omega)]p, \quad a.e. \text{ on } J.$$

Thus we have $\alpha_k(t,\omega) \leq \alpha_{k+1}(t,\omega) \leq \beta_k(t,\omega)$ on $J \times \Omega$. Similar arguments yield $\alpha_k(t,\omega) \leq \beta_{k+1}(t,\omega) \leq \beta_k(t,\omega)$ on $J \times \Omega$. Now to show that $\alpha_{k+1}' \leq f(t,\alpha_{k+1},\omega) + g(t,\alpha_{k+1},\omega)$, we proceed as before. Utilizing (2.2), (2.7) and (A_2), we get

$$\begin{aligned} \alpha_{k+1}' &\leq f(t,\alpha_{k+1},\omega) + g(t,\alpha_{k+1},\omega) - g_u(t,\alpha_{k+1},\omega)(\alpha_{k+1} - \alpha_k) + g_u(t,\beta_k,\omega)(\alpha_{k+1} - \alpha_k) \\ &= f(t,\alpha_{k+1},\omega) + g(t,\alpha_{k+1},\omega) + [g_u(t,\beta_k,\omega) - g_u(t,\alpha_{k+1},\omega)](\alpha_{k+1} - \alpha_k) \\ &\leq f(t,\alpha_{k+1},\omega) + g(t,\alpha_{k+1},\omega), \; a.e. \text{ on } J. \end{aligned}$$

In a similar manner, we can prove that $\beta_{k+1}' \geq f(t,\beta_{k+1},\omega) + g(t,\beta_{k+1},\omega)$, *a.e.* on J and hence Theorem 1.1 [11] shows that $\alpha_{k+1}(t,\omega) \leq \beta_{k+1}(t,\omega)$ on $J \times \Omega$ which proves (2.6) is true. Hence by induction we have for all n,

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_2 \leq \beta_1 \leq \beta_0 \text{ on } J \times \Omega.$$

Let us note that for each fixed $t \in J$,

$$a(t,\omega) = \sup_{n \geq 0} \alpha_n(t,\omega) = \lim_{n \to \infty} \alpha_n(t,\omega),$$
$$b(t,\omega) = \inf_{n \geq 0} \beta_n(t,\omega) = \lim_{n \to \infty} \beta_n(t,\omega)$$

exist and $a(t,\omega)$, $b(t,\omega)$ are measurable functions in ω for each $t \in J$. We obtain, from (2.7) and (2.8),

$$\alpha_{n+1}(t,\omega) = u_0(\omega) + \int_0^t [f(s,\alpha_k(s,\omega),\omega) + f_u(s,\alpha_k(s,\omega),\omega)(\alpha_{k+1}(s,\omega) - \alpha_k(s,\omega))$$
$$+ g(s,\alpha_k(s,\omega),\omega) + g_u(s,\beta_k(s,\omega),\omega)(\alpha_{k+1}(s,\omega) - \alpha_k(s,\omega))]ds$$

and

$$\beta_{k+1}(t,\omega) = u_0(\omega) + \int_0^t [f(s,\beta_k(s,\omega),\omega) + f_u(s,\alpha_k(s,\omega),\omega)(\beta_{k+1}(s,\omega) - \beta_k(s,\omega))$$
$$+ g(\mathrm{s},\beta_k(s,\omega),\omega) + g_u(s,\beta_k(s,\omega),\omega)(\beta_{k+1}(s,\omega) - \beta_k(s,\omega))]ds.$$

By standard arguments, it is easily seen that $\{\alpha_{n+1}(t,\omega)\}$ and $\{\beta_{n+1}(t,\omega)\}$ are sample bounded and equicontinuous and consequently, (2.3) together with Lebesgue dominated convergence theorem yields that

$$a(t,\omega) = u_0(\omega) + \int_0^t \{f(s,a(s,\omega),\omega) + g(s,a(s,\omega),\omega)\}ds$$

and

$$b(t,\omega) = u_0(\omega) + \int_0^t \{f(s,b(s,\omega),\omega) + g(s,b(s,\omega),\omega)\}ds.$$

In view of (2.3), it is clear that $a = b$ on $J \times \Omega$, and as a result, $a = b = u$ on $J \times \Omega$ is the unique sample solution of (2.1).

Next we shall show that the convergence of the sequences $\{\alpha_n(t,\omega)\}$, $\{\beta_n(t,\omega)\}$ to $u(t,\omega)$ is quadratic. Let $p_n(t,\omega) = u(t,\omega) - \alpha_n(t,\omega) \geq 0$, $q_n(t,\omega) = \beta_n(t,\omega) - u(t,\omega) \geq 0$, and note that $p_n(0,\omega) = 0$, $q_n(0,\omega) = 0$. From (2.7) and the mean value theorem together with (A_2), we obtain successively,

$$p_n' = f(t,u,\omega) + g(t,u,\omega) - [f(t,\alpha_{n-1},\omega) + f_u(t,\alpha_{n-1},\omega)(\alpha_n - \alpha_{n-1})$$
$$+ g(t,\alpha_{n-1},\omega) + g_u(t,\beta_{n-1},\omega)(\alpha_n - \alpha_{n-1})]$$
$$= f_u(t,\delta,\omega)p_{n-1} + g_u(t,\sigma,\omega)p_{n-1} + f_u(t,\alpha_{n-1},\omega)(-p_{n-1} + p_n)$$
$$+ g_u(t,\beta_{n-1},\omega)(-p_{n-1} + p_n)$$
$$\leq [f_u(t,u,\omega) - f_u(t,\alpha_{n-1},\omega)]p_{n-1} - [g_u(t,\beta_{n-1},\omega) - g_u(t,\alpha_{n-1},\omega)]p_{n-1}$$
$$+ [f_u(t,\alpha_{n-1},\omega) + g_u(t,\beta_{n-1},\omega)]p_n$$
$$= f_{uu}(t,\delta_1,\omega)p_{n-1}^2 - g_{uu}(t,\sigma_1,\omega)(\beta_{n-1} - \alpha_{n-1})p_{n-1}$$
$$+ [f_u(t,\alpha_{n-1},\omega) + g_u(t,\beta_{n-1},\omega)]p_n,$$

where $\alpha_{n-1} < \delta$, $\sigma < u$ and $\alpha_{n-1} < \delta_1 < u$, $\alpha_{n-1} < \sigma_1 < \beta_{n-1}$.

But

$$\begin{aligned} -g_{uu}(t,\sigma_1,\omega)[\beta_{n-1}-\alpha_{n-1}]p_{n-1} &\leq N_2(t,\omega)[q_{n-1}+p_{n-1}]p_{n-1} \\ &= N_2(t,\omega)[p_{n-1}^2 + p_{n-1}q_{n-1}] \\ &\leq 2N_2(t,\omega)p_{n-1}^2 + N_2(t,\omega)q_{n-1}^2. \end{aligned}$$

Thus

$$p_n' \leq M(t,\omega)p_n + [N_1(t,\omega)+2N_2(t,\omega)]p_{n-1}^2 + N_2(t,\omega)q_{n-1}^2,$$

where $|f_u(t,u,\omega)| \leq M_1(t,\omega)$, $|g_u(t,u,\omega)| \leq M_2(t,\omega)$, $|f_{uu}(t,u,\omega)| \leq N_1(t,\omega)$, $|g_{uu}(t,u,\omega)| \leq N_2(t,\omega)$,

$$M = M_1 + M_2, \int_0^T M(t,\omega)dt = Q(\omega) < \infty, \int_0^T N_1(t,\omega)dt = R(\omega) < \infty$$

and

$$\int_0^T N_2(t,\omega)dt = S(\omega) < \infty.$$

Thus, by Gronwall's lemma, we get

$$\begin{aligned} 0 &\leq p_n(t,\omega) \\ &\leq \int_0^t \{exp\int_s^t M(s,\omega)ds\}[\{N_1(s,\omega)+2N_2(s,\omega)\}p_{n-1}^2(s,\omega) + N_2(s,\omega)q_{n-1}^2(s,\omega)]ds \\ &\leq \int_0^T \{exp\int_0^T M(s,\omega)ds\}[\{N_1(s,\omega)+2N_2(s,\omega)\}p_{n-1}^2(s,\omega) + N_2(s,\omega)q_{n-1}^2(s,\omega)]ds. \end{aligned}$$

It therefore follows that

$$\begin{aligned} \max_J |u(t,\omega)-\alpha_n(t,\omega)| \leq e^{Q(\omega)}[&\{R(\omega)+2S(\omega)\}\max_J |u(t,\omega)-\alpha_{n-1}(t,\omega)|^2 \\ &+ S(\omega)\max_J |\beta_{n-1}(t,\omega)-u(t,\omega)|^2], \end{aligned}$$

for almost all $\omega \in \Omega$.

Similarly we can proved that

$$\begin{aligned} \max_J |\beta_n(t,\omega)-u(t,\omega)| \leq e^{Q(\omega)}[&\{S(\omega)+2R(\omega)\}\max_J |u(t,\omega)-\beta_{n-1}(t,\omega)|^2 \\ &+ R(\omega)\max_J |\alpha_{n-1}(t,\omega)-u(t,\omega)|^2], \end{aligned}$$

for almost all $\omega \in \Omega$. This completes the proof. $\square$

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