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OPTIMAL CONTROLLABILITY OF IMPULSIVE CONTROL SYSTEMS

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ABSTRACT

The problem of optimal controllability of a nonlinear impulsive control system is studied using the method of vector Lyapunov functions and the generalized comparison principle.

Key words: Impulsive control systems, optimal control, vector Lyapunov functions.

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1. INTRODUCTION

Many evolutionary processes are subject to short term perturbations which act instantaneously in the form of impulses. Thus impulsive differential equations provide a natural description of observed evolutionary processes of several real world processes [1].

Control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems [8]. A central problem in this area is the optimal control problem, that is, the problem of controlling a system in some "best" possible manner by minimizing some function of the trajectories.

In this paper, the problem of optimal controllability of a nonlinear impulsive control system is studied, using the method of vector Lyapunov functions and the generalized comparison principle [3, 4]. An example is provided to illustrate the results.

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2. MAIN RESULTS

We shall consider the following impulsive control system

$$\begin{cases} x' = f(t,x,u), & t \neq t_k, \quad k = 1,2,\dots \\ x(t_k^+) = x(t_k) + I_k(x,u), & k = 1,2,\dots, \\ x(t_0) = x_0 \end{cases} \tag{2.1}$$

where $0 < t_1 < t_2 \dots < t_k < \dots$ and $t_k \to \infty$ as $k \to \infty$, $f \in PC[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n]$, $I_k \in C[\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n]$ for every k, $k = 1,2,\dots,$ and $u = u(t)$ is a control vector. Let Ω be the control prescribed. Corresponding to any control function $u = u(t)$, we shall denote a solution of (2.1) by $x(t) = x(t; t_0, x_0, u)$, with $x(t_0) = x_0$.

The following result deals with the optimal stabilization of (2.1).

Theorem 2.1: *Assume that*

(*i*) $0 < \lambda < A$ *are given,*

(*ii*) $V \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+^N]$, $V(t,x)$ *is locally Lipschitzian in* x, $Q \in \mathcal{K}[\mathbb{R}_+^N, \mathbb{R}_+]$, $g \in PC[\mathbb{R}_+ \times \mathbb{R}_+^N \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^N]$, $g(t,w,x,u)$ *is quasimonotone nondecreasing in* w *and* $\psi_k: \mathbb{R}_+^N \to \mathbb{R}^N$ *is nondecreasing for* $k = 1,2,\dots,$

(*iii*) $\Omega \subset \mathbb{R}^m$ *is a convex, compact set and for* $u^0(t) \in \Omega$, *the system* (2.1) *admits unique solutions for* $t \geq t_0$ *and for* $(t,x) \in \mathbb{R}_+ \times S(A)$,

$$b(\|x\|) \leq Q(V(t,x)) \leq a(\|x\|), \; a, b \in \mathcal{K}[\mathbb{R}_+, \mathbb{R}_+]$$

and

$$\|x + I_k(x)\| < \rho \text{ whenever } \|x\| < A, \; \rho > A.$$

(*iv*) $B[V,t,x,u^0,g] \equiv V_t(t,x) + V_x^T(t,x) f(t,x,u^0) + g(t, V(t,x), x, u^0) \leq 0$, $t \neq t_k$,

(*v*) $C_k[V, t_k, x, \psi_k] \equiv \Delta V + \psi_k(V(t_k, x(t_k))) = 0$, $k = 1,2,\dots,$ *where* $\Delta V = V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))$,

(*vi*) $B[V,t,x,u,g] \geq 0$ *for any* $u \in \Omega$, $t \neq t_k$,

(*vii*) $a(\lambda) < b(A)$ *holds,*

(*viii*) *any solution* $w(t, t_0, w_0)$ *of*

$$\begin{cases} w' = -g(t, w, x^0(t), u^0(t)), & t \neq t_k \\ \Delta w = \psi_k(w(t_k)) & k = 1,2,\dots, \\ w(t_0) = w_0 \geq 0 \end{cases} \tag{2.2}$$

exists on $[t_0, \infty)$ *and satisfies*

$$Q(w_0) < a(\lambda) \textit{ implies } Q(w(t,t_0,w_0) < b(A),\ t \geq t_0 \tag{2.3}$$

and

$$\lim_{t \to \infty} w(t;t_0,w_0) = 0. \tag{2.4}$$

Then, the control system (2.1) is practically asymptotically stable and the inequality

$$\begin{cases} \int_{t_0}^{\infty} g(s,V(s,x^0(s)),x^0(s),u^0(s))ds + \sum_{k=1}^{\infty} \psi_k(V(t_k,x^0(t_k))) \\ = \min_u \int_{t_0}^{\infty} g(s,V(s,x(s)),x(s),u(s))ds + \sum_{k=1}^{\infty} \psi_k(V(t_k)x^0(t_k)) \\ \leq V(t_0,x_0) \textit{ holds.} \end{cases} \tag{2.5}$$

i.e. $u^0 \in \Omega$ *assures optimal stabilization.*

Proof: To prove this theorem, we have to show two things:

(1) the control $u^0(t) \in \Omega$ assures practical asymptotic stability,

(2) the relation (2.5) holds.

Let $x^0(t) = x(t;t_0,x_0,u^0)$ be the solution of (2.1) corresponding to the control $u^0(t) \in \Omega$. Then, setting $m(t) = V(t,x^0(t))$, $w_0 = V(t_0,x_0)$ and using assumptions (i)-(v), (vii) and $(viii)$, we can prove that the system (2.1) is practically stable following the standard arguments of [1, 5, 7]. Then, we also have

$$V(t,x^0(t)) \leq w(t;t_0,w_0),\ t \geq t_0. \tag{2.6}$$

Consequently, (2.4) implies that $\lim_{t \to \infty} x^0(t) = 0$, which proves practical asymptotic stability.

Now, to prove (2.5), let us suppose that another control $u^*(t) \in \Omega$ also assures practical asymptotic stability of (2.1). Then, the corresponding solution $x^*(t)$ also satisfies $\| x^*(t) \| < A$, $t \geq t_0$, provided $\| x_0 \| < \lambda$, and $\lim_{t \to \infty} x^*(t) = 0$. This implies that

$$\lim_{t \to \infty} V(t,x^*(t)) = 0 \tag{2.7}$$

and we also have from (2.6)

$$\lim_{t \to \infty} V(t,x^0(t)) = 0. \tag{2.8}$$

Then, by (iv), we get

$$\int_{t_0}^{\infty} g(s,V(s,x^0x)),x^0(s),u^0(s))ds + \sum_{k=1}^{\infty} \psi_k(V(t_k,x^0(t_k)) \leq V(t_0,x_0). \tag{2.9}$$

But by (2.7) and (vi), we get

$$\int_{t_0}^{\infty} g(s, V(s, x^*(s)), x^*(s), u^*(s))ds + \sum_{k=1}^{\infty} \psi_k(V(t_k, x^*(t_k)) \geq V(t_0, x_0). \tag{2.10}$$

The inequalities (2.9) and (2.10) prove the desired relation (2.5) and the proof is complete.

Q.E.D.

The following simple example illustrates this result.

Example 2.1: Consider the following impulsive control system

$$\begin{cases} x' = F(t,x) + R(t,x)u & t \neq t_k, \\ x(t_k^+) = b_k x_k & k = 1, 2, \ldots, \\ x(t_0) = x_0 \end{cases} \tag{2.11}$$

where $F \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $R(t,x)$ is an $n \times m$ matrix and u is a control.

We shall base the solution of the problem on the consideration of the function $V(t,x)$ given by

$$V(t,x) = \sum_{i=1}^{N} a_i V_i(t,x),\ a_i = const > 0$$

where $V_i(t,v)$ are the components of Lyapunov's vector function.

Suppose we have

$$V_t(t,x) + V_x^T(t,x)F(t,x) \equiv p(t,x) \leq \lambda'(t)V(t,x) \tag{2.12}$$

where $\lambda'(t) \geq 0$, $t \geq t_0$ and $\lambda \in C^1[\mathbb{R}_+, \mathbb{R}_+]$.

Define, for $t \neq t_k$,

$$B[V,t,x,u] = p(t,x) + V_x^T R(t,x)u + w(t,x) + u^T Du \tag{2.13}$$

where D is an $m \times m$ non-singular matrix.

We shall find the control $u^0 = u^0(t) \in \Omega$ from the condition of the minimum of B:

$$B[V,t,x,u] = 0 \text{ at } u = u_0 \tag{2.14}$$
$$\frac{\partial B}{\partial u}[V,t,x,u] = 0 \text{ at } u = u_0. \tag{2.15}$$

Thus we obtain

$$R^T(t,x)V_x(t,x) + 2Du^0 = 0 \tag{2.16}$$

and it then follows that

$$u^0(t) = -\tfrac{1}{2}D^{-1}R^T(t,x)V_x(t,x). \tag{2.17}$$

To discuss the problem of minimization of $\int_0^\infty g(s, V(s,x(s)), x(s)), u(s,x(s)))ds$, we obtain from (2.13), (2.14) and (2.16) the relation

$$w(t,v) + p(t,x) - u^{0T}Du^0 = 0$$

which yields

$$w(t,x) = -p(t,x) + u^{0T}Du^0.$$

Thus

$$\begin{aligned} -g(t,v,x,u) &= p(t,x) - u^{0T}Du^0 - u^TDu \\ &\le \lambda'(t)V - u^{0T}Bu^0 - u^TDu, \ t \neq t_k. \end{aligned} \tag{2.18}$$

For $t = t_k$, we want

$$V(t_k^+, x(t_k^+)) \le d_kV(t_k, x(t_k)) \tag{2.19}$$

where $\alpha d_k \le e^{\lambda(t_k) - \lambda(t_{k+1})}, \alpha > 1$. Thus, $u^0 = -\frac{1}{2}D^{-1}R^T(t,x)V_x(t,x)$ assures optimal stabilization of (2.1).

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