RANDOM FIXED POINT THEOREMS ON PRODUCT SPACES[1]

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ABSTRACT

The existence of random fixed point of a locally contractive random operator in first variable on product spaces is proved. The concept "continuous random height-selection" is discussed. Some random fixed point theorems for nonexpansive self and nonself maps are also obtained in product spaces.

Key words: Product space, random operator, random fixed point, weakly inward operator.

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I. INTRODUCTION

The study of the fixed points of random operators of various types is a lively and fascinating field of research lying at the intersection of nonlinear analysis and probability theory. A wide spread interest in the domain and a vast amount of mathematical activity have led to many remarkable new results and viewpoints yielding insight even into traditional question.

Random fixed point theorems are stochastic generalizations of classical fixed point theorems. In Polish spaces, random fixed point theorems for contraction mappings were proved by Spacek [24] and Hans [7, 8]. For a complete survey, we refer to Bharucha-Reid [2, 3]. Itoh [10, 11, 12] gave several random fixed point theorems for various single and multivalued random operators, that is, α-condensing or nonexpansive random operators. He also gave several common random fixed point theorems for commuting random operators. Afterwards, Sehgal and Singh [23], Papageorgiou [22] and Lin [20] gave different stochastic versions of a very interesting approximation theorem of Fan [5]. Recently, Beg and Shahzad

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[1] studied the structure of common random fixed points and random coincidence points of a pair of compatible random multivalued operators in Polish spaces. They also proved a random fixed point theorem for contractive random operators in ϵ-chainable Polish spaces. An interesting application to random approximation is also given. The aim of this paper is to prove several random fixed point theorems for various self and nonself random operators, that is, contractive and nonexpansive random operators in product spaces. The concept "continuous random height-selection" is discussed and its relation to the existence of random fixed points for a function is shown. Section 2 contains definitions and preliminary material. In section 3, the existence of random fixed point of a contractive random operator in first variable on product spaces is proved. Other results are proved concerning the random fixed point theorem for product spaces. Section 4 contains some random fixed point theorems for nonexpansive self or nonself random operators.

2. PRELIMINARIES

Throughout this paper, let (X,d) be a Polish space, that is, a separable complete metric space, and $(\Omega, \mathcal{A})$ be a measurable space. Let 2^X be the family of all subsets of X and $WK(X)$ the family of all weakly compact subsets of X. A mapping $T:\Omega\to2^X$ is called *measurable* if for any open subset C of X, $T^{-1}(C) = \{\omega \in \Omega: T(\omega)\cap C \neq \phi\} \in \mathcal{A}$. This type of measurability is usually called weak measurability (cf. Himmelberg [9]), but in this paper since we only use this type of measurability, thus we omit the term "weak" for simplicity. A mapping $\zeta:\Omega\to X$ is said to be the *measurable selector* of a measurable mapping $T:\Omega\to2^X$ if ζ is measurable and for any $\omega \in \Omega$, $\zeta(\omega) \in T(\omega)$. A mapping $f:\Omega\times X\to X$ is called a *random operator* if for any $x \in X$, $f(\cdot,x)$ is measurable. A measurable mapping $\zeta:\Omega\to X$ is called *random fixed point* of a random operator $f:\Omega\times X\to X$ if for every $\omega \in \Omega$, $f(\omega,\zeta(\omega)) = \zeta(\omega)$. A random operator $f:\Omega\times X\to X$ is called continuous if for each $\omega \in \Omega$, $f(\omega,\cdot)$ is continuous. A random operator $T:\Omega\times X\to X$ is said to be nonexpansive if for any $\omega \in \Omega$, $\| T(\omega,u) - T(\omega,v) \| \leq \| u-v \|$ for all $u,v \in X$.

A separable metric space Y has a random fixed point property for nonexpansive (continuous) random operators if every nonexpansive (continuous) random operator $T:\Omega\times Y\to Y$ must have a random fixed point. In what follows, $P_1:X\times Y\to X$ will denote the first projection mapping defined by $P_1(x,y) = x$, while $P_2:X\times Y\to Y$ will denote the second projection mapping defined by $P_2(x,y) = y$.

3. RANDOM FIXED POINT THEOREMS AND CONTINUOUS RANDOM HEIGHT-SELECTION

In 1968, Nadler [21] established some fixed point theorems for uniformly continuous functions on product metric spaces. Subsequently, Fora [6] further generalized these results. Recently, Kuczumow [18] proved a fixed point theorem in product spaces using the generic fixed point property for nonexpansive mapping. The aim of this section is to prove some random fixed point theorems using a recent result of Beg and Shahzad [1].

Let X be a Polish space and Y be any space. A random operator $f: \Omega \times (X \times Y) \to (X \times Y)$ is said to be a locally contractive random operator in the first variable if and only if for any $x_* \in X$ there exists $\epsilon > 0$ and a measurable map $\lambda: \Omega \to (0,1)$ such that $p, q \in S(x_*, \epsilon) = \{z \in X : d(x_*, z) < \epsilon\}$ and for any $\omega \in \Omega$

$$d(P_1 of(\omega, p, y),\ P_1 of(\omega, q, y)) \leq \lambda(\omega)\ d(p,q)$$

for any $y \in Y$.

A random operator $f: \Omega \times (X \times Y) \to (X \times Y)$ is called a contraction mapping in the first variable if and only if there exists a measurable map $\lambda: \Omega \to (0,1)$ such that for any $y \in Y$, we have for any $\omega \in \Omega$

$$d(P_1 of(\omega, x, y),\ P_1 of(\omega, x_*, y)) \leq \lambda(\omega)\ d(x, x_*)$$

for all $x, x_* \in X$.

A metric space (X, d) is said to be ϵ-chainable if and only if given x, y in X, there is an ϵ-chain from x to y (i.e., a finite set of points $x = z_0, z_1, \ldots, z_n = y$ such that $d(z_{j-1}, z_j) < \epsilon$ for $j = 1, 2, \ldots, n$).

Theorem 3.1: *Let X be an ϵ-chainable Polish space, let Y be a separable metric space with the random fixed point property and let $f: \Omega \times (X \times Y) \to X \times Y$ be a continuous random operator. If f is a locally contractive operator in the first variable, then f has a random fixed point.*

Proof: Let $\xi_0: \Omega \to X$ be a fixed measurable mapping. Define the random operator $g: \Omega \times Y \to Y$ as follows. Let $y \in Y$, $\omega \in \Omega$. In order to define $g(\omega, y)$, we first define a sequence of measurable mappings $\{\eta_n(\omega, y)\}$ as follows

$$\eta_0(\omega, y) = \xi_0(\omega), \eta_n(\omega, y) = P_1 of(\omega, \eta_{n-1}(\omega, y), y),$$

$n = 1, 2, \ldots$. For simplicity, we shall write $\eta_n(\omega)$ to stand for $\eta_n(\omega, y)$. Since f is a locally contractive random operator in the first variable, there exists a measurable map $\lambda: \Omega \rightarrow (0, 1)$ such that

$$d(P_1 \circ f(\omega, \eta_{n-1}(\omega) y), P_1 \circ f(\omega, \eta_n(\omega), y)) \leq \lambda(\omega) d(\eta_{n-1}(\omega), \eta_n(\omega)), n \geq 1.$$

By induction we find that

$$d(\eta_n(\omega), \eta_{n+1}(\omega)) \leq \lambda^n(\omega) d(\eta_0(\omega), \eta_1(\omega)).$$

It further implies that $\{\eta_n(\omega)\}$ is a Cauchy sequence. Since X is complete, there exists $\eta_y(\omega) \in X$ such that $\eta_n(\omega) \rightarrow \eta_y(\omega)$ for each $\omega \in \Omega$. [The mapping $\eta_y: \Omega \rightarrow X$ is a point-wise limit of measurable mappings $\{\eta_n\}$, therefore measurable]. Thus we obtain $P_1 \circ f(\omega, \eta_y(\omega), y) = \eta_y(\omega)$ for each $\omega \in \Omega$ (see Beg and Shahzad [1]). Now define for any $\omega \in \Omega$, $g(\omega, y) = P_2 \circ f(\omega, \eta_y(\omega), y)$. Then g is continuous [6, Lemma 3]. Since Y has the random fixed point property, there is a measurable map $\xi: \Omega \rightarrow Y$ such that $g(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$. But $\xi(\omega) = g(\omega, \xi(\omega)) = P_2 \circ f(\omega, \eta_{\xi(\omega)}(\omega), \xi(\omega))$ for every $\omega \in \Omega$. We also have $P_1 \circ f(\omega, \eta_{\xi(\omega)}(\omega), \xi(\omega)) = \eta_{\xi(\omega)}(\omega)$. Hence for $\omega \in \Omega$, $f(\omega, \eta_{\xi(\omega)}(\omega), \xi(\omega)) = (\eta_{\xi(\omega)}(\omega), \xi(\omega))$.

Corollary 3.2: *Let X be a Polish space, let Y be a separable metric space with random fixed point property and let $f: \Omega \times (X \times Y) \rightarrow (X \times Y)$ be a continuous random operator. If f is a contraction random operator in the first variable, then f has a random fixed point.*

Let $f: \Omega \times (X \times Y) \rightarrow (X \times Y)$ be a continuous random operator. A measurable mapping $\eta_x: \Omega \rightarrow Y$ for which $P_2 \circ f(\omega, x, \eta_x(\omega)) = \eta_x(\omega)$ for each $\omega \in \Omega$ is called a random fixed f-height of x. The set $\{\eta_x: \eta_x: \Omega \rightarrow Y \text{ is a random } f\text{-height of } x\}$ is called the random fixed f-height of x and is denoted by $F(f, x)$. The set $U\{F(f, x): x \in X\}$ is called the random fixed f-height of X and is denoted by $F(f, X)$. A continuous random height-selection of f is a continuous random operator $g: \Omega \times X \rightarrow F(f, X)$ such that $g(\cdot, x): \Omega \rightarrow Y$ is a random f-height of x.

Theorem 3.3: *Let X and Y be separable metric spaces with the random fixed point property, and let $f: \Omega \times (X \times Y) \rightarrow (X \times Y)$ be a continuous random operator. If f has a continuous random height-selection, then f has a random fixed point.*

Proof: Let $x \in X$. Define a random operator $g': \Omega \times Y \rightarrow Y$ by the rule $g'(\omega, y) = P_2 \circ f(\omega, x, y)$. Then g' is continuous because f is continuous. But Y has the random fixed point property, therefore there exists a measurable mapping $\xi_x: \Omega \rightarrow Y$ such that $\xi_x(\omega) = g'(\omega, \xi_x(\omega))$ for each $\omega \in \Omega$. Thus $P_2 \circ f(\omega, x, \xi_x(\omega)) = \xi_x(\omega)$, that is, $F(f, x) \neq \phi$. Let

$g:\Omega \times X \rightarrow F(f,X)$ be a continuous random height selection of f. Define the continuous random operator $h:\Omega \times X \rightarrow X$ by the rule $h(\omega,x) = P_1 o f(\omega,x,g(\omega,x))$ for each $\omega \in \Omega$ and $x \in X$. Let $\phi:\Omega \rightarrow X$ be a random fixed point of h. Then $h(\omega,\phi(\omega)) = \phi(\omega)$ for $\omega \in \Omega$, that is, $P_1 o f(\omega,\phi(\omega), g(\omega,\phi(\omega))) = \phi(\omega)$. Therefore $f(\omega,\phi(\omega), g(\omega,\phi(\omega))) = (\phi(\omega),v)$ for some $v \in Y$. But $g(\omega,\phi(\omega)) \in F(f,X)$ for each $\omega \in \Omega$. Therefore $P_2 o f(\omega,\phi(\omega),g(\omega,\phi(\omega))) = g(\omega,\phi(\omega))$, that is, $f(\omega,\phi(\omega),g(\omega,\phi(\omega))) = (u,g(\omega,\phi(\omega)))$ for some $u \in X$. Hence for every $\omega \in \Omega$, $f(\omega,\phi(\omega), g(\omega,\phi(\omega))) = (\phi(\omega),g(\omega,\phi(\omega)))$.

Using the proof of Theorem 3.1 and the technique used in the proof of Theorem 3.2, one can obtain the following theorem.

Theorem 3.4:   *Let X be an ϵ-chainable Polish space, let Y be a separable metric space with the random fixed point property and let $f:\Omega \times (Y \times X) \rightarrow (Y \times X)$ be a continuous random operator. If f is a locally contractive random operator in the second variable, then f has a continuous random height-selection and hence f has a random fixed point.*

4. RANDOM FIXED POINT THEOREMS OF NONEXPANSIVE RANDOM OPERATORS IN PRODUCT SPACES

Let E and F be two Banach spaces with $X \subset E$ and $Y \subset F$. For $1 \leq P < \infty$ and $(x,y) \in E \oplus F$, set

$$\| (x,y) \|_p = (\| x \|_E^P + \| y \|_F^P)^{1/P}$$

and for $P = \infty$, $\| (x,y) \|_\infty = max\{ \| x \|_E, \| y \|_F \}$.

It was shown in Kirk and Sternfeld [17] that if E is uniformly convex (or reflexive with the B-G property), if X is bounded closed convex, and if Y is bounded closed and separable, then the assumption that Y has the fixed point property for nonexpansive mappings assures the same is true of $(X \oplus Y)_\infty$. Subsequently, various results on fixed point theorems in product spaces were given by many authors (cf. Khamsi [13], Kirk [14, 15], Kirk and Yanez [16] and their references).

Recently Tan and Xu [25] proved some fixed point theorems for nonexpansive self and nonself mappings in product spaces. They also generalized and improved some results of Kirk [15] and Kirk and Sternfeld [17]. In this section we give a stochastic version of the results of [14, 16, 17, 25].

A subset K of a Banach space E has the Browder-Göhde (B-G) property [14] if for each nonexpansive mapping $T: K \to E$, the mapping (I-T) is demiclosed on K (i.e., for each sequence $\{u_j\}$ in K, the condition $u_j \to u$ weakly and $u_j - T(u_j) \to p$ strongly implies $u \in K$ and $u - T(u) = p$). It is known (Browder [4]) that all closed convex subsets of uniformly convex Banach spaces have this property.

Theorem 4.1:   *Let E and F be two separable Banach spaces with $X \subset E$ and $Y \subset F$. Suppose that X is weakly compact, convex, and has the B-G property. Suppose also that Y has the random fixed point property for nonexpansive random operators. Then $(X \oplus Y)_\infty$ has the random fixed point property for nonexpansive random operators.*

Proof: Let P_1 and P_2 be the natural projections of $(E \oplus F)$ onto E and F, respectively. For each y in Y, we define $T_y: \Omega \times X \to X$ by $T_y(\omega, x) = P_1 o T(\omega, x, y), x \in X$. Then T_y is a nonexpansive random operator. Let $S_y = (I + T_y)/2$ (I denotes the identity operator on E). Let $\xi_0: \Omega \to X$ be a fixed measurable mapping. We have

$$\lim_{n \to \infty} \| S_y(\omega, \xi_{y,n}(\omega)) - \xi_{y,n}(\omega) \|_E = 0 \tag{1}$$

where $\xi_{y,n} = S_y^n(\omega, \xi_0(\omega)) = S_y(\omega, S_y^{n-1}(\omega, \xi_0(\omega)))$, for each $\omega \in \Omega$. For each n, define $F_{y,n}: \Omega \to WK(X)$ by

$$F_{y,n}(\omega) = w^{-cl}\{\xi_{y,i}(\omega): i \geq n\},$$

where $w^{-cl}(C)$ is the weak closure of C.

Let $F_y: \Omega \to WK(X)$ be a mapping defined by $F_y(\omega) = \bigcap_{n=1}^{\infty} F_{y,n}(\omega)$. Then, since the weak-topology is a metric topology, F_y is w-measurable by [9, Theorem 4.1]. Thus there is a w-measurable selector of F_y [19]. For any $x^* \in E^*$ (the dual space of E), $x^*(\xi_y(\cdot))$ is measurable as a numerically-valued function on Ω. Since E is separable, ξ_y is measurable. Fix $\omega \in \Omega$ arbitrarily. Then some subsequences $\{\xi_{y,m}(\omega)\}$ of $\{\xi_{y,n}(\omega)\}$ converges weakly to $\xi_y(\omega)$. Then by the B-G property of X and (1), it follows that $\xi_y(\omega)$ is a random fixed point of S_y and hence of T_y, that is, $P_1 o T(\omega, \xi_y(\omega), y) = \xi_y(\omega)$. Also,

$$\| \xi_{u,n}(\omega) - \xi_{v,m}(\omega) \|_E \leq \| u - v \|_F, \quad n, m = 1, 2, \dots \tag{2}$$

for u, v in Y and $\omega \in \Omega$.

It further implies

$$\| \xi_u(\omega) - \xi_v(\omega) \|_E \leq \lim_{n \to \infty} \sup[\lim_{m \to \infty} \sup \| \xi_{u,n}(\omega) - \xi_{v,m}(\omega) \|_E] \leq \| u - v \|_F.$$

Now let $f\colon \Omega \times Y \to Y$ defined by $f(\omega, y) = P_2 oT(\omega, \xi_y(\omega), y)$, $y \in Y$. Then for u, v in Y, we have for any $\omega \in \Omega$

$$\| f(\omega, u) - f(\omega, v) \|_F = \| P_2 oT(\omega, \xi_u(\omega), u) - P_2 oT(\omega, \xi_v(\omega), v) \|_F$$

$$\leq \| T(\omega, \xi_u(\omega), u) - T(\omega, \xi_v(\omega), v) \|_\infty$$

$$\leq max\{ \| \xi_u(\omega) - \xi_v(\omega) \|_E, \| u - v \|_F \}$$

$$= \| u - v \|_F.$$

Therefore f is nonexpansive on Y and thus has a random fixed point $\eta\colon \Omega \to Y$. It follows that $T(\omega, \xi_{\eta(\omega)}(\omega), \eta(\omega)) = (\xi_{\eta(\omega)}(\omega), \eta(\omega))$ for each $\omega \in \Omega$.

For a subset K of a separable Banach space, a random operator $T\colon \Omega \times K \to K$ is said to be strictly contractive if

$$\| T(\omega, x) - T(\omega, y) \| < \| x - y \|, \quad x, y \in K, \; x \neq y, \; \omega \in \Omega.$$

Theorem 4.2: *Let E and F be two Banach spaces with $X \subset E$ and $Y \subset F$ separables. Suppose that X is a closed bounded convex subset of E. Suppose also that Y has the random fixed point property for strictly contractive random operators. Then for each $1 \leq P \leq \infty$, $(X \oplus Y)_P$ has the random fixed point property for strictly contractive random operators.*

Proof: For a fixed p, $1 \leq p \leq \infty$, suppose $T\colon \Omega \times (X \oplus Y)_P \to (X \oplus Y)_P$ is strictly contractive. As before, for each $y \in Y$, we define $T_y\colon \Omega \times X \to X$ by

$$T_y(\omega, x) = P_1 oT(\omega, x, y), \quad \omega \in \Omega, \; x \in X.$$

Then it is easily checked that in any case of P, T_y is strictly contractive and hence has a random fixed point $\xi_y\colon \omega \to X$ [11, Theorem 2.1]. Now define $f\colon \Omega \times Y \to Y$ be

$$f(\omega, x) = P_2 oT(\omega, \xi_y(\omega), y), \quad \omega \in \Omega, \; y \in Y.$$

As in [25, Theorem 2.2], for any case of p, $f\colon \Omega \times Y \to Y$ is strictly contractive. Thus f has a random fixed point $\eta\colon \Omega \to Y$. It follows that,

$$T(\omega, \xi_{\eta(\omega)}(\omega), \eta(\omega)) = (\xi_{\eta(\omega)}(\omega), \eta(\omega))$$

for each $\omega \in \Omega$.

Finally, we prove random fixed point theorems for nonself random operators in product spaces. Recall that for a closed subset C of a separable Banach space E, the inward set of C at a point x in C, $I_C(x)$, is defined by

$$I_C(x) = \{x + c(z - x) \mid z \in C, \; c > 0\}.$$

A random operator $f: \Omega \times C \rightarrow E$ is said to be weakly inward if for each $\omega \in \Omega$, $f(\omega, x) \in cl(I_C(x))$ for $x \in C$. We will say that C has the random fixed point property for a nonexpansive (continuous) weakly inward random operator if every nonexpansive (continuous) weakly inward random operator $T: \Omega \times C \rightarrow E$ has a random fixed point.

Theorem 4.3: *Let E and F be two separable Banach spaces with $X \subset E$ and $Y \subset F$. Let $E \oplus F$ be a product space with an L^P norm, $1 \leq P < \infty$. Suppose that both X and Y have the random fixed point property for nonexpansive weakly inward random operators. Then $(X \oplus Y)_P$ also has the fixed point property for nonexpansive weakly inward random operators.*

Proof: Let $T: \Omega \times (X \oplus Y)_P \rightarrow (E \oplus F)_P$ be a nonexpansive and weakly inward random operator. For a fixed y in Y, we define $T_y: \Omega \times X \rightarrow E$ by

$$T_y(\omega, x) = P_1 o T(\omega, x, y), \quad \omega \in \Omega, \; x \in X.$$

Then T_y is a nonexpansive random operator. It is also weakly inward. Indeed, for each $\omega \in \Omega$. Since

$$T(\omega, x, y) \in cl(I_{X \oplus Y}(x, y)) \text{ for } (x, y) \in X \times Y,$$

we have

$$T_y(\omega, x) = P_1(cl \; I_{X \oplus Y}(x, y)) \subseteq cl \; I_x(x).$$

Hence there exists a measurable map $\xi_y: \Omega \rightarrow X$ such that for each $\omega \in \Omega$, $T_y(\omega, \xi_y(\omega)) = \xi_y(\omega)$. Now define $f: \Omega \times Y \rightarrow F$ by

$$f(\omega, x) = P_2 o T(\omega, \xi_y(\omega), y), \quad \omega \in \Omega, \; y \in Y.$$

Then it is easy to see that f is nonexpansive. Also, for each $\omega \in \Omega$, $f(\omega, y) \in P_2(cl \; I_{X \oplus Y}(\xi_y(\omega), y)) \subseteq cl \; I_y(y)$ for $y \in Y$. Therefore, f has a random fixed point $\eta: \Omega \rightarrow Y$ and hence

$$T(\omega, \xi_{\eta(\omega)}(\omega), \eta(\omega)) = (\xi_{\eta(\omega)}(\omega), \eta(\omega)) \text{ for } \omega \in \Omega.$$

In the next theorem, we assume that X has the net B-G property, that is, if $T: X \to E$ is nonexpansive and if $\{x_\alpha\}_{\alpha \in A}$ (we also assume that A is countable) is a net in X for which $x_\alpha \to x$ weakly and $x_\alpha - T(x_\alpha) \to p$ strongly, then $x \in X$ and $x - T(x) = p$.

Theorem 4.4:   *Let E and F be two separable Banach spaces with $X \subset E$ and $Y \subset F$. Suppose that X is weakly compact, convex, and has the net B-G property. Suppose also that Y has the random fixed point property for nonexpansive weakly inward random operators. Then $(X \oplus Y)_\infty$ has the random fixed point property for nonexpansive weakly inward random operators.*

Proof: Let $T: \Omega \times (X \oplus Y)_\infty \to (E \oplus F)_\infty$ be a nonexpansive and weakly inward random operator. then for each $y \in Y$, the operator $T_y: \Omega \times X \to E$ defined by $T_y(\omega, x) = P_1 o T(\omega, x, y)$, $(\omega \in \Omega, x \in X)$ is nonexpansive and weakly inward. For a fixed $z \in X$ and $t \in (0, 1)$, the contraction operator $(1 - t)z + tT_y$ is weakly inward and has a unique random fixed point [Han's, 7] which we denote by $\xi_{y,t}$. Thus for $\omega \in \Omega$,

$$\xi_{y,t}(\omega) = (1 - t)z + t\, T_y(\omega, \xi_{y,t}(\omega)).$$

Now let $\{t_\alpha : \alpha \in A\}$ be a universal subnet of the net $\{t : 0 < t < 1\}$ in $[0, 1]$. It follows that $\{\xi_{y,t_\alpha}(\omega)\}$ is a universal net in X $(\xi_{y,t_\alpha} : \Omega \to X$ is a measurable map). Also, since $t \to 1$, $t_\alpha \overset{\alpha}{\to} 1$ and for each $\omega \in \Omega$,

$$\| \xi_{y,t_\alpha}(\omega) - T_y(\omega, \xi_{y,t_\alpha}(\omega)) \|_E = (1 - t) \| z - T_y(\omega, \xi_{y,t_\alpha}(\omega)) \|_E \overset{\alpha}{\to} 0. \qquad (*)$$

For each α, define $F_{y,\alpha} : \Omega \to WK(X)$ by

$$F_{y,\alpha}(\omega) = w^{-cl}\{\xi_{y,t_{\alpha^*}}(\omega) : \alpha^* \geq \alpha\}.$$

Define $F_y : \Omega \to WK(X)$ by $F_y(\omega) = \bigcap_{\alpha \in A} F_{y,\alpha}(\omega)$. Then, as in the proof of Theorem 4.1, F_y is w-measurable and has a measurable selector ξ_y. Furthermore, since X is weakly compact, $\{\xi_{y,t_\alpha}(\omega)\}$ converges weakly, say to $\xi_y(\omega)$. Combining this with $(*)$, the B-G property implies

$$T_y(\omega, \xi_y(\omega)) = \xi_y(\omega) \text{ for } \omega \in \Omega.$$

Now let $u, v \in Y$, $\omega \in \Omega$

$$\| \xi_{u,t}(\omega) - \xi_{v,t}(\omega) \|_E = t \| T_u(\omega, \xi_{u,t}(\omega)) - T_v(\omega, \xi_{v,t}(\omega)) \|_E$$

$$\leq t \| T(\omega, \xi_{u,t}(\omega), u) - T(\omega, \xi_{v,t}(\omega), v) \|_\infty$$

$$\leq t \| (\xi_{u,t}(\omega), u) - (\xi_{v,t}(\omega), v) \|$$

$$= t\, max.\{ \| \xi_{u,t}(\omega) - \xi_{v,t}(\omega) \|_E, \| u - v \|_F\}$$

$$\leq t \, \| \, u - v \, \|_F.$$

Since $\{\xi_{u,t_\alpha}(\omega) - \xi_{v,t_\alpha}(\omega)\}$ converges weakly to $\xi_u(\omega) - \xi_v(\omega)$, it follows that

$$\| \, \xi_u(\omega) - \xi_v(\omega) \, \|_E \leq \| \, u - v \, \|_F \text{ for any } \omega \in \Omega.$$

Now it is easily checked that the random operator $f : \Omega \times Y \to F$ defined by $f(\omega, y) = P_2 o T(\omega, \xi_y(\omega), y)$, $y \in Y$ and $\omega \in \Omega$ is nonexpansive and weakly inward. Thus f has a random fixed point $\eta : \Omega \to Y$. It follows that $T(\omega, \xi_{\eta(\omega)}(\omega), \eta(\omega)) = (\xi_{\eta(\omega)}(\omega), \eta(\omega))$ for each $\omega \in \Omega$.

Theorem 4.5:   Let E and F be two separable Banach spaces with $X \subset E$ a closed ball with center at origin and radius r, $Y \subset F$. Let $(E \oplus F)$ be a product space with an L^P norm $1 \leq P \leq \infty$. Suppose that Y has the random fixed point property for strictly contractive weakly inward random operators. Then $(X \oplus Y)_P$ also has the random fixed point property for strictly contractive weakly inward random operators.

Proof: Let $T : \Omega \times (X \oplus Y)_P \to (E \oplus F)_P$ be a strictly contractive and weakly inward random operator. For a fixed y in Y, we define the random operator $T_y : \Omega \times X \to E$ by $T_y(\omega, x) = P_1 o T(\omega, x, y)$, $\omega \in \Omega$, $x \in X$. Then T_y is strictly contractive and weakly inward random operator. Hence T_y has a random fixed point $\xi_y : \Omega \to X$ [20, Theorem 4]. Now define $f : \Omega \times Y \to F$ by $f(\omega, y) = P_2 o T(\omega, \xi_y(\omega), y)$, for $\omega \in \Omega$, $y \in Y$. Then it is easy to see that f is strictly contractive and weakly inward random operator. Therefore, f has a random fixed point $\eta : \Omega \to Y$. Hence $T(\omega, \xi_{\eta(\omega)}(\omega), \eta(\omega)) = (\xi_{\eta(\omega)}(\omega), \eta(\omega))$ for every $\omega \in \Omega$.

Corollary 4.6:   Let E and F be two Hilbert spaces with $X \subset E$ a separable closed bounded convex and $Y \subset F$ separable. Suppose Y has the random fixed point property for nonexpansive and weakly inward random operators. Then $(X \oplus Y)_2$ also has the random fixed point property for nonself nonexpansive and weakly inward random operators.

Proof: Similar to the proof of Theorem 4.5. Only need to notice that the corresponding T_y is nonexpansive and has the random fixed point $\xi_y : \Omega \to X$ from [20, Theorem 6].

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