ON SOLVABILITY OF MIXED MONOTONE OPERATOR EQUATIONS WITH APPLICATIONS TO MIXED QUASIMONOTONE DIFFERENTIAL SYSTEMS INVOLVING DISCONTINUITIES[1]

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ABSTRACT

In this paper we shall first study solvability of mixed monotone systems of operator equations in an ordered normed space by using a generalized iteration method. The obtained results are then applied to prove existence of coupled extremal quasisolutions of the systems of first and second order mixed quasimonotone differential equations with discontinuous right hand sides. Most of the results deal with systems in a Banach space ordered by a regular order cone.

Key words: Operator equations, systems of differential equations in ordered Banach spaces, discontinuous nonlinearity, mixed quasimonotonicity, generalized iteration method.

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1. INTRODUCTION

In [6] existence of coupled minimal and maximal quasisolutions of the system

$$u_i'(t) = f_i(t, u_i(t), [u]_{p_i}(t), [u]_{q_i}(t)), \quad t \in [0, T], \quad u_i(0) = u_{0_i}, \quad i = 1, \ldots, n \qquad (1)$$

is studied by assuming that the functions f_i are continuous, mixed quasimonotone and satisfy one-sided Lipschitz condition with respect to u_i. The last assumption is replaced in [5] by g_i-monotonicity of the functions f_i.

In this paper we shall extend the above mentioned results to the case when the functions f_i are allowed to be discontinuous in all of their variables. These extensions will be

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obtained as applications of a result proved first for operator equations of mixed monotone type in an ordered normed space. This result will be applied also to the second order mixed quasimonotone system

$$u_i''(t) = f_i(t, u_i(t), [u]_{p_i}(t), [u]_{q_i}, u_i'(t)), \quad t \in [0, T],$$

$$u_i(0) = u_{0_i}, \quad u_i'(0) = u_{1_i}$$

(2)

in an ordered Banach space.

2. ON MIXED MONOTONE OPERATOR EQUATIONS

In this section we shall study solvability of the operator equations

$$v = A(v, w), \quad w = A(w, v)$$

(2.1)

in an ordered normed space X.

Theorem 2.1:   *Given a subset Z of X possessing the least and the greatest element, and a mapping $A: Z \times Z \rightarrow Z$, assume that*

(A1) $A(\cdot, z)$ *is increasing and $A(z, \cdot)$ is decreasing for each $z \in Z$.*

(A2) $(A(v_j, w_j))_{j=1}^{\infty}$ *converges in Z whenever $(v_j)_{j=1}^{\infty}$ and $(w_j)_{j=1}^{\infty}$ are sequences in Z, one being increasing and the other decreasing.*

Then the system (2.1) has a solution (v, w) such that

(i) $v \leq w$ *and $\bar{v}, \bar{w} \in [v, w]$ for any other solution $(\bar{v}, \bar{w})$ of (2.1).*

Proof: Note first that $X \times X$, equipped with componentwise addition and scalar multiplication, is an ordered normed space with respect to the norm and the partial ordering given by

$$\| (v, w) \| = \| v \| + \| w \| \quad \text{and} \quad (v, w) \leq (\bar{v}, \bar{w}) \text{ if and only if } v \leq \bar{v} \text{ and } \bar{w} \leq w. \quad (a)$$

Denoting $Y = Z \times Z$ it follows from (A1) that the equation

$$F(v, w) = (A(v, w), A(w, v))$$

(b)

defines an increasing mapping $F: Y \rightarrow Y$. If $(u_j)_{j=1}^{\infty} = ((v_j, w_j))_{j=1}^{\infty}$ is an increasing sequence in Y, then $(v_j)_{j=1}^{\infty}$ is increasing and $(w_j)_{j=1}^{\infty}$ decreasing in Z. The hypothesis (A2) implies that the sequences $(A(v_j, w_j))_{j=1}^{\infty}$ and $(A(w_j, v_j))_{j=1}^{\infty}$ converge in Z. Thus, by (a) and (b), the sequence $(Fu_j)_{j=1}^{\infty}$ converges in Y.

The above proof shows that the mapping $F: Y \rightarrow Y$ given by (b) is increasing and right compact. Moreover, since $v_0 = minZ$ and $w_0 = maxZ$ exist, then $a = (v_0, w_0)$ is the least

element of Y. Thus, F has by Theorem 3.1 of [2] the least fixed point $u = (v, w)$ in Y. Moreover, $u = sup\{Fu_\alpha \mid \alpha \in \Lambda\}$, where $(u_\alpha)_{\alpha \in \Lambda}$ is the longest transfinite sequence, indexed by ordinals, which satisfies

$$u_0 = a, \text{ and if } 0 < \alpha \in \Lambda \text{ then } u_\alpha = sup_{\beta < \alpha} Fu_\beta \text{ and } u_\alpha < Fu_\alpha.$$

The above results imply by (a) and (b) that (v, w) is a solution of (2.1), and that $v = sup\{A(v_\alpha, w_\alpha) \mid \alpha \in \Lambda\}$ and $w = inf\{A(w_\alpha, v_\alpha) \mid \alpha \in \Lambda\}$, where $(v_\alpha)_{\alpha \in \Lambda}$ and $(w_\alpha)_{\alpha \in \Lambda}$ are the longest transfinite sequences which satisfy

(I) If $0 < \alpha \in \Lambda$ then $v_\alpha = sup_{\beta < \alpha} A(v_\beta, w_\beta)$, $w_\alpha = inf_{\beta < \alpha} A(w_\beta, v_\beta)$, and the

inequalities $v_\alpha \le A(v_\alpha, w_\alpha)$, $A(w_\alpha, v_\alpha) \le w_\alpha$ hold, at least one of them being strict.

If $(\overline{v}, \overline{w})$ is a solution of (2.1), it follows from (2.1) and (b) that $(\overline{v}, \overline{w})$ is a fixed point of F in Y. Because (v, w) is the least one, then $(v, w) \le (\overline{v}, \overline{w})$, i.e. $v \le \overline{v}$ and $\overline{w} \le w$. In particular, we can choose $\overline{v} = w$ and $\overline{w} = v$, whence $v \le w$. Since $(\overline{w}, \overline{v})$ is also a solution of (2.1), it follows that $\overline{v}, \overline{w} \in [v, w]$, so that (i) holds.

Remark 2.1:   If A in Theorem 2.1 is continuous, then also F is continuous, whence the sequence $(u_\alpha)_{\alpha \in \Lambda}$ is reduced to a finite or infinite sequence of ordinary iterations $F^j a$, $j \in \mathbb{N}$. In particular, $v = \lim_{j \to \infty} v_j$ and $w = \lim_{j \to \infty} w_j$, where $(v_j)_{j=0}^\infty$ and $(w_j)_{j=0}^\infty$ are defined by

$$v_{j+1} = A(v_j, w_j), \ w_{j+1} = A(w_j, v_j), \ j \in \mathbb{N}. \tag{2.2}$$

3. APPLICATIONS TO FIRST ORDER DIFFERENTIAL SYSTEMS

Given an ordered Banach space E and $J = [0, T]$, consider the differential system

$$u_i'(t) = f_i(t, u_i(t), [u]_{p_i}(t), [u]_{q_i}(t)) \text{ for a.e. } t \in J, \ u_i(0) = u_{0_i}, \ i = 1, \ldots, n, \tag{3.1}$$

where $f_i: J \times E^n \to E$, and p_i, $q_i \in \mathbb{N}$, $i = 1, \ldots, n$ satisfy $p_i + q_i = n - 1$. In the notation $u = (u_i, [u]_{p_i}, [u]_{q_i})$ for $u \in E^n$ the term $[u]_{p_i}$ is formed by p_i coordinates of u, different from u_i, and $[u]_{q_i}$ contains the remaining ones.

The absolutely continuous and a.e. differentiable functions $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ from J to E^n are called *coupled lower and upper quasisolutions* of (3.1), if

$$v_i'(t) \le f_i(t, v_i(t), [v]_{p_i}(t), [w]_{q_i}(t)) \text{ for a.e. } t \in J, \ v_i(0) \le u_{0_i}, \ i = 1, \ldots, n, \tag{3.2}$$
and

$$w_i'(t) \geq f_i(t, w_i(t), [w]_{p_i}(t), [v]_{q_i}(t)) \text{ for a.e. } t \in J, \ w_i(0) \geq u_{0_i}, \ i = 1,\ldots,n. \tag{3.3}$$

If the equalities hold in (3.2) and in (3.3), we say that v, w are *coupled quasisolutions* of (3.1). If $v = w = u$, we say that u is a lower, an upper or an ordinary solution of (3.1), respectively. In the following, we shall equip $C(J, E^n)$ with the maximum norm and pointwise ordering, and L_p-spaces with p-norms and *a.e.* pointwise orderings.

3.1. g-monotone and mixed quasimonotone real systems.

Consider first the case when $E = \mathbb{R}$. Given $f_i: J \times \mathbb{R}^n \to \mathbb{R}$ and $u_{0_i} \in \mathbb{R}$, $i = 1,\ldots,n$, assume that

($f1$) (3.1) has coupled lower and upper quasisolutions v_0, w_0 such that $v_0 \leq w_0$.

($f2$) $f_i(\cdot, u(\cdot))$ is Lebesgue measurable on J and whenever $u \in [v_0, w_0]$ is absolutely continuous.

($f3$) $f_i(t, u_i[u]_{p_i}, [u]_{q_i})$ is increasing with respect to $[u]_{p_i}$ on $[[v_0]_{p_i}(t), [w_0]_{p_i}(t)]$ and decreasing with respect to $[u_{q_i}]$ on $[[v_0]_{q_i}(t), [w_0]_{q_i}(t)]$ for all $u_i \in [v_{0_i}(t), w_{0_i}(t)]$ and for *a.e.* $t \in J$.

($f4$) There is a Borel measurable function $g_i \in L_\infty(\mathbb{R}, \mathbb{R}_+)$ with positive essential infimum such that $u_i \mapsto \dfrac{f_i(t, u_1, \ldots, u_i, \ldots, u_n)}{g_i(u_i)}$ is increasing in $[v_{0_i}(t), w_{0_i}(t)]$ for all $u = (u_1, \ldots, u_n) \in [v_0(t), w_0(t)]$ and for *a.e.* $t \in J$.

Theorem 3.1: *If the hypotheses* $(f1)-(f4)$ *hold for each* $i = 1,\ldots,n$, *then* (3.1) *has coupled quasisolutions* v, w *satisfying* $v_0 \leq v \leq w \leq w_0$, *such that* $\bar{v}, \bar{w} \in [v, w]$ *whenever* $\bar{v}, \bar{w} \in [v_0, w_0]$ *are coupled quasisolutions of* (3.1).

Proof: Denote $Z = \{z \in [v_0, w_0] \mid z \text{ is absolutely continuous}\}$, and let $i \in \{1,\ldots,n\}$ be given. The hypotheses $(f1)-(f4)$ imply that for all $v, w \in Z$

$$\int_0^t \frac{f_i(x, v_i(x), [v]_{p_i}(x), [w]_{q_i}(x))}{g_i(v_i(x))} dx \leq \int_0^t \frac{f_i(x, w_{0_i}(x), [w_0]_{p_i}(x), [v_0]_{q_i}(x))}{g_i(w_{0_i}(x))} dx, \ t \in J.$$

From (3.3) it follows (see also [7], V. 38.3) that

$$\int_0^t \frac{f_i(x, w_{0_i}(x), [w_0]_{p_i}(x), [v_0]_{q_i}(x))}{g_i(w_{0_i}(x))} dx \leq \int_0^t \frac{w_{0_i}'(x)}{g_i(w_{0_i}(x))} dx \leq \int_{u_{0_i}}^{w_{0_i}(t)} \frac{dy}{g_i(y)}, \ t \in J.$$

The reverse inequalities hold when w_{0_i} is replaced by v_{0_i} and vice versa.

Thus, the equation

$$\int_{u_{0_i}}^{A_i(v,w)(t)} \frac{dy}{g_i(y)} = \int_0^t \frac{f_i(x,v_i(x),[v]_{p_i}(x),[w]_{q_i}(x))}{g_i(v_i(x))} dx, \ t \in J \qquad (a)$$

defines a mapping $A_i: Z \times Z \to C(J,\mathbb{R})$ such that $v_{0_i} \leq A_i(v,w) \leq w_{0_i}$. From (a) it follows that $A_i(v,w)$ is absolutely continuous on J for each $(v,w) \in Z \times Z$. The hypotheses $(f3)$ and $(f4)$ ensure that $A_i(\cdot,z)$ is increasing and $A_i(z,\cdot)$ is decreasing for each $z \in Z$.

The above considerations hold for each $i = 1,\ldots,n$, so that the equations (a) define a mapping $A = (A_1,\ldots,A_n): Z \times Z \to Z$ which satisfies condition $(A1)$ of Theorem 2.1. To prove that condition $(A2)$ holds, let $(v_j)_{j=1}^{\infty}$ be an increasing and $(w_j)_{j=1}^{\infty}$ a decreasing sequence in Z. For given $i \in \{1,\ldots,n\}$ the functions

$$h_j^i(x) = \frac{f_i(x,v_{j_i}(x),[v_j]_{p_i}(x),[w_j]_{q_i}(x))}{g_i(v_{j_i}(x))}, \ x \in J \qquad (b)$$

form an increasing sequence $(h_j^i)_{j=1}^{\infty}$ in $L^1(J,\mathbb{R})$, bounded above by a function

$$h_0^i(x) = \frac{f_i(x,w_{0_i}(x),[w_0]_{p_i}(x),[v_0]_{q_i}(x))}{g_i(w_{0_i}(x))}, \ x \in J.$$

Thus, the sequence $(h_j^i)_{j=1}^{\infty}$ converges by the Monotone Convergence Theorem to a function $h^i \in L_1(J,\mathbb{R})$, and

$$\lim_{j\to\infty} \int_0^t h_j^i(x)dx = \int_0^t h^i(x)dx, t \in J. \qquad (c)$$

Defining $u_i: J \to \mathbb{R}$ by

$$\int_{u_{0_i}}^{u_i(t)} \frac{dy}{g_i(y)} = \int_0^t h^i(x)dx, \ t \in J, \qquad (d)$$

and denoting $M_i = ess \ sup \ g_i$, it follows from (a), (b) and (d) that

$$0 \leq \frac{1}{M_i}(u_i(t) - A_i(v_j,w_j)(t)) \leq \int_{A_i(v_j,w_j)(t)}^{u_i(t)} \frac{dy}{g_i(y)} \leq \int_0^T (h^i(x) - h_j^i(x))dx, \ t \in J.$$

This and (c) imply that the sequence $(A_i(v_j,w_j))_{j=1}^{\infty}$ converges uniformly on J to u_i. Thus, denoting $u = (u_1,\ldots,u_n)$, then $A(v_j,w_j) \to u$ in $X = C(J,\mathbb{R}^n)$ with respect to the uniform norm. Moreover, it is easy to see that $u \in Z$, whence $(A(v_j,w_j))_{j=1}^{\infty}$ converges in Z. This convergence can be proved similarly also in the case $(v_j)_{j=1}^{\infty}$ is a decreasing and $(w_j)_{j=1}^{\infty}$ an increasing sequence in Z. Thus, A satisfies the condition $(A2)$ of Theorem 2.1. Noting also that v_0 is the least and w_0 is the greatest element of Z, it follows from Theorem 2.1 that (2.1) has a solution (v,w) such that (i) holds.

Denoting $v = (v_1,...,v_n)$, and $w = (w_1,...,w_n)$, it follows from (2.1) that $v_i = A_i(v,w)$ for each $i = 1,...,n$. Thus, (a) is equivalent to

$$\int_{u_{0_i}}^{v_i(t)} \frac{dy}{g_i(y)} = \int_0^t \frac{v_i'(x)dx}{g_i(v_i(x))} = \int_0^t \frac{f_i(x,v_i(x),[v]_{p_i}(x),[w]_{q_i}(x))}{g_i(v_i(x))}dx, \quad t \in J.$$

This is in turn equivalent to

$$\frac{v_i'(t)}{g_i(v_i(t))} = \frac{f_i(t,v_i(t),[v]_{p_i}(t),[w]_{q_i}(t))}{g_i(v_i(t))}, \quad \text{for a.e. } t \in J, \ v_i(0) = u_{0_i},$$

and hence to

$$v_i'(t) = f_i(t,v_i(t),[v]_{p_i}(t),[w]_{q_i}(t)) \text{ for a.e. } t \in J, \ v_i(0) = u_{0_i}, \ i = 1,...,n.$$

Similarly, it can be shown that

$$w_i'(t) = f_i(t,w_i(t),[w]_{p_i}(t),[v]_{q_i}(t)) \text{ for a.e. } t \in J, \ w_i(0) = u_{0_i}, \ i = 1,...,n.$$

Thus, v and w are coupled quasisolutions of (3.1) and $v_0 \le v \le w \le w_0$.

If $\bar{v}$, $\bar{w}$ are coupled quasisolutions of (3.1) in $[v_0,w_0]$, then they belong to Z, whence the above equivalences imply that $(\bar{v},\bar{w})$ is a solution of (2.1) when the components of A are defined by (a). This and the result (i) of Theorem 2.1 imply that $\bar{v}$, $\bar{w} \in [v,w]$.

As a consequence of Theorem 3.1 we obtain,

Proposition 3.1: *Given the functions* $g_i \in L_\infty(\mathbb{R},\mathbb{R}_+)$ *and* $h_i: J \times \mathbb{R}^n \to \mathbb{R}$, $i = 1,...,n$, *assume that for each* $i = 1,...,n$, g_i *has positive essential infimum, that* h_i *is Borel measurable, that* $h_i(t,\cdot)$ *is increasing on* $\mathbb{R}^n$ *for a.e.* $t \in J$, *and that there is* $m_i \in L_1(J,\mathbb{R}_+)$ *such that*

$$|h_i(t,u)| \le m_i(t) \text{ for all } u \in \mathbb{R}^n \text{ and for a.e. } t \in J. \tag{3.4}$$

Then the system

$$u_i'(t) = g_i(u_i(t))h_i(t,u(t)) \text{ for a.e. } t \in J, \ u_i(0) = u_{0_i}, \ i = 1,...,n \tag{3.5}$$

has for each $(u_{0_1},...,u_{0_n}) \in \mathbb{R}_+^n$ *the least and the greatest solution.*

Proof: The assumptions given for g_i and condition (3.4) imply that the equations

$$\int_{u_{0_i}}^{w_{0_i}(t)} \frac{dy}{g_i(y)} = \int_0^t m_i(x)dx, \quad \int_{u_{0_i}}^{v_{0_i}(t)} \frac{dy}{g_i(y)} = -\int_0^t m_i(x)dx \tag{a}$$

define an upper solution $w_0 = (w_{0_1},...,w_{0_n})$ and a lower solution $v_0 = (v_{0_1},...,v_{0_n})$ of the system (3.5). Moreover, it is easy to see that the functions $f_i = g_i h_i$ satisfy the conditions

$(f2) - (f4)$ when $p_i = n - 1$ and $q_i = 0$, $i = 1,...,n$. (Borel measurability of g_i is not needed since $\frac{f_i}{g_i} = h_i$. As for the proof of $(f2)$ see [8]). Thus, Theorem 3.1, applied to this special case, implies that the system (3.5) has the least solution v and the greatest solution w between v_0 and w_0.

If $u = (u_1,...,u_n)$ is any solution of (3.5), then its components are absolutely continuous in J. Thus, it follows from (3.5) that

$$\int_{u_{0_i}}^{u_i(t)} \frac{dy}{g_i(y)} = \int_0^t h_i(x,u(x))dx, \ t \in J, \ i = 1,...,n. \tag{b}$$

From (3.4), (a) and (b) we obtain

$$v_{0_i}(t) \le u_i(t) \le w_{0_i}(t) \text{ for all } t \in J.$$

Thus, $v_0 \le u \le w_0$, so that v is the least and w is the greatest of all the solutions of (3.5).

3.2. Mixed quasimonotone systems in an ordered Banach space.

Next we shall give an extension of Theorem 1.4.1 of [6] to the case when $f = (f_1,...,f_n)$ is a mapping from $J \times E^n$ to E^n, where E is an ordered Banach space, by allowing f also to possess discontinuities.

Theorem 3.2: *Let E be an ordered Banach space with regular order cone, and let $f_i: J \times E^n \to E$ and $u_{0_i} \in E$, $i = 1,...,n$ satisfy the conditions $(f1)$, $(f3)$,*

$(f5)$ $f_i(\cdot, u(\cdot))$ is strongly measurable whenever $u: J \to E^n$ is absolutely continuous and a.e. differentiable, and

$(f6)$ there is $\mu_i \in L_1(J, \mathbb{R}_+)$ such that $u_i \mapsto f_i(t, u_1,..., u_i,...,u_n) + \mu_i(t)u_i$ is increasing on $[v_{0_i}(t), w_{0_i}(t)]$ for all $u = (u_1,...,u_n) \in [v_0(t), w_0(t)]$ and for a.e. $t \in J$.

Then the results of Theorem 3.1 hold.

Proof: Denote $Z = \{z \in [v_0, w_0] \,|\, z$ is absolutely continuous and a.e. differentiable$\}$. The given hypotheses imply that the equation

$$\lambda_i(v,w)(t) = f_i(t, v_i(t), [v]_{p_i}(t), [w]_{q_i}(t)) + \mu_i(t)v_i(t), \ t \in J \tag{a}$$

defines for each $i = 1,...,n$ a mapping $\lambda_i: Z \times Z \to L_1(J,E)$, which is increasing in the first argument and decreasing in the second one. Thus, the initial value problem

$$u_i'(t) + \mu_i(t)u_i(t) = \lambda_i(v,w)(t) \text{ for a.e. } t \in J, \ u_i(0) = u_{0_i} \tag{b}$$

has for each $i = 1,\ldots,n$ and $v,w \in Z$ a unique absolutely continuous solution $u_i = A_i(v,w)$ given by

$$A_i(v,w)(t) = exp(-\int_0^t \mu_i(x)dx)(u_{0_i} + \int_0^t exp(\int_0^x \mu_i(s)ds)\lambda_i(v,w)(x)dx), \quad t \in J. \qquad (c)$$

The so obtained functions $A_i: Z \times Z \to C(J,E)$ have absolutely continuous and $a.e.$ differentiable values, and they possess the mixed monotonicity conditions stated for the functions λ_i above. Moreover, from (3.2), (a) and (b) it follows when $v = v_0$ and $w = w_0$ that $r_i = u_i - v_{0_i}$ satisfies for each $i = 1,\ldots,n$

$$r_i'(t) + \mu_i(t)r_i(t) \geq 0 \text{ for } a.e. \ t \in J, \ r_i(0) \geq 0,$$

which implies that

$$r_i(t) \geq r_i(0)exp(-\int_0^t \mu_i(x)dx) \geq 0 \text{ for each } t \in J.$$

Thus, $v_{0_i} \leq A_i(v_0,w_0)$, $i = 1,\ldots,n$. By the analogous reasoning it can be shown that $A_i(w_0,v_0) \leq w_{0_i}$, $i = 1,\ldots,n$.

The above results imply that $A = (A_1,\ldots,A_n)$ is a mapping from $Z \times Z$ to Z, and satisfies condition $(A1)$ of Theorem 2.1.

Assume now that $(v_j)_{j=1}^\infty$ is an increasing and $(w_j)_{j=1}^\infty$ a decreasing sequence in Z. The sequences $(\lambda_i(v_j,w_j)(t))_{j=1}^\infty$, $t \in J$ are increasing and are contained in the order interval $[\lambda_i(v_0,w_0)(t),\lambda_i(w_0,v_0)(t)]$, whence they converge, because the order cone K of E is regular. Denote

$$h^i(t) = \lim_{j\to\infty} \lambda_i(v_j,w_j)(t), \quad t \in J.$$

Since K is also normal, there is a positive constant c such that

$$\| \lambda_i(v_j,w_j)(t) \| \leq (1+c)(\| \lambda_i(v_0,w_0)(t) \| + \| \lambda_i(w_0,v_0)(t) \|) \text{ for } a.e. \ t \in J.$$

Thus, $h^i \in L_1(J,E)$ by the Dominated Convergence Theorem for Bochner Integrals (cf. [4]), and

$$\lim_{j\to\infty} \int_0^t \lambda_i(v_j,w_j)(x)dx = \int_0^t h^i(x)dx, \quad t \in J. \qquad (d)$$

Defining $\nu_i: J \to E$ by

$$\nu_i(t) = exp(-\int_0^t \mu_i(x)dx)(u_{0_i} + \int_0^t exp(\int_0^x \mu_i(s)ds)h^i(x)dx), \quad t \in J, \qquad (e)$$

it follows from (c), (d) and (e) that

$$\lim_{j\to\infty} A_i(v_j,w_j)(t) = \nu_i(t) \ t \in J. \qquad (f)$$

Moreover,

$$0 \leq \nu_i(t) - A_i(v_j, w_j)(t) \leq exp(\int\limits_0^T \mu_i(x)ds) \int\limits_0^T (h^i(x) - \lambda_i(v_j, w_j)(x))dx, \; t \in J,$$

which implies by (d) and the normality of K that the convergence in (f) is uniform on J. Thus, denoting $\nu = (\nu_1,...,\nu_n)$, then $A(v_j, w_j) \to \nu$ in $C(J, E^n)$. Obviously, $\nu \in Z$, whence $(A(v_j, w_j))_{j=1}^\infty$ converges in Z. This convergence in the case when $(v_j)_{j=1}^\infty$ is decreasing and $(w_j)_{j=1}^\infty$ is increasing is proved similarly. Thus, A satisfies the condition $(A2)$ of Theorem 2.1, whence (2.1) has a solution (v, w) such that (i) holds.

Denoting $v = (v_1,...,v_n)$, and $w = (w_1,...,w_n)$, it follows from (2.1) and (c) that

$$v_i(t) = exp(-\int\limits_0^t \mu_i(x)dx)(u_{0_i} + \int\limits_0^t exp(\int\limits_0^x \mu_i(s)ds)\lambda_i(v, w)(x)dx), \; t \in J.$$

This is easily seen to be equivalent to

$$v_i'(t) + \mu_i(t)v_i(t) = \lambda_i(v, w)(t) \text{ for a.e. } t \in J, \; v_i(0) = u_{0_i},$$

and hence to

$$v_i'(t) = f_i(t, v_i(t), [v]_{p_i}(t), [w]_{q_i}(t)) \text{ for a.e. } t \in J, \; v_i(0) = u_{0_i}, \; i = 1,...,n.$$

Similarly, it can be shown that

$$w_i'(t) = f_i(t, w_i(t), [w]_{p_i}(t), [v]_{q_i}(t)) \text{ for a.e. } t \in J, \; w_i(0) = u_{0_i}, \; i = 1,...,n.$$

Thus, v and w are coupled quasisolutions of (3.1) and $v_0 \leq v \leq w \leq w_0$.

If $\bar{v}, \bar{w}$ are coupled quasisolutions of (3.1) in $[v_0, w_0]$, then they belong to Z, whence the above equivalences imply that $(\bar{v}, \bar{w})$ is a solution of (2.1) when the components of A are defined by (c). Thus, the result (i) of Theorem 2.1 ensures that $\bar{v}, \bar{w} \in [v, w]$.

Lemma 3.1: *The hypothesis $(f1)$ holds in Theorem 3.2 for all fixed $u_{0_1},...,u_{0_n} \in E$, if for each $i = 1,...,n$ there exist $a_i^j \in L_1(J, \mathbb{R})$ and $b_i^j \in L_1(J, E)$, $j = 1, 2$ such that*

$$a_i^1(t)u_i + b_i^1(t) \leq f_i(t, u) \leq a_i^2(t)u_i + b_i^2(t) \text{ for a.e. } t \in J \text{ and for all } u \in E^n. \quad (3.6)$$

Proof: Let $u_{0_1},...,u_{0_n} \in E$ be given. Direct calculations show that the equations

$$v_{0_i}(t) = exp(\int\limits_0^t a_i^1(x)dx)(u_{0_i} + \int\limits_0^t exp(-\int\limits_0^x a_i^1(s)ds)b_i^1(x)dx), \; t \in J, \; i = 1,...,n \quad (3.7)$$

and

$$w_{0_i}(t) = exp(\int_0^t a_i^2(x)dx)(u_{0_i} + \int_0^t exp(-\int_0^x a_i^2(s)ds)b_i^2(x)dx), \ t \in J, \ i = 1,...,n \qquad (3.8)$$

define coupled lower and upper quasisolutions $v_0 = (v_{0_1},...,v_{0_n})$ and $w_0 = (w_{0_1},...,w_{0_n})$ of the system (3.1) such that $v_0 \le w_0$.

In the special case when $p_i = n - 1$ and $q_i = 0$, $i = 1,...,n$ we obtain from Theorem 3.2 the following result for the system

$$u_i'(t) = f_i(t, u(t)) \text{ for a.e. } t \in J, \ u_i(0) = u_{0_i}, \ i = 1,...,n, \qquad (3.9)$$

which generalizes Theorems 3.1 and 3.2 of [1].

Proposition 3.2:   *Given an ordered Banach space E with regular order cone, assume that the functions $f_i: J \times E^n \to E$ satisfy the hypotheses of Lemma 3.1, that $f_i(\cdot, u(\cdot))$ is strongly measurable whenever $u: J \to E^n$ is absolutely continuous and a.e. differentiable, and that there is $\mu_i \in L_1(J, \mathbb{R}_+)$ such that $u \mapsto f_i(t, u) + \mu_i(t)u_i$ is increasing for a.e. $t \in J$ and for each $i = 1,...,n$. Then the system (3.9) has for each choice of $u_{0_1},...,u_{0_n} \in E$ the least and the greatest solution.*

Proof: Given $u_{0_1},...,u_{0_n} \in E$, it is easy to see that hypotheses of Theorem 3.2 hold with v_0, w_0 defined by (3.7) − (3.8). Thus, the system (3.9) has the least solution v and the greatest solution w between v_0 and w_0.

If $u = (u_1,...,u_n)$ is any solution of (3.9), it follows from (3.6), (3.7) and (3.9) that $r_i = u_i - v_{0_i}$ satisfies for each $i = 1,...,n$

$$r_i'(t) - a_i^1(t)r_i(t) \ge 0 \text{ for a.e. } t \in J, \ r_i(0) = 0,$$

which implies that

$$r_i(t) \ge r_i(0)exp(\int_0^t a_i^1(x)dx) = 0 \text{ for each } t \in J.$$

Thus, $v_0 \le u$. Similarly it can be shown that $u \le w_0$, whence all the solutions of (3.9) are between v_0 and w_0. Consequently, v is the least and w the greatest of all the solutions of the system (3.9).

Remarks 3.1:   Condition $(f2)$ holds, for instance when f_i is a "standard function" in the sense defined in [8], and in particular, when f_i is Borel measurable. If each h_i in Proposition 3.1 is standard, then the assumption on the Borel measurability of h_i is not needed.

Condition $(f5)$ of Theorem 3.2 holds, if E is separable and each f_i is a standard

function (cf. [8]).

If in Theorems 3.1 and 3.2 the functions g_i are continuous and each f_i is a Carathéodory function, it follows from Remark 2.1 that the asserted coupled quasisolutions v and w of (3.1) are obtained as the uniform limits of the successive approximations given by (2.2). Moreover, if the functions f_i and μ_i are continuous then v, w are coupled extremal quasisolutions of the system (1) given in Introduction. Similar special cases are obtained also for Propositions 3.1 and 3.2.

Theorem 3 in [9] contains the special case of Theorem 3.2 where $E = \mathbb{R}$ and the functions f_i are mixed monotone, i.e. each μ_i is the zero function in the condition ($f6$).

4. APPLICATIONS TO SECOND ORDER DIFFERENTIAL SYSTEMS

Given an ordered Banach space E, consider the differential system

$$u_i''(t) = f_i(t, u_i(t), [u]_{p_i}(t), [u]_{q_i}(t), u_i'(t)) \text{ for a.e. } t \in J,$$
$$u_i(0) = u_{0_i}, \ u_i'(0) = u_{1_i}, \ i = 1,\dots,n, \tag{4.1}$$

where $f_i: J \times E^{n+1} \to E$, $i = 1,\dots,n$. The functions $v = (v_1,\dots,v_n)$ and $w = (w_1,\dots,w_n): J \to E^n$ are said to be *coupled lower and upper quasisolutions* of (4.1) if their components possess absolutely continuous and *a.e.* differentiable first derivatives, and if

$$v_i''(t) \le f_i(t, v_i(t), [v]_{p_i}(t), [w]_{q_i}(t), v_i'(t)) \text{ for a.e. } t \in J,$$
$$v_i(0) \le u_{0_i}, \ v_i'(0) \le u_{1_i}, \ i = 1,\dots,n, \tag{4.2}$$

and

$$w_i''(t) \ge f_i(t, w_i(t), [w]_{p_i}(t), [v]_{q_i}(t), w_i'(t)) \text{ for a.e. } t \in J,$$
$$w_i(0) \ge u_{0_i}, \ w_i'(0) \ge u_{1_i}, \ i = 1,\dots,n. \tag{4.3}$$

If the equalities hold in (4.2) and in (4.3), we say that v, w are *coupled quasisolutions* of (4.1). In particular, if $v = w = u$, we say that u is a lower, an upper or an ordinary solution of (4.1), respectively.

Our considerations are based on Theorem 2.1 and the following Lemma, which is a consequence of Example 2.1 and Lemma 6.1 of [3].

Lemma 4.1: *If $\mu \in L_1(J, \mathbb{R})$ and $C \in L_1(J, E)$ then the IVP*

$$u''(t) + \mu(t)u'(t) = C(t) \text{ for a.e. } t \in J, \ u(0) = u_0, \ u'(0) = u_1 \tag{4.4}$$

has for each choice of u_0, $u_1 \in E$ *the unique solution*

$$u(t) = u_0 + \int_0^t exp\left(-\int_0^x \mu(y)dy\right)u_1 dx + \int_0^t \int_0^x exp\left(\int_x^y \mu(s)ds\right)C(y)dydx. \qquad (4.5)$$

Moreover, if v, w *are lower and upper solutions of* (4.4) *then* $v \leq u \leq w$ *and* $v' \leq u' \leq w'$.

Assume now that the functions $f_i: J \times E^{n+1} \to E$ satisfy for each $i = 1,...,n$

(fa)   (4.1) has coupled lower and upper quasisolutions v_0, w_0, such that $v_0 \leq w_0$ and $v_0' \leq w_0'$.

(fb) $f_i(\cdot, u(\cdot), v(\cdot))$ is strongly measurable on J whenever $u \in [v_0, w_0]$, $v \in [v_{0_i}', w_{0_i}']$, and u' and v are absolutely continuous and a.e. differentiable.

(fc) $f_i(t, u_i, [u]_{p_i}, [u]_{q_i}, v)$ is increasing with respect to u_i on $[v_{0_i}(t), w_{0_i}(t)]$ and to $[u]_{p_i}$ on $[[v_0]_{p_i}(t), [w_0]_{p_i}(t)]$, and decreasing with respect to $[u]_{q_i}$ on $[[v_0]_{q_i}(t), [w_0]_{q_i}(t)]$ for all $v \in [v_{0_i}'(t), w_{0_i}'(t)]$ and for a.e. $t \in J$.

(fd)   There is $\mu_i \in L_1(J, \mathbb{R}_+)$ such that $v \mapsto f_i(t, u, v) + \mu_i(t)v$ is increasing on $[v_{0_i}'(t), w_{0_i}'(t)]$ for all $u \in [v_0(t), w_0(t)]$ and for a.e. $t \in J$.

Theorem 4.1:   *Let* E *be an ordered Banach space with regular order cone* K. *If the functions* $f_i: J \times E^{n+1} \to E$ *satisfy the hypotheses* $(fa) - (fd)$, *then the system* (4.1) *has coupled quasisolutions* v, w *satisfying* $(v_0, v_0') \leq (v, v') \leq (w, w') \leq (w_0, w_0')$, *such that* $\bar{v}$, $\bar{w} \in [v, w]$ *and* $\bar{v}'$, $\bar{w}' \in [v', w']$ *whenever* $\bar{v}$, $\bar{w} \in [v_0, w_0]$ *are coupled quasisolutions of* (4.1) *such that* $\bar{v}'$, $\bar{w}' \in [v_0', w_0']$.

Proof: Denote $Z = \{u \in [v_0, w_0] \mid u'$ is absolutely continuous a.e. differentiable and $u' \in [v_0', w_0']\}$, and define a norm and a partial ordering in $X = C_1(J, E^n)$ by

$$\| u \| = sup\{ \| u(t) \| + \| u'(t) \| \mid t \in J\}$$

and

$$u \leq v \text{ if and only if } u(t) \leq v(t) \text{ and } u'(t) \leq v'(t) \text{ for all } t \in J.$$

The given hypotheses imply that the equation

$$C_i(v, w)(t) = f_i(t, v_i(t), [v]_{p_i}(t), [w]_{q_i}(t), v_i'(t)) + \mu_i(t)v_i'(t), \; t \in J \qquad (a)$$

defines for each $i = 1,...,n$ a mapping $C_i: Z \times Z \to L_1(J, E)$, which is increasing in the first argument and decreasing in the second one. Thus, the initial value problem

$$u_i''(t) + \mu_i(t)u_i'(t) = C_i(v, w)(t) \text{ for a.e. } t \in J, \; u_i(0) = u_{0_i}, \; u_i'(0) = u_{1_i} \qquad (b)$$

has by Lemma 4.1 for each $i = 1,...,n$ and $v, w \in Z$ a unique absolutely continuous solution,

$u_i = A_i(v, w)$, given by

$$A_i(v,w)(t) = u_{0_i} + \int_0^t exp(-\int_0^x \mu_i(y)dy)u_{1_i}dx + \int_0^t \int_0^x exp(\int_x^y \mu_i(s)ds)C_i(v,w)(y)dydx. \quad (c)$$

The so obtained functions $A_i(v, w): J \to E$ have absolutely continuous and *a.e.* differentiable first derivatives, and they possess the mixed monotonicity conditions stated for the functions C_i above. Moreover, from (4.2), (a) and (b) it follows when $v = v_0$ and $w = w_0$ that $r = v_{0_i} - u_i$ is for each $i = 1, \ldots, n$ a lower solution of

$$r''(t) + \mu_i(t)r'(t) = 0 \text{ for } a.e. \ t \in J, \ r(0) = r'(0) = 0.$$

Since the zero function is the only solution of this *IVP*, it follows from Lemma 4.1 that $v_{0_i} \le A_i(v_0, w_0)$ and $v'_{0_i} \le A_i(v_0, w_0)'$. Similarly, it follows from Lemma 4.1 that $A_i(w_0, v_0) \le w_{0_i}$ and $A_i(w_0, v_0)' \le w'_{0_i}$ for each $i = 1, \ldots, n$.

The above results imply that $A = (A_1, \ldots, A_n)$ is a mapping from $Z \times Z$ to Z, and satisfies condition $(A1)$ of Theorem 2.1.

Assume now that $(v_j)_{j=1}^{\infty}$ is an increasing and $(w_j)_{j=1}^{\infty}$ a decreasing sequence in Z. The sequences $(C_i(v_j, w_j)(t))_{j=1}^{\infty}$, $t \in J$ are increasing and are contained in the order interval $[C_i(v_0, w_0)(t), C_i(w_0, v_0)(t)]$, whence they converge, because the order cone K of E is regular. Denote

$$h^i(t) = \lim_{j \to \infty} C_i(v_j, w_j)(t), \ t \in J.$$

Since K is also normal, there is a positive constant c such that

$$\| C_i(v_j, w_j)(t) \| \le (1+c)(\| C_i(v_0, w_0)(t) \| + \| C_i(w_0, v_0)(t) \|).$$

Thus, $h^i \in L_1(J, E)$ by Dominated Convergence Theorem for Bochner integrals and

$$\lim_{j \to \infty} \int_0^t C_i(v_j, w_j)(x)dx = \int_0^t h^i(x)dx, \ t \in J. \quad (d)$$

Defining $\nu_i: J \to E$ by

$$\nu_i(t) = u_{0_i} + \int_0^t exp(-\int_0^x \mu_i(y)dy)u_{1_i}dx + \int_0^t \int_0^x exp(\int_x^y \mu_i(s)ds)h^i(y)dydx, \ t \in J,$$

then

$$\nu'_i(t) = exp(-\int_0^t \mu_i(y)dy)u_{1_i} + \int_t^t exp(\int_t^y \mu_i(s)ds)h^i(y)dy, \ t \in J.$$

These equations, together with (c) and (d) imply that

$$\lim_{j \to \infty} A_i(v_j, w_j)(t) = \nu_i(t), \ \lim_{j \to \infty} A_i(v_j, w_j)'(t) = \nu'_i(t) \ t \in J. \quad (e)$$

Moreover,

$$0 \le \nu_i(t) - A_i(v_j, w_j)(t) \le T exp(\int_0^T \mu_i(x)dx) \int_0^T (h^i(x) - C_i(v_j, w_j)(x))dx, \quad t \in J,$$

and

$$0 \le \nu_i'(t) - A_i(v_j, w_j)'(t) \le exp(\int_0^T \mu_i(x)dx) \int_0^T (h^i(x) - C_i(v_j, w_j)(x))dx, \quad t \in J.$$

These inequalities, (d) and the normality of K imply that the convergence in (e) is uniform on J. Thus, denoting $\nu = (\nu_1, ..., \nu_n)$, then $A(v_j, w_j) \to \nu$ in $C_1(J, E^n)$. Obviously, $\nu \in Z$, whence $(A(v_j, w_j))_{j=1}^\infty$ converges in Z. This convergence in the case when $(v_j)_{j=1}^\infty$ is decreasing and $(w_j)_{j=1}^\infty$ is increasing is proved similarly. Thus, A satisfies the condition $(A2)$ of Theorem 2.1, whence (2.1) has a solution (v, w) such that (i) holds.

Denoting $v = (v_1, ..., v_n)$, and $w = (w_1, ..., w_n)$, it follows from (2.1) and (c) that for each $i = 1, ..., n$

$$v_i(t) = u_{0_i} + \int_0^t exp(- \int_0^x \mu_i(y)dy)u_{1_i}dx + \int_0^t \int_0^x exp(\int_x^y \mu_i(s)ds)C_i(v, w)(y)dydx. \quad (g)$$

By Lemma 4.1 this holds if and only if

$$v_i''(t) + \mu_i(t)v_i'(t) = C_i(v, w)(t) \text{ for a.e. } t \in J, \ v_i(0) = u_{0_i}, \ v_i'(0) = u_{1_i}$$

or equivalently,

$$v_i''(t) = f_i(t, v_i(t), [v]_{p_i}(t), [w]_{q_i}(t), v_i'(t)) \text{ for a.e. } t \in J$$

$$v_i(0) = u_{0_i}, \ v_i'(0) = u_{1_i}$$

for each $i = 1, ..., n$. Similarly, it can be shown that

$$w_i''(t) = f_i(t, w_i(t), [w]_{p_i}(t), [v]_{q_i}(t), w_i'(t)) \text{ for a.e. } t \in J$$

$$w_i(0) = u_{0_i}, \ w_i'(0) = u_{1_i}$$

for each $i = 1, ..., n$. Thus, v and w are coupled quasisolutions of (4.1) in Z. The above equivalences imply in turn that such coupled quasisolutions of (4.1) form also a solution of (2.1) with the components of A given by (c). From the definitions of Z and the partial ordering in $X = C_1(J, E^n)$ and from the result (i) in Theorem 2.1, it then follows that $(v_0, v_0') \le (v, v') \le (w, w') \le (w_0, w_0')$, $\bar{v}, \bar{w} \in [v, w]$ and $\bar{v}', \bar{w}' \in [v, w']$ whenever $\bar{v}, \bar{w} \in [v_0, w_0]$ are coupled quasisolutions of (4.1) such that $\bar{v}', \bar{w}' \in [v_0', w_0']$.

Next we shall give some special cases of the above result. Given an ordered Banach

space E with order cone K, denote by $L(E)$ the Banach space of all bounded linear mappings $A: E \to E$ with the norm $\| A \| = sup\{ \| Ax \| \mid \| x \| \le 1\}$. We say that $A \in L(E)$ is *positive* if $A[K] \subset K$. Denote by I the identity mapping of E.

Lemma 4.2:   The hypothesis (fa) holds, for each choice of u_{0_i}, $u_{1_i} \in E$, $i = 1,\ldots,n$, if for each $i = 1,\ldots,n$ there exist A_i^j, $B_i^j \in L_1(J, L(E)), C_i^j \in L_1(J, E)$ and $h_i^j \in L_1(J,\mathbb{R})$, $j = 1,2$ such that each $A_i^j(t) + h_i^j(t)I$ and $B_i^j(t)$ is positive for a.e. $t \in J$, and that

$$A_i^1(t)v + B_i^1(t)u_i + C_i^1(t) \le f_i(t,u,v) \le A_i^2(t)v + B_i^2(t)u_i + C_i^2(t) \text{ for a.e. } t \in J. \quad (4.6)$$

Proof: Let u_{0_i}, $u_{1_i} \in E$, $i = 1,\ldots,n$ be given. From Lemma 6.1 of [3] it follows that the IVP

$$u''(t) = A_i^j(t)u'(t) + B_i^j(t)u(t) + C_i^j(t) \text{ for a.e. } t \in J, \ u(0) = u_{0_i}, \ u'(0) = u_{1_i} \quad (4.7)$$

has for each $i = 1,\ldots,n$ the uniquely determined solution $u = v_{0_i}$ when $j = 1$, and $u = w_{0_i}$ when $j = 2$, and that $v_{0_i} \le w_{0_i}$ and $v'_{0_i} \le w'_{0_i}$ for each $i = 1,\ldots,n$. From (4.6) it follows that $v_0 = (v_{0_1},\ldots,v_{0_n})$ and $w_0 = (w_{0_1},\ldots,w_{0_n})$ are coupled lower and upper quasisolutions of (4.1).

In the case when $p_i = n - 1$ and $q_i = 0$ for each $i = 1,\ldots,n$ we obtain the following result concerning the existence of extremal solutions of the system

$$u_i''(t) = f_i(t,u(t),u_i'(t)) \text{ for a.e. } t \in J,$$
$$u_i(0) = u_{0_i}, \ u_i'(0) = u_{1_i}, \ i = 1,\ldots,n. \quad (4.8)$$

Proposition 4.2:   Given an ordered Banach space E with regular order cone and $f_i: J \times E^{n+1} \to E$, $i = 1,\ldots,n$, assume that the hypotheses of Lemma 4.2 hold, that $f(\cdot, u(\cdot), v(\cdot))$ is strongly measurable whenever u' and v are absolutely continuous and a.e. differentiable on J, and that there is $\mu_i \in L_1(J,\mathbb{R}_+)$ such that $(u,v) \mapsto f_i(t,u,v) + \mu_i(t)v$ is increasing for a.e. $t \in J$ and for each $i = 1,\ldots,n$. Then for each choice of u_{0_i}, $u_{1_i} \in E$, $i = 1,\ldots,n$ the system (4.8) has solutions v and w such that $(v,v') \le (u,u') \le (w,w')$ for all solutions u of (4.8).

Proof: Let u_{0_i}, $u_{1_i} \in E$, $i = 1,\ldots,n$ be given. It is easy to show that the hypotheses of Theorem 4.1 hold with v_0, w_0 given as in the proof of Lemma 4.2. If $u = (u_1,\ldots,u_n)$ is any solution of (4.8), it follows from (4.6) that u_i is for each $i = 1,\ldots,n$ an upper solution of (4.7) when $j = 1$, and a lower solution when $j = 2$. This implies by Lemma

6.1 of [3] that $(v_0, v_0') \leq (u, u') \leq (w_0, w_0')$. This holds for each solution u of (4.8), so that the assertions follow from the results of Theorem 4.1.

Remark 4.1:   If E is separable and each f_i is standard then the condition (fb) and the corresponding hypothesis of Proposition 4.2 hold (cf. [8]). These assumptions hold also without separability of E in the special cases when each f_i is a Carathéodory function, or all the functions in question are continuous.   In these special cases we obtain analogous consequences for systems (4.1), (4.8) and (2) as stated in Remarks 3.1 for first order systems.

The regularity of the order cone K of E is essentially used in the proofs of Theorems 3.2 and 4.1. This holds if, for instance, E is finite-dimensional, or E is a real Hilbert space, and $(x \mid y) \geq 0$ for all $x, y \in K$, or E is reflexive and K is normal. In particular, the nonnegative elements form a regular cone in the L^p-spaces of real-valued functions, defined on any measured space Ω, if $1 \leq p < \infty$. More generally, if K is a regular order cone in E, then the cone $L^p(\Omega, K)$ of a.e. K-valued functions of $L^p(\Omega, E)$ is regular in $L^p(\Omega, E)$. The nonnegative sequences form a regular order cone in l^p-spaces with $1 \leq p < \infty$, and also in c_0.

The above examples of ordered Banach spaces with regular order cone imply that the results of Theorems 3.2 and 4.1 can be applied to finite and infinite systems of first and second order initial value problems, as well as to finite systems of first and second order stochastic initial value problems.

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