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ON THE PARABOLIC POTENTIALS IN DEGENERATE-TYPE HEAT EQUATION[1]

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ABSTRACT

Using distributions theory technique we introduce parabolic potentials for the heat equation with one time-dependent coefficient (not everywhere positive and continuous) at the highest space-derivative, discuss their properties, and apply obtained results to three illustrative problems. Presented technique allows to deal with some equation of the degenerate/mixed type.

Key words: parabolic potentials, variable coefficient, boundary value problems, equations of degenerate/mixed type.

AMS subject classifications: 35K65, 35R05.

1. INTRODUCTION

In this paper we shall study the properties of "parabolic" potentials associated with the boundary value problems in a semi-infinite domain of the following type:

(1) $$L_\alpha u = \frac{\partial u}{\partial t} - \alpha(t)\frac{\partial^2 u}{\partial x^2} = f(x,t),\ x > 0,\ t > 0;$$

(2) $$u(x,0) = \varphi(x),\quad x \geq 0;$$

(3) $$u(0,t) = r(t),\ t \geq 0; \qquad (\varphi(0) = r(0)).$$

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Throughout the paper the coefficient $\alpha(t) \in L_1[0, T]$, is not necessarily positive (which implies that (1) may be of degenerate/mixed type), is defined everywhere in [0,T] and satisfies one of the following conditions:

(i) $\alpha(t) \geq 0$, with equality allowed only at isolated points that do not cluster anywhere in [0, T] ;

(ii) $\alpha_1(t)$ defined by the formula

$$\alpha_1(t) = \int_0^t \alpha(z)\,dz, \quad (\alpha_1(0) = 0)$$

is positive for all $t > 0$, which allows $\alpha(t)$ to be even negative in some intervals. Obviously, any function satisfying (i) is a function of the (ii) type.

It should be noted that in neither case (for different reasons) (1) is reducible to a standard heat operator $u_t - u_{xx}$. The realization of this comes from the relatively obvious substitution of variables [1]:

$$\tau = \int_0^t \alpha(z)\,dz,$$

which in case of (i) implies existence of inverse function $t(\tau)$ with a finite derivative $t'_\tau = 1/\alpha(t)$ at the points where $\alpha(t) \neq 0$. In (ii) case inversion is not possible at all. To get around this obstacle, we derive the fundamental solution, potentials and their properties, and solution of (1)-(3) directly from (1) in its original form.

The boundary S of the domain consists of two parts denoted throughout by $S_1 = \{x \geq 0,\ t = 0\}$ and $S_2 = \{x = 0, t \geq 0\}$. And, finally, M denotes the class of bounded in any strip $(-\infty < x < \infty) \times [0, T]$ functions, vanishing at $t < 0$.

Under condition (ii) the fundamental solution of (1) can be found by applying Fourier transform in x in the form [1]:

$$(4) \qquad E_\alpha(x,t) = E(x, \alpha_1(t)) = \frac{H(t)}{2\sqrt{\pi\,\alpha_1(t)}} \exp(-x^2/4\,\alpha_1(t)),$$

(were H(t) is Heaviside function), provided that $\alpha_1(t) > 0$.

Function (4) has the properties similar to those of standard fundamental solution of heat operator [2], such as

$$(5) \qquad \int_{-\infty}^{+\infty} E_\alpha(x,t)\,dx = 1\,; \quad E_\alpha(x,t) \to \delta(x) \text{ with } t \to 0^+ .$$

Denoting f, u, etc. the functions in (1)-(3) extended as $\equiv 0$ for $x < 0$, $t < 0$, the initial-boundary value problem can be put into generalized form

$$(6) \qquad L_\alpha \bar{u} = \bar{f}(x,t) + [\bar{u}]_{s_1} \cos(\bar{n}, \bar{e}_1)\,\delta_{s_1} - \alpha(t)\left[\frac{\partial \bar{u}}{\partial x}\right]_{s_2} \cos(\bar{n}, \bar{e}_2)\,\delta_{s_2}$$
$$- \frac{\partial}{\partial x}\big(\alpha(t)\,[\bar{u}]_{s_2} \cos(\bar{n}, \bar{e}_2)\,\delta_{s_2}\big) \equiv F(x,t),$$

where $[u]_S$ is a jump of u on $S = S_1 \cup S_2$, n is an external normal to S, e_1, e_2 are unit vectors along t, x -axis respectively and distributions in the form $\mu\delta_S$, $-(\mu\delta_S)'_x$ are single and double layers in terms of [2].

Since the operator L_α contains a non-constant coefficient, it is not immediately clear whether solution of (6) can be found in the form $u = E_\alpha * F$, as in the case of a constant coefficient. However, we still have the following

LEMMA. Under the condition (i) the distributional solution of (6) is unique and can be represented as a convolution of the fundamental solution E_β ("dual" to E_α) with the right-hand side of (6), that is $u = E_\beta * F$, where, as in [1],

$$(7) \qquad E_\beta(x-\xi, t-\tau) = E(x-\xi,\ \beta_1(t-\tau)),$$

and

$$\beta_1(t-\tau) = \int_\tau^t \alpha(z)\,dz = \alpha_1(t) - \alpha_1(\tau)\,; \quad \beta_1(t) = \alpha_1(t).$$

In other words, we treat $\alpha_1(t)$ as if it were time variable in a standard case. Obviously, β_1 is continuous and, due to (i) $\beta_1(t-\tau) > 0$ for $t-\tau > 0$.

Proof. Let condition (i) hold. Then $E_\beta(x-\xi, t-\tau)$ from (7) is a distributional solution of

$$L_\alpha E_\beta(x-\xi, t-\tau) = \frac{\partial E}{\partial t} - \alpha(t)\frac{\partial^2 E}{\partial x^2} = \delta(x-\xi, t-\tau) \quad \text{in } x, t,$$

and

$$L_\alpha^+ E_\beta(x-\xi, t-\tau) = -\frac{\partial E}{\partial \tau} - \alpha(\tau)\frac{\partial^2 E}{\partial \xi^2} = \delta(x-\xi, t-\tau) \quad \text{in } \xi, \tau.$$

Verification can be easily done by the Fourier transform technique. Then, using integration by parts we find that $u = E_\beta * L_\alpha u$, and by the direct differentiation $u = L_\alpha(E_\beta * u)$, which leads to:

$$L_\alpha(E_\beta * \bar{u}) = (L_\alpha E_\beta) * \bar{u} = E_\beta * L_\alpha \bar{u},$$

and the uniqueness of the distributional solution follows immediately, since

$$L_\alpha u = 0 \Rightarrow E_\beta * L_\alpha u = L_\alpha E_\beta * u = \delta * u = u = 0. \qquad \blacksquare$$

Later we also find that in case of f and r in (1)-(3) being zero, the condition (i) here can be relaxed into (ii).

As a result of Lemma, we obtain the following integral representation for the solution of (6) $(x > 0, t > 0)$:

$$\text{(8)} \quad u(x,t) = \int_0^t d\tau \int_0^\infty f(\xi,\tau)\, E_\beta(x-\xi, t-\tau)\, d\xi + \int_0^\infty u(\xi,0)\, E_\beta(x-\xi, t)\, d\xi$$

$$+ \int_0^t \alpha(\tau)\, u(0,\tau)\, \frac{\partial}{\partial \xi}(E_\beta(x-\xi, t-\tau))\Big|_{\xi=0} d\tau - \int_0^t \alpha(\tau)\, \frac{\partial u}{\partial \xi}(0,\tau)\, E_\beta(x, t-\tau)\, d\tau.$$

Formula (8) (see also [2]) motivates the following definition of parabolic potentials associated with the boundary value problem (1)-(3):

a) volume potential

$$(9) \qquad V(x,t) = E_\beta * \tilde{f} = \int_0^t d\tau \int_0^\infty f(\xi,\tau)\, E_\beta(x-\xi, t-\tau)\, d\xi ;$$

b) single-layer potential concentrated on $S_1 = \{ x \geq 0,\ t = 0 \}$

$$(10) \qquad V^{(0)}(x,t) = E_\beta * (\tilde{\varphi}\, \delta_{s_1}) = \int_0^\infty \varphi(\xi)\, E_\beta(x-\xi, t)\, d\xi ;$$

c) single-layer potential concentrated on $S_2 = \{ x = 0, t \geq 0 \}$

$$(11) \qquad V^{(1)}(x,t) = E_\beta * (\alpha\, \mu\, \delta_{s_2}) = \int_0^t \alpha(\tau)\, \mu(\tau)\, E_\beta(x, t-\tau)\, d\tau ;$$

d) double-layer potential concentrated on S_2

$$(12) \qquad W(x,t) = -\frac{\partial}{\partial x}(\alpha\, r\, \delta_{s_2}) * E_\beta = \int_0^t \alpha(\tau)\, r(\tau)\, \frac{\partial}{\partial \xi}(E_\beta(x-\xi, t-\tau))\Big|_{\xi=0} d\tau .$$

2. VOLUME POTENTIAL

Volume potential $V(x,t)$ given by (9) - is a part of a boundary value problem solution that corresponds to the source-function $f(x,t)$.

THEOREM 1. Let $\alpha(t) \in L_1[0,T]$ and satisfy condition (i). Then: (a) for $f \in M$, $V(x,t) \in M$; (b) for $x \geq 0, t \geq 0$ $V(x,t)$ is a distributional solution of (1), satisfying zero initial condition as $t \to 0^+$; (c) if extension $f \in C^2$ for all x and $t \geq 0$ (which in particular implies that $f(0,t) = f_x(0,t) = 0$) and all its derivatives up to the second order belong to M, then $V_{xx}(x,t)$ is continuous in $\{ x \geq 0, t \geq 0 \}$, V_t exists for all x and t , is continuous in x, and its smoothness in t is determined by that of $\alpha(t)$ itself; thus, if in addition $\alpha(t) \in C(R_+)$, then $V(x,t)$ satisfies (1) in the classical sense.

Proof. Introducing in (9) a new variable y ($\beta_1(t-\tau) > 0$ for $t-\tau > 0$)

$$x-\xi = 2y\sqrt{\beta_1(t-\tau)},$$

for $x \geq 0, t \geq 0$ we express $V(x,t)$ in the form

$$\text{(13)}\quad V(x,t) = \frac{1}{\sqrt{\pi}}\int_0^t d\tau \int_{-\infty}^{\frac{x}{2\sqrt{\beta_1(t-\tau)}}} f\left(x-2y\sqrt{\beta_1(t-\tau)};\tau\right)e^{-y^2}\,dy,$$

and its time-derivative ($t > 0$):

$$\text{(14)}\quad \frac{\partial V}{\partial t} = f(x,t)$$

$$-\frac{\alpha(t)}{\sqrt{\pi}}\int_0^t d\tau \int_{-\infty}^{\frac{x}{2\sqrt{\beta_1(t-\tau)}}} f'_{\text{arg.1}}\left(x-2y\sqrt{\beta_1(t-\tau)};\tau\right)\frac{y}{\sqrt{\beta_1(t-\tau)}}\,e^{-y^2}\,dy.$$

Using properties of integrals with parameters, it follows from (13)-(14) that $V(x,t) \in C^2(x \geq 0, t > 0) \cap C^1(x \geq 0, t \geq 0)$ for f and α satisfying conditions (c). At the same time V, being a distributional solution of $L_\alpha V = f$ and sufficiently smooth, is its classical solution (Du Bois Reimond theorem).

Then, since $f \in M$ and E_β (as E_α) satisfies (5),

$$|V(x,t)| \leq \|f\| \int_0^t d\tau \int_{-\infty}^{+\infty} E_\beta\, d\xi \leq t\,\|f\|.$$

It follows immediately that $V \in M$ and satisfies zero initial condition. The rest of (b) can be obtained as in Lemma, since

$$\bar{f} = \delta * \bar{f} = L_\alpha E_\beta * \bar{f} = L_\alpha(E_\beta * \bar{f}) = L_\alpha V. \qquad \blacksquare$$

3. SINGLE-LAYER POTENTIALS

(A) Single-layer potential $V^{(0)}(x,t)$, given by (10), is a part of a solution corresponding to the initial condition (2).

THEOREM 2. Let now the condition (ii) hold. Then: (a) for $\varphi \in M$, $V^{(0)} \in M$; (b) $V^{(0)}$ is a distributional solution of the equation $L_\alpha u = \varphi\delta_{S1}$ and satisfies the initial condition $V^{(0)}(x,t) \to \varphi(x)$ as $t \to 0^+$ for $x > 0$; (c) if extension $\varphi \in C^2$ (which implies that $\varphi(0) = \varphi'(0) = 0$) and its derivatives up to the second order belong to M, then $V^{(0)}{}_{xx}(x, t)$ is continuous in $\{ x \geq 0, t \geq 0 \}$ and $V^{(0)}{}_t$ exists is continuous in x, and its smoothness in t is determined by that of $\alpha(t)$ itself; (d) if in addition $\alpha \in C(R_+)$, then $V^{(0)}(x,t) \in C^2(x \geq 0, t > 0) \cap C(x \geq 0, t \geq 0)$ and, since the support of the distribution $\varphi\delta_{S1}$ is S_1, it follows that $V^{(0)}(x,t)$ is a classical solution of the problem (1)-(2) (with $f \equiv 0$).

Proof is similar to that of Theorem 1 with the substitution of variables in the form: $x - \xi = 2(\alpha_1(t))^{1/2} y$. ■

(B) Single-layer potential $V^{(1)}(x,t)$, given by (11), is a part of a solution, corresponding to the boundary values $u'_x(0, t)$.

THEOREM 3. Let again condition (i) hold. Then: (a) for $\mu \in M$, $V^{(1)}(x,t) \in M$; (b) $V^{(1)}(x,t)$ is a distributional solution of the equation $L_\alpha u = \mu\alpha\delta_{S2}$, $x \geq 0$, $t \geq 0$; satisfies zero initial condition as $t \to 0^+$; (c) if in addition $\alpha \in C(R_+)$ and $\mu' \in M$, then $V^{(1)}(x,t) \in C^{\infty}$ in x and C^1 in t for $x > 0$, $t \geq 0$ and is a classical solution of (1) with $f = \varphi = 0$; (d) $V^{(1)}(x,t)$ is continuous at $x = 0$ for all $t \geq 0$.

Proof. Let us introduce a new variable in (11):

(15) $\quad y = 1/4\,\beta_1(t-\tau)$.

Since $y'_\tau \geq 0$ (= 0 only at isolated points), (15) gives an implicit function $\tau = \tau(t, y)$ with $1/(4\,\beta_1(t)) \leq y < +\infty$ and $\tau = 0$ for $y = 1/(4\,\beta_1(t))$. Then, since $\alpha_1(t) = \beta_1(t)$, (11) can be rewritten in the form:

$$(16)\quad V^{(1)}(x,t) = \frac{1}{4\sqrt{\pi}} \int_{1/4\alpha_1(t)}^{\infty} \mu(\tau(t,y))\, y^{-3/2}\, e^{-x^2 y}\, dy\,.$$

(a) immediately follows from (16) since

$$|\, V^{(1)}(x,t)\,| \leq \frac{1}{\sqrt{\pi}} \,\|\mu\|\, (\alpha_1(t))^{1/2}; \qquad ((\alpha_1(0) = 0).$$

Part (b) can be proved in the way similar to that of Theorem 1, and since

(17) $\quad (V^{(1)}(x,t))'_t = 1/2\, \pi^{-1/2} \mu(0)\, (\alpha_1(t))^{-1/2}\, \alpha(t) \exp(-x^2/4\alpha_1(t)) + V^{(1)}(x,t;\mu'_t)$

(where $V^{(1)}(x,t;\mu'_t)$ is the potential (16) with density $[\mu(\tau(t,y))]'_t$), part (c) of this theorem is an immediate consequence of (16) and (17). For $x > 0$ $V^{(1)}(x,t)$ satisfies equation $L_\alpha V^{(1)} = 0$ since the support of the distribution $\mu\alpha\delta_{S2}$ is S_2, i.e. $\mu\alpha\delta_{S2}$ is equal to 0 for $x \notin S_2$.

Statement (d) is obtained by comparison of the convergent integral

$$V^{(1)}(0,t) = \frac{1}{4\sqrt{\pi}} \int_{1/4\alpha_1(t)}^{\infty} \mu(\cdot)\, y^{-3/2}\; dy$$

with $V^{(1)}(x,t)$, given by (16), for x close to 0. This, and formulae 3.383(3), 8.359(3) from [3], leads to the estimate:

$$| V^{(1)}(x,t) - V^{(1)}(0,t) | \leq \frac{1}{2} \| \mu \| \, | x | \, (1 - \Phi(| x^* | / 2(\alpha_1(t))^{1/2})),$$

where $0 \leq x^* \leq x$ and Φ is the probability integral. ■

4. DOUBLE-LAYER POTENTIAL

Double-layer potential $W(x,t)$, given by (12), is a part of a solution corresponding to the boundary condition (3).

THEOREM 4. Let α satisfy condition (i). Then: (a) for $r \in M$, $W(x,t) \in M$; (b) $W(x,t)$ is a distributional solution of the equation $L_\alpha u = -(\alpha r \delta_{S2})'_x$ and satisfies zero initial condition as $t \to 0+$; (c) for $x > 0$, $t \geq 0$ if $\alpha, r \in C(R_+)$ and $r' \in M$, then $W(x,t) \in C^\infty$ in x and C^1 in t, and it is a classical solution of (1)-(2) with $f = \varphi = 0$; (d) given that $r(t) \in C^1(R_+)$ W satisfies the following "jump formulae":

$$(18) \quad \lim_{x \to \pm 0} W(x,t) = \pm \frac{1}{2} r(t).$$

Proof. Parts (a)-(c) of this theorem are proved in the same way as those in Theorem 3. We introduce a new variable (15) and express W in the form:

$$(19) \quad W(x,t) = \frac{x}{2\sqrt{\pi}} \int_{1/4\alpha_1(t)}^{\infty} r(\tau)\, y^{-1/2} e^{-x^2 y} \, dy,$$

(where $\tau = \tau(t, y)$, as in Theorem 3) and its time-derivative:

$$(20)\qquad \frac{\partial W}{\partial t} = \frac{x}{\sqrt{\pi}}\, r(0)\, (\alpha_1(t))^{-3/2}\, \alpha(t) \exp(-x^2/4(\alpha_1(t)) + W(x, t; r_t'),$$

where $W(x, t; r_t')$ is the potential (19) with density $[r(\tau(t,y))]'_t$. Now part (b) can be proved applying the same technique as in Theorem 2, and (a), (c) follow from (19)-(20) as in Theorem 3.

Let us consider part (d) in more detail. First we let $r(\tau) \equiv r(t)$ for all $0 \le \tau \le t$, and denote the double layer potential in this case by W_0. Then, it follows from (19) and [3] (3.381, 8.359), that for $x \neq 0$

$$(21)\qquad W_0 = \frac{x}{2\sqrt{\pi}}\, r(t) \int_{1/4\alpha_1(t)}^{\infty} y^{-1/2} e^{-x^2 y}\, dy = \pm \frac{r(t)}{2}\left(1 - \Phi\!\left(\frac{x}{2\sqrt{\alpha(t)}}\right)\right),$$

($\pm$ depending on the sign of x), and, since $\Phi(0) = 0$,

$$\lim_{x \to 0^{\pm}} W_0(x, t) = \pm \frac{1}{2}\, r(t)\,.$$

Then, we consider the difference $W_0 - W$ for $x > 0$, performing integration in two steps (over $(0, t - \Delta)$ and $(t - \Delta, t)$ intervals), and separately studying cases where point t is "regular" (i.e., $\alpha(t) > 0$) and "irregular" (i.e., $\alpha(t) = 0$). Let

$$W(x, t) - W_0(x, t) = I_1 + I_2\,,$$

where

$$I_1 = \frac{x}{4\sqrt{\pi}} \int_0^{t-\Delta} (r(t) - r(\tau))\, \frac{\alpha(\tau)}{\beta_1^{3/2}(t-\tau)} \exp\!\left(-\frac{x^2}{4\beta_1(t-\tau)}\right) d\tau\,,$$

$$I_2 = \frac{x}{4\sqrt{\pi}} \int_{t-\Delta}^{t} (r(t) - r(\tau))\, \frac{\alpha(\tau)}{\beta_1^{3/2}(t-\tau)} \exp\!\left(-\frac{x^2}{4\beta_1(t-\tau)}\right) d\tau\,,$$

and, as in (21), for both types of t

$$|\, I_1 \,| \le \|r\| \left[\, \Phi\!\left(\frac{x}{2\sqrt{\alpha(t) - \alpha(t-\Delta)}}\right) - \Phi\!\left(\frac{x}{2\sqrt{\alpha(t)}}\right)\right] \to 0$$

with $x \to 0$ and fixed but arbitrary $\Delta > 0$.

I_2 should be estimated separately for different types of t. Thus, for t "regular", that is $\alpha(t) > 0$, Δ can be chosen sufficiently small so that $\alpha(\tau) > 0$ over the entire interval $[t - \Delta, t]$. Then, from $\beta_1(t - \tau) = \alpha(\tau *)(t - \tau)$ in $[t - \Delta, t]$ and the substitution of variables $y = (t - \tau)^{-1}$, we obtain:

$$| I_2 | \le \frac{\| \alpha \| \, | x | \, \| r' \|}{4\sqrt{\pi}\, \alpha_\Delta^{3/2}} \int_{1/\Delta}^{\infty} y^{-3/2} \exp\left(- \frac{x^2}{4(\alpha(t) - \alpha(t - \Delta))} \right) dy$$

$$= \frac{\| \alpha \| \, | x | \, \| r' \|}{2\sqrt{\pi}\, \alpha_\Delta^{3/2}} \sqrt{\Delta} \exp\left(- \frac{x^2}{4(\alpha(t) - \alpha(t - \Delta))} \right),$$

where $0 < \alpha_\Delta = \min_{\tau \in [t - \Delta, t]} | \alpha(\tau^*) | \to \alpha(t)$ with $\Delta \to 0$. As a result, $I_2 \to 0$ with either x or $\Delta \to 0$. For t "irregular", the fact that $\alpha(t) = 0$, requires a different approach. Using (15), we can show that

$$| I_2 | \le \frac{1}{2} \max_{\tau \in [t - \Delta, t]} | r(t) - r(\tau) | \left(1 - \Phi\left(\frac{x}{2\sqrt{\alpha(t) - \alpha(t - \Delta)}} \right) \right)$$

$$\le \frac{1}{2} \max_{\tau \in [t - \Delta, t]} | r(t) - r(\tau) | < \varepsilon$$

for arbitrarily small $\varepsilon > 0$. These estimates imply that $W_0 - W \to 0$ with $x \to 0$, hence the formula (18). ■

5. EXAMPLES

(a) Let's consider the problem (1)-(3) and $\alpha(t)$ satisfying (i). Then we introduce odd extension of all functions into the region $x < 0$. Then since the jumps at $x = 0$ are

$[u]_{x=0} = -2\,r(t)$ and $[u'_x]_{x=0} = 0$, from (8) we obtain the integral representation for the solution of initial-boundary value problem (1)-(3) for $x \geq 0, t \geq 0$:

$$u(x,t) = \int_0^t d\tau \int_0^\infty f(\xi,\tau)\,(E_\beta(x-\xi, t-\tau) - E_\beta(x+\xi, t-\tau))\,d\xi$$

$$+ \int_0^\infty \varphi(\xi)\,(E_\beta(x-\xi, t) - E_\beta(x+\xi, t))\,d\xi + 2\int_0^t \alpha(\tau)\,r(\tau)\,\frac{\partial}{\partial\xi}(E_\beta(-\xi, t-\tau))\Big|_{\xi=0}\,d\tau\,.$$

Function $u(x,t)$ satisfies the equation (1) and initial and boundary conditions (2)-(3), given that the functions α, r, φ, f satisfy restrictions discussed in Theorems 1-4.

(b) As in a), considering the problem (1)-(3) for $0 < x < b$ with additional condition $u(b,t) = h(t)$, we find solution $u(x,t)$ in the form (with $\alpha(t)$ still satisfying (i)):

(22) $u(x,t) = V(x,t) + V^{(0)}(x,t) + W_1(x,t) + W_2(x,t)$,

where double-layer potentials W_1 (the same as in (12)) and W_2 have density functions 2 r(t) and $\mu(t)$ respectively. W_2 is concentrated on the $x = 1$ part of the boundary and is given by the formula:

$$W_2(x,t) = \int_0^t \alpha(\tau)\mu(\tau)\,\frac{\partial}{\partial\xi}[E_\beta(x-\xi, t-\tau) - E_\beta(x+\xi, t-\tau)]_{\xi=b}\,d\tau\,.$$

Using (18) for W_1 we find that u (22) satisfies the conditions (2)-(3) (note that $W_2(0,t) = 0$). Applying then the boundary condition $u(b,t) = h(t)$ to (22) and using the "jump formula" for W_2 we obtain:

$$h(t) = V(b,t) + V^{(0)}(b,T) + W_1(b,t) - \frac{1}{2}\mu(t)$$

$$+\frac{1}{2\sqrt{\pi}}\int_0^t \alpha(\tau)\,\mu(\tau)\,(\beta_1(t-\tau))^{-3/2}\exp(-b^2/4\beta_1(t-\tau))\,d\tau\,.$$

The density $\mu(t)$ has to be found from the linear Volterra integral equation of the second kind:

$$(23)\qquad \mu(t)=\int_0^t k(t,\tau)\,\mu(\tau)\,d\tau+F(t)\;\equiv\;K[\mu],$$

with continuous $F(t)$ (Theorems 1-4) and a kernel

$$k(t,\tau)=\frac{1}{\sqrt{\pi}}\,\alpha(\tau)\,(\beta_1(t-\tau))^{-3/2}\exp(-b^2/4\beta_1(t-\tau))\,.$$

The unique solvability of the equation (23) can be obtained by methods discussed in [3], or it can be proved that some power K^m of the operator K is a contraction on $C[0,T]$. So, equation (23) has a unique solution, which can be found by the method of successive approximations, and formula (22) gives its integral representation .

(c) Considering (1)-(2) with $\alpha(t)$ satisfying (ii), $f=0$ and φ being an odd extension into $x<0$, we can find the solution in the form

$$u(x,t)\;=\;E*\tilde{\varphi}\,\delta_{S_1}=\int_0^\infty \varphi(\xi)\,(E_\beta(x-\xi,t)-E_\beta(x+\xi,t))\,d\xi$$

$$=\int_0^\infty \frac{\varphi(\xi)}{2\sqrt{\pi\,\alpha_1(t)}}\left[\exp\left(-\frac{(x-\xi)^2}{4\,\alpha_1(t)}\right)-\exp\left(-\frac{(x+\xi)^2}{4\,\alpha_1(t)}\right)\right]d\xi\,.$$

Verification is straightforward. As an example of $\alpha(t)$ satisfying (ii) $1/2+\cos(t)$ may do. Under the condition (ii) equation (1), not being of parabolic type, still can be solved in the form of a convolution of its fundamental solution with a single layer (Theorem 2).

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