*Journal of Applied Mathematics and Stochastic Analysis* 4, Number 1, Spring 1991, 1-27

# ON A PROBABILITY PROBLEM CONNECTED WITH RAILWAY TRAFFIC[1]

LAJOS TAKÁCS
*Case Western Reserve University*
*Cleveland, Ohio*[2]

**ABSTRACT**

Let $F_n(x)$ and $G_n(x)$ be the empirical distribution functions of two independent samples, each of size $n$, in the case where the elements of the samples are independent random variables, each having the same continuous distribution function $V(x)$ over the interval $(0,1)$. Define a statistic $\theta_n$ by

$$\theta_n/n = \int_0^1 [F_n(x) - G_n(x)]dV(x) - \min_{0 \le x \le 1}[F_n(x) - G_n(x)].$$

In this paper the limits of $E\{(\theta_n/\sqrt{2n})^r\}$ $(r = 0,1,2,\ldots)$ and $P\{\theta_n/\sqrt{2n} \le x\}$ are determined for $n \to \infty$. The problem of finding the asymptotic behavior of the moments and the distribution of $\theta_n$ as $n \to \infty$ has arisen in a study of the fluctuations of the inventory of locomotives in a randomly chosen railway depot.

**Key words:** Storage problem, Bernoulli excursion, Limit theorems.

**AMS (MOS) Subject Classification:** 60J15, 60F05, 60K30.

## 1. INTRODUCTION

In the mid 1950's Hungarian State Railway's researchers proposed the following problem for investigation to the Research Institute of Mathematics of the Hungarian Academy of Sciences where the author had a position at that time. A timetable indicates that each day $n$ trains arrive and also $n$ trains depart from a railway depot. There are numerous railway depots and we choose one at random. Suppose that the arrival instants and the departure instants are considered to be independent sequences of independent and identically distributed random variables. For each departing train one locomotive should be provided, and each arriving train adds one locomotive to the inventory of the depot. Let us suppose that the inventory of locomotives is managed in an optimal way, that is, no more locomotives are kept in the inventory than the bare minimum necessary to satisfy every demand. Denote by $\theta_n$ the total time spent at the depot by all the

[1]Received: December 1990. Revised: January 1991.

[2]Postal Address: 2410 Newbury Drive, Cleveland Heights, OH 44118

 

locomotives during the course of a day. The problem was to find the expectation, the variance and possibly the asymptotic distribution of $\theta_n$ as $n \to \infty$.

Let us denote by $V(x)$ the distribution function of an arrival (departure) time in the time interval $(0,1)$. Let $F_n(x)$ and $G_n(x)$ be the empirical distribution functions of two independent samples, each of size $n$, in the case where the elements of the two samples have a common distribution function $V(x)$. Then the random variable $\theta_n$ is determined by the following formula

$$\theta_n/n = \int_0^1 [F_n(x) - G_n(x)]dV(x) - \min_{0 \le x \le 1}[F_n(x) - G_n(x)]. \tag{1}$$

If $V(x)$ is a continuous distribution function, then the distribution of $\theta_n$ is independent of $V(x)$, and so in finding the distribution of $\theta_n$ we may assume that $V(x) = x$ for $0 \le x \le 1$, that is, the arrival times and the departure times are independent sequences of mutually independent random variables, each having a uniform distribution over the interval $(0,1)$.

K. Sarkadi [15] proved that

$$E\{\theta_n\} = \frac{1}{2}\left[\frac{4^n}{\binom{2n}{n}} - 1\right], \tag{2}$$

and L. Takács [18] proved that

$$E\{\theta_n^2\} = \frac{5n}{6} - E\{\theta_n\}. \tag{3}$$

In this paper an account is given of some recent progress in the solution of the aforementioned problem. The main results are stated in the following two theorems.

***Theorem 1:*** *If* $r = 0, 1, 2, \ldots$, *then*

$$\lim_{n \to \infty} E\{(\theta_n/\sqrt{2n})^r\} = M_r \tag{4}$$

*exists and*

$$M_r = K_r \frac{4\sqrt{\pi}\, r!}{\Gamma(\frac{3r-1}{2})2^{r/2}} \tag{5}$$

*where* $K_0 = -1/2, K_1 = 1/8$ *and*

$$K_r = \frac{(3r-4)}{4}K_{r-1} + \sum_{j=1}^{r-1} K_j K_{r-j} \tag{6}$$

*for* $r = 2, 3, \ldots$.

***Theorem 2:*** *There exists a distribution function $W(x)$ such that*

$$\lim_{n\to\infty} P\{\theta_n/\sqrt{2n} \le x\} = W(x) \tag{7}$$

*and $W(x) = 0$ for $x \le 0$. The distribution function $W(x)$ is uniquely determined by its moments*

$$\int_0^\infty x^r dW(x) = M_r \tag{8}$$

*for $r = 0,1,2,\ldots$ where $M_r$ is defined by (5) and (6). If $x > 0$, we have*

$$W(x) = \frac{\sqrt{6}}{x} \sum_{k=1}^{\infty} e^{-v_k} v_k^{2/3} U(1/6, 4/3, v_k) \tag{9}$$

*and*

$$W'(x) = \frac{2\sqrt{6}}{x^2} \sum_{k=1}^{\infty} e^{-v_k} v_k^{2/3} U(-5/6, 4/3, v_k) \tag{10}$$

*where $U(a,b,x)$ is the confluent hypergeometric function,*

$$v_k = 2a_k^3/(27x^2), \tag{11}$$

*and $z = -a_k (k = 1,2,\ldots)$ are the zeros of the Airy function $Ai(z)$ arranged so that $0 < a_1 < a_2 < \ldots < a_k < \ldots$.*

For the definitions of the confluent hypergeometric function and the Airy function we refer to L.J. Slater [16], J.C.P. Miller [13] and M. Abramowitz and I.A. Stegun [1].

We shall prove the above theorems in three stages.

First, we consider a Bernoulli excursion $\{\eta_0^+, \eta_1^+, \ldots, \eta_{2n}^+\}$ in which $\eta_{2n}^+ = \eta_0^+ = 0$ and $\eta_i^+ \ge 0$ for $0 \le i \le 2n$ and define $\omega_n$ for $n \ge 1$ by the following equation

$$2n\omega_n = \sum_{i=1}^{n} \eta_i^+ . \tag{12}$$

We shall prove that

$$\lim_{n\to\infty} E\{(\omega_n/\sqrt{2n})^r\} = M_r \tag{13}$$

for $r = 0,1,2,\ldots$ where $M_r$ is defined by (5) and (6), and that

$$\lim_{n\to\infty} P\{\omega_n/\sqrt{2n} \le x\} = W(x) \tag{14}$$

where $W(x)$ is defined in Theorem 2.

Then we shall consider a tied-down random walk $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ in which $\eta_{2n} = \eta_0 = 0$ and define $\rho_n$ by the following equation

$$2n\rho_n = \sum_{i=1}^{2n} (\eta_i + \delta_{2n}) \tag{15}$$

where

$$\delta_{2n} = -\min(\eta_0, \eta_1, \ldots, \eta_{2n}). \tag{16}$$

We shall prove that

$$\lim_{n\to\infty} E\{(\rho_n/\sqrt{2n})^r\} = M_r \tag{17}$$

for $r = 0, 1, 2, \ldots$ where $M_r$ is defined by (5) and (6), and that

$$\lim_{n\to\infty} P\{\rho_n/\sqrt{2n} \le x\} = W(x) \tag{18}$$

where $W(x)$ is defined in Theorem 2.

Finally, we express $\theta_n$ in the following way

$$\theta_n = \sum_{i=1}^{2n} \eta_i \xi_i + \delta_{2n} \tag{19}$$

where $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ is the tied-down random walk defined above and the random variables $\xi_0, \xi_1, \ldots, \xi_{2n}$ are independent of the random walk $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ and are defined in the following way: We choose $2n$ points at random in the interval $(0,1)$. We assume that the $2n$ points are distributed independently and each point has a uniform distribution over the interval $(0,1)$. These $2n$ points divide the interval $(0,1)$ into $2n+1$ subintervals. Denote by $\xi_0, \xi_1, \ldots, \xi_{2n}$ their lengths. The random variables $\xi_0, \xi_1, \ldots, \xi_{2n}$ are interchangeable and obviously $\xi_0 + \xi_1 + \ldots + \xi_{2n} = 1$.

We shall prove that

$$E\{(\theta_n - \rho_n)^2/(2n)\} = 1/(24n). \tag{20}$$

Thus

$$\lim_{n\to\infty} (\theta_n - \rho_n)/\sqrt{2n} = 0 \tag{21}$$

in probability. Consequently, (18) and (21) imply (7).

Also, we shall prove that

$$\lim_{n\to\infty} [E\{\theta_n^r\} - E\{\rho_n^r\}]/n^{r/2} = 0 \tag{22}$$

for $r = 1, 2, \ldots$. Hence (4) follows from (17).

## 2. THE BERNOULLI EXCURSION

Let us arrange $n$ white and $n$ black balls in a row in such a way that for every $i = 1, 2, \ldots, 2n$ among the first $i$ balls there are at least as many white balls as black. The total number of such arrangements is given by the $n$-th Catalan number,

$$C_n = \binom{2n}{n}\frac{1}{n+1}. \tag{23}$$

We have $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, \ldots$.

Let us suppose that all the possible $C_n$ sequences are equally probable and denote by $\eta_i^+$ the difference between the number of white balls and the number of black balls among the first $i$ balls in a sequence chosen at random. We have $\eta_{2n}^+ = \eta_0^+ = 0$ and $\eta_i^+ \geq 0$ for $i = 1, 2, \ldots, 2n$. The sequence $\{\eta_0^+, \eta_1^+, \ldots, \eta_{2n}^+\}$ is called a Bernoulli excursion.

We can imagine that a particle performs a random walk on the $x$-axis. It starts at $x = 0$ and takes $2n$ steps. In the $i$-th step the particle moves either a unit distance to the right or a unit distance to the left according as the $i$-th ball in the row is white or black respectively. At the end of the $i$-th step the position of the particle is $x = \eta_i^+$ for $i = 1, 2, \ldots, 2n$.

As an alternative we can assume that the particle starts at time $t = 0$ at the origin and in the time interval $(i-1, i], i = 1, 2, \ldots, 2n$, it moves with a unit velocity to the right or to the left according as the $i$-th ball in the row is white or black respectively. Denote by $\eta_n^+(t)$ the position of the particle at time $2nt$ where $0 \leq t \leq 1$. Then $\eta_i^+ = \eta_n^+(i/2n)$ for $i = 1, 2, \ldots, 2n$.

Define $\omega_n$ by the following equations

$$2n\omega_n = \sum_{i=0}^{2n} \eta_i^+ = 2n \int_0^1 \eta_{[2nt]}^+ dt = 2n \int_0^1 \eta_n^+(t) dt \tag{24}$$

if $n = 1, 2, \ldots$ and $\omega_0 = 0$.

The random variable $2n\omega_n$ is a discrete random variable with possible values $n + 2j$ $(j = 0, 1, \ldots, \binom{n}{2})$. Denote by $f_n(n+2j)$ the number of sequences $\{\eta_0^+, \eta_1^+, \ldots, \eta_{2n}^+\}$ in which $2n\omega_n = n + 2j$. Then we have

$$P\{2n\omega_n = n + 2j\} = f_n(n+2j)/C_n \tag{25}$$

for $j = 0, 1, \ldots, \binom{n}{2}$.

The distribution of $2n\omega_n$ is determined by the generating function

$$\phi_n(z) = \sum_{j=0}^{\binom{n}{2}} f_n(n+2j)z^j \tag{26}$$

which can be obtained by the following theorem.

***Theorem 3:*** *We have*

$$\phi_n(z) = \sum_{i=1}^{n} \phi_{i-1}(z)\phi_{n-i}(z)z^{i-1} \tag{27}$$

*for* $n = 1, 2, \ldots$ *and* $\phi_0(z) = 1$.

***Proof:*** If $i = 1, 2, \ldots, n$ is the smallest positive integer for which $\eta_{2i}^{+} = 0$, then in the representation

$$\eta_1^{+} + \ldots + \eta_{2n}^{+} = 2i - 1 + (\eta_1^{+} - 1) + \ldots + (\eta_{2i-1}^{+} - 1) + \eta_{2i}^{+} + \ldots + \eta_{2n}^{+} \tag{28}$$

the sum $(\eta_1^{+} - 1) + \ldots + (\eta_{2i-1}^{+} - 1)$ has the same distribution as $2(i-1)\omega_{i-1}$ and the sum $(\eta_{2i}^{+} + \ldots + \eta_{2n}^{+})$ has the same distribution as $2(n-i)\omega_{n-i}$ and these two random variables are independent. If we use the notation (26), then by (28) we obtain (27) which was to be proved. By (27) we obtain that $\phi_1(z) = 1, \phi_2(z) = 1 + z, \phi_3(z) = 1 + 2z + z^2 + z^3$ and

$$\phi_4(z) = 1 + 3z + 3z^2 + 3z^3 + 2z^4 + z^5 + z^6. \tag{29}$$

Now let us define

$$\Phi(z, w) = \sum_{n=0}^{\infty} \phi_n(z)w^n. \tag{30}$$

Since $|\phi_n(z)| \le \phi_n(1) = C_n$ if $|z| \le 1$ and since

$$w\sum_{n=0}^{\infty} C_n w^n = [1 - (1 - 4w)^{1/2}]/2 \tag{31}$$

for $|w| \le 1/4$, the series (30) is convergent if $|z| \le 1$ and $|w| \le 1/4$.

Multiplying (27) by $w^n$ and forming the sum for $n = 1, 2, \ldots$ we obtain that

$$\Phi(z, w) = 1 + w\Phi(z, w)\Phi(z, zw) \tag{32}$$

for $|z| \le 1$ and $|w| \le 1/4$.

If we define

$$F(z,w) = w\Phi(z,zw) = w\sum_{n=0}^{\infty} \phi_n(z)(zw)^n, \tag{33}$$

then by (32)

$$\Phi(z,w) = 1/[1 - F(z,w)] \tag{34}$$

and

$$F(z,w) = w\Phi(z,zw) = w/[1 - F(z,zw)] \tag{35}$$

or

$$F(z,w) = w + F(z,w)F(z,zw) \tag{36}$$

for $|z| \le 1$ and $|w| \le 1/4$. The repeated application of (35) leads to the continued fraction

$$F(z,w) = \cfrac{w}{1 - \cfrac{zw}{1 - \cfrac{z^2 w}{1 - \ddots}}}. \tag{37}$$

The continued fraction (37) has been encountered by S. Ramanujan [14] in the theory of partitions. (See e.g., G.H. Hardy and E.M. Wright [10] p. 295, and G. Szekeres [17].)

By (26) and (33) we have

$$F(z,w) = w\sum_{n=0}^{\infty} \sum_{j=0}^{\binom{n}{2}} f_n(n+2j)z^{n+j}w^n \tag{38}$$

for $|z| \le 1$ and $|w| \le 1/4$.

## 3. THE MOMENTS OF $\omega_n$

The equation (36) makes it possible to determine explicitly the moments of $\omega_n$. Let us define

$$b_r(n) = \sum_{j=0}^{\binom{n}{2}} \binom{n+j}{r} f_n(n+2j) \tag{39}$$

for $r = 0, 1, 2, \ldots$ and $n \geq 0$. Then the $r$-th binomial moment of $(2n\omega_n + n)/2$ is given by

$$E\left\{\binom{(2n\omega_n + n)/2}{r}\right\} = b_r(n)/b_0(n) \tag{40}$$

for $r = 0, 1, 2, \ldots$, where $b_0(n) = C_n$. If we know (40) for $r \leq m$, then the moments $E\{\omega_n^r\}$ can readily be determined for $r \leq m$.

The generating function

$$B_r(w) = w\sum_{n=0}^{\infty} b_r(n) w^n \tag{41}$$

can also be expressed as

$$B_r(w) = \frac{1}{r!}\left(\frac{\partial^r F(z,w)}{\partial z^r}\right)_{z=1} \tag{42}$$

for $r = 0, 1, 2, \ldots$ and $|w| < 1/4$. Moreover, we have

$$\frac{1}{r!}\left(\frac{\partial^r F(z,zw)}{\partial z^r}\right)_{z=1} = \sum_{i=0}^{r} B_{r-i}^{(i)}(w) w^i / i! \tag{43}$$

for $r, = 0, 1, 2, \ldots$.

If we form the first $m$ derivatives of (36) with respect to $z$ at $z = 1$, we can determine $B_r(w)$ step by step for $r = 1, 2, \ldots, m$.

If we put $z = 1$ in (36), we obtain that

$$B_0(w) = w + [B_0(w)]^2 \tag{44}$$

for $|w| \leq 1/4$. Hence

$$B_0(w) = [1 - (1-4w)^{\frac{1}{2}}]/2 = w\sum_{n=0}^{\infty} C_n w^n, \tag{45}$$

that is, $b_0(n) = C_n$ for $n \geq 0$.

In what follows we shall use the abbreviation

$$R = 1 - 4w. \tag{46}$$

Then

$$B_0(w) = [1 - R^{\frac{1}{2}}]/2. \tag{47}$$

If $r = 0, 1, 2, \ldots$ and if we form the $r$-th derivative of (36) with respect to $z$ at $z = 1$, we obtain that

$$B_r(w) = \delta_{r0} w + \sum_{j=0}^{r} B_{r-j}(w) \sum_{i=0}^{j} B_{j-i}^{(i)}(w) w^i / i! \tag{48}$$

where $\delta_{r0} = 1$ for $r = 0$ and $\delta_{r0} = 0$ for $r > 0$.

If $r = 1$ in (48), we obtain that

$$B_1(w) = \frac{w B_0(w) B_0'(w)}{1 - 2B_0(w)} = w(R^{-1} - R^{\frac{1}{2}})/2 \tag{49}$$

for $|w| < 1/4$. Hence

$$b_1(n) = \tfrac{1}{2}[4^n - \tbinom{2n}{n}]. \tag{50}$$

If $r = 2$ in (48), we obtain that

$$B_2(w) = w(5R^{-5/2} - 4R^{-2} - 2R^{-3/2} - 4R^{-1} + 5R^{-1/2})/16. \tag{51}$$

Hence

$$b_2(n) = \frac{5n^2 + 7n + 6}{12}\binom{2n}{n} - \frac{(n+2)}{4}4^n. \tag{52}$$

If we use the following expansions

$$R^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} w^n, \tag{53}$$

$$R^{-s} = \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} 4^n w^n, \tag{54}$$

and

$$R^{-s-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(2n+1)(2n+3)\ldots(2n+2s-1)}{1.3\ldots(2s-1)} w^n \tag{55}$$

where $s = 1, 2, 3, \ldots$ and $|w| < 1/4$, then step by step we can determine $b_r(n)$ for every $r = 0, 1, 2, \ldots$, and the moments of $\omega_n$ can be obtained by (40).

By the recurrence formula (48) we can draw the conclusion that $B_r(w)/(wR^{1/2})$ is a polynomial of degree $3r$ in $R^{-1/2}$ if $r \geq 1$. Consequently, if $r = 0, 1, 2, \ldots$, then

$$B_r(w) = \frac{K_r}{(1-4w)^{(3r-1)/2}} + \ldots \tag{56}$$

where the neglected terms are constant multiples of $R^{-j/2}$ for $-3 \leq j \leq 3r-2$. If in (48) we retain only those terms which contribute to the determination of $K_r$ in (56), we obtain that

$$R^{1/2}B_r(w) = \sum_{j=1}^{r-1} B_{r-j}(w)B_j(w) + B_0(w)B_{r-1}^{(1)}(w)w + \ldots \tag{57}$$

for $r \geq 2$. Here $w = (1-R)/4$, and if we form the coefficient of $R^{-(3r-2)/2}$ on both sides of (57), we obtain that

$$K_r = \sum_{j=1}^{r-1} K_j K_{r-j} - K_0 K_{r-1}\left(\frac{3r-4}{2}\right) \tag{58}$$

for $r \geq 2$. By (47) and (49) we have $K_0 = -1/2$ and $K_1 = 1/8$. This proves (6).

By (56)

$$b_r(n) = K_r \binom{n+1+3(r-1)/2}{n+1} 4^{n+1} + \ldots \tag{59}$$

where the neglected terms are of smaller order than the displayed one as $n \to \infty$. Hence

$$b_r(n) \sim K_r \frac{4^{n+1} n^{3(r-1)/2}}{\Gamma(\frac{3r-1}{2})} \tag{60}$$

for $r = 0, 1, 2, \ldots$ as $n \to \infty$. By (40) and (60)

$$E\{(n\omega_n)^r\} \sim r! \frac{b_r(n)}{b_0(n)} \sim K_r \frac{4r!\sqrt{\pi}}{\Gamma(\frac{3r-1}{2})} n^{3r/2} \tag{61}$$

for $r = 0, 1, 2, \ldots$ as $n \to \infty$. Here we used that $b_0(n) = C_n$ and

$$C_n n\sqrt{n\pi} \sim 4^n \tag{62}$$

as $n \to \infty$. This proves (13). Accordingly,

$$\lim_{n \to \infty} E\{(\omega_n/\sqrt{2n})^r\} = M_r \tag{63}$$

for $r = 0, 1, 2, \ldots$ where $M_r$ is defined by (5) and (6).

Alternatively, we can calculate $K_r (r = 1, 2, \ldots)$ by the following recurrence formula

$$K_r = \frac{(6r+1)}{2(6r-1)}\alpha_r - \sum_{j=1}^{r} \alpha_j K_{r-j} \tag{64}$$

for $r \geq 1$ where

$$\alpha_j = \frac{\Gamma(3j+\frac{1}{2})}{\Gamma(j+\frac{1}{2})(36)^j j!} = \frac{(2j+1)(2j+3)\ldots(6j-1)}{(144)^j j!} \tag{65}$$

for $j \geq 1$ and $\alpha_0 = 1$.

Clearly,

$$\alpha_j = \frac{3}{4}(j - 1 + \frac{5}{36j})\alpha_{j-1} \tag{66}$$

for $j \geq 1$ and $\alpha_0 = 1$.

To prove (64) let us introduce the formal generating function

$$y(z) = \sum_{r=0}^{\infty} (-1)^r K_r z^r. \tag{67}$$

By (58) we obtain that

$$3z^2 y'(z) - zy(z) = 4[y(z)]^2 - 1. \tag{68}$$

The appropriate solution of (68) is given by

$$y(z) = Ai'(z^{-2/3})z^{1/3}/[2Ai(z^{-2/3})] \tag{69}$$

where

$$Ai(z) = \frac{1}{\pi}\int_0^{\infty} cos(\frac{t^3}{3} + tz)dt \tag{70}$$

is the Airy function which satisfies the differential equation

$$Ai''(z) = zAi(z). \tag{71}$$

See J.C.P. Miller [13] and M. Abramowitz and I.A. Stegun [1]. The function (69) satisfies (68) and has an asymptotic expansion in $1/z$ as $|z| \to \infty$ and $|arg z| < \pi$. This expansion is in agreement with (67). By equating coefficients of similar powers of $z$ in both sides of the equation

$$2Ai(z^{-2/3})y(z) = Ai'(z^{-2/3})z^{1/3}, \tag{72}$$

we obtain (64).

By (58)

$$K_r \geq K_{r-1}\frac{(3r-4)}{4} + 2K_{r-1}\frac{1}{8} = \frac{3(r-1)}{4}K_{r-1} \tag{73}$$

if $r \geq 2$ and $K_1 = 1/8$. Accordingly, $(4/3)^r K_r/(r-1)!$ $(r = 2, 3, \ldots)$ is an increasing sequence. By (64)

$$K_r \leq [6r/(6r-1)]\alpha_r \tag{74}$$

for $r \geq 1$ and by (65)

$$\lim_{r\to\infty}(\tfrac{4}{3})^r \frac{\alpha_r}{(r-1)!} = \frac{1}{2\pi}. \tag{75}$$

Thus

$$\lim_{r\to\infty}(\tfrac{4}{3})^r \frac{K_r}{(r-1)!} = \gamma \tag{76}$$

exists and $0 < \gamma \leq 1/(2\pi)$. Actually, we can prove that $\gamma = 1/(2\pi)$.

By (5) we have

$$M_r \sim 2\pi\gamma \frac{6r}{\sqrt{2}}(\frac{2r}{3e\pi})^{r/2} \tag{77}$$

as $r \to \infty$.

## 4. THE ASYMPTOTIC DISTRIBUTION OF $\omega_n$

We shall prove the following result.

***Theorem 4:*** *There exists a distribution function $W(x)$ such that*

$$\lim_{n\to\infty} P\{\omega_n/\sqrt{2n} \leq x\} = W(x) \tag{78}$$

*in every continuity point of $W(x)$. The distribution function $W(x)$ is uniquely determined by its moments*

$$\int_0^\infty x^r dW(x) = M_r \tag{79}$$

*for $r = 0, 1, 2, \ldots$ where $M_r$ is defined by (5) and (6).*

***Proof:*** It follows from (77) that

$$\sum_{r=1}^{\infty} M_r^{-1/r} = \infty. \tag{80}$$

By (63) the sequence $M_r (r = 0, 1, 2, \ldots)$ is a moment sequence. Since the condition (80) is satisfied, we can conclude from a theorem of T. Carleman [3], [4] that there exists one and only one distribution function $W(x)$ such that $W(0) = 0$ and (79) holds for $r = 0, 1, 2, \ldots$. By the moment convergence theorem of M. Fréchet and J. Shohat [7], (63) implies (78). This completes the proof of the theorem.

To determine $W(x)$ let us define

$$\psi(s) = \int_0^{\infty} e^{-sx} dW(x) \tag{81}$$

as the Laplace-Stieltjes transform of $W(x)$. We have

$$\psi(s) = \sum_{r=0}^{\infty} (-1)^r M_r s^r / r! \tag{82}$$

and the series (82) is convergent on the whole complex plane. This follows from (77). In (5) we can write

$$\frac{1}{\Gamma(\frac{3r-1}{2})} = \frac{1}{2\pi i} \int_C e^t t^{-(3r-1)/2} dt \tag{83}$$

for $r = 0, 1, 2, \ldots$ where $C$ denotes integration along a contour which starts at infinity on the negative real $t$-axis, encircles the origin counter-clockwise, and returns its starting point. Accordingly,

$$\frac{M_r}{r!} = \frac{4\sqrt{\pi} K_r}{2\pi i} \int_C e^t (t 2^{1/3})^{-(3r/2)} t^{1/2} dt \tag{84}$$

for $r = 0, 1, 2, \ldots$. Now we shall sketch a heuristic proof for the determination of $\psi(s)$. If we use the notation (69) and interchange summation and integration, then by (82) we obtain that

$$\psi(s) = \frac{4\sqrt{\pi}}{2\pi i} \int_C y(s(t2^{1/3})^{-3/2}) t^{1/2} e^t dt =$$

$$\frac{4\sqrt{\pi}}{2\pi i} \int_C \frac{Ai'(2^{1/3} s^{-2/3} t)}{Ai(2^{1/3} s^{-2/3} t)} s^{1/3} 2^{-1/3} e^t dt = \tag{85}$$

$$\frac{s\sqrt{2\pi}}{2\pi i}\int_{C^*}\frac{Ai'(z)}{Ai(z)}e^{zs^{2/3}/2^{1/3}}dz$$

where we used the substitution $z = ts^{-2/3}2^{1/3}$ and $C^*$ is the new contour in the $z$-plane. The function $Ai(z)$ has zeros only on the negative real axis; $z = -a_k (k = 1, 2, \ldots)$ where $0 < a_1 < a_2 < \ldots$. By the theorem of residues we obtain from (85) that

$$\psi(s) = s\sqrt{2\pi}\sum^{\infty} e^{-a_k s^{2/3}/2^{1/3}} \tag{86}$$

for $Re(s) > 0$. Hence we obtain (9) and (10) by inversion. It remains to find a rigorous proof for (86). We shall see that $\psi(s)$ can also be interpreted in several other ways and formula (86) can be deduced either from the results of D.A. Darling [6] or from the results of G. Louchard [12].

We note that the process $\{\eta_n^+(t)/\sqrt{2n}, 0 \le t \le 1\}$, where $\eta_n^+(t)$ is defined at the beginning of Section 2, converges weakly to the Brownian excursion $\{\eta^+(t), 0 \le t \le 1\}$. If we define

$$\omega^+ = \int_0^1 \eta^+(t)dt, \tag{87}$$

then, because the integral is a continuous functional of the process, we can conclude that

$$\lim_{n\to\infty} P\{\omega_n/\sqrt{2n} \le x\} = P\{\omega^+ \le x\}. \tag{88}$$

Thus by (78) it follows that

$$P\{\omega^+ \le x\} = W(x), \tag{89}$$

by (79) $E\{(\omega^+)^r\} = M_r$ for $r = 0, 1, 2, \ldots$, and by (81)

$$E\{e^{-s\omega^+}\} = \psi(s) \tag{90}$$

for $Re(s) \ge 0$.

The Laplace-Stieltjes transform (90) has been studied by R.K. Getoor and M.J. Sharpe [8], J.W. Cohen and G. Hooghiemstra [5], Ph. Biane and M. Yor [2], P. Groeneboom [9] and G. Louchard [11], [12]. G. Louchard [12] has found formula (86) for (90) and determined the moments $E\{(\omega^+)^r\} (r = 0, 1, 2, \ldots)$.

## 5. THE TIED-DOWN RANDOM WALK

Let us suppose that a box contains $n$ white and $n$ black balls. We draw all the $2n$ balls from the box one by one at random without replacement. There are $\binom{2n}{n}$ possible results and they are supposed to be equally probable. Define $\eta_i (i = 0, 1, \ldots, 2n)$ as the difference between the number of white balls and the number of black balls among the first $i$ balls drawn ($\eta_{2n} = \eta_0 = 0$). We can interpret the sequence $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ as a tied-down random walk on the real line.

Let us define $\rho_n$ for $n \geq 1$ by

$$2n\rho_n = \sum_{i=1}^{2n} (\eta_i + \delta_{2n}) \tag{91}$$

where

$$\delta_{2n} = -\min(\eta_0, \eta_1, \ldots, \eta_{2n}), \tag{92}$$

and write $\rho_0 = 0$.

The random variable $2n\rho_n$ is a discrete random variable with possible values $n + 2j (j = 0, 1, \ldots, \binom{n}{2})$. Denote by $h_n(n+2j)$ the number of random walks $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ in which $2n\rho_n = n + 2j$. Then we have

$$P\{2n\rho_n = n + 2j\} = h_n(n+2j)/\binom{2n}{n} \tag{93}$$

for $j = 0, 1, \ldots, \binom{n}{2}$. The distribution of $2n\rho_n$ is determined by the generating function

$$\psi_n(z) = \sum_{j=0}^{\binom{n}{2}} h_n(n+2j) z^j. \tag{94}$$

This generating function can be obtained by the following theorem.

***Theorem 5:*** *We have $\psi_0(z) = 1$, and*

$$\psi_n(z) = 2\sum_{i=1}^{n} i\phi_{i-1}(z)\phi_{n-i}(z)z^{i-1} \tag{95}$$

*for $n = 1, 2, \ldots$ where $\phi_n(z) (n = 0, 1, 2, \ldots)$ are determined by the recurrence formula* (27).

***Proof:*** Let $n \geq 1$. In the random walk $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ it may happen that $\delta_{2n} = 0$, that is, $\eta_s \geq 0$ for $0 \leq s \leq 2n$. Then $2n\rho_n$ simply has the same distribution as $2n\omega_n$. If $\delta_{2n} \geq 1$, then let $s = i$ be the first subscript for which $\eta_s = -\delta_{2n}$ and let $s = i + 2(n-1-k)$ be the last subscript for which $\eta_s = -\delta_{2n}$. Then $i$ may be $0, 1, \ldots, 2k$ and $k$ may be $0, 1, \ldots, n-1$. Now let us consider a new random walk defined by

$$\{\eta_i + \delta_{2n}, \ldots, \eta_{2n} + \delta_{2n}, \eta_0 + \delta_{2n}, \ldots, \eta_i + \delta_{2n}\}. \tag{96}$$

That is, in the original random walk we transfer the first $i$ steps from the beginning to the end and shift the zero level to $-\delta_{2n}$. For fixed $i$ and $k$, the random walk (96) has the same stochastic properties as a Bernoulli excursion $\{\eta_0^+, \eta_1^+, \ldots, \eta_{2n}^+\}$ in which $\eta^+_{2(n-1-k)} = 0$ and $\eta_s^+ > 0$ for $2(n-1-k) < s < 2n$, and $2n\rho_n$ has the same distribution as $\eta_0^+ + \eta_1^+ + \ldots + \eta_{2n}^+$. Under the aforementioned conditions $\eta_0^+ + \ldots + \eta^+_{2(n-1-k)}$ has the same distribution as $2(n-1-k)\omega_{n-1-k}$ and $\eta^+_{2(n-1-k)} + \ldots + \eta^+_{2n-1}$ has the same distribution as $2k+1+2k\omega_k$ and these two sums are independent random variables. Obviously, $\eta_{2n}^+ = 0$. By the above considerations

$$\psi_n(z) = \phi_n(z) + \sum_{k=0}^{n-1} \sum_{i=0}^{2k} \phi_k(z)\phi_{n-1-k}(z)z^k = 2\sum_{k=0}^{n-1}(k+1)\phi_k(z)\phi_{n-1-k}(z)z^k \tag{97}$$

for $n \geq 1$ where $\phi_n(z)$ is defined by (26) and determined by (27). By definition, $\psi_0(z) = 1$. This completes the proof of (95).

It is worthwhile to point out the significance of formula (95). To find the distribution of $2n\omega_n$ we should determine the generating functions $\phi_0(z), \phi_1(z), \ldots, \phi_n(z)$. If these functions are known, then the distribution of $2n\rho_n$ can immediately be calculated by (95). No extra calculations are needed, although the random variable $2n\rho_n$ is much more complicated than $2n\omega_n$.

By (95) we obtain that $\psi_0(z) = 1, \psi_1(z) = 2, \psi_2(z) = 2 + 4z$ and

$$\psi_3(z) = 2 + 6z + 6z^2 + 6z^3. \tag{98}$$

Let us define

$$\Psi(z,w) = \sum_{n=0}^{\infty} \psi_n(z)z^n = \sum_{n=0}^{\infty} \sum_{j=0}^{\binom{n}{2}} h_n(n+2j)z^j w^n. \tag{99}$$

Since $|\psi_n(z)| \leq \psi_n(1) = \binom{2n}{n}$ for $|z| \leq 1$, and since

$$\sum_{n=0}^{\infty} \binom{2n}{n} w^n = (1-4w)^{-1/2} \tag{100}$$

for $|w| < 1/4$, the infinite series (99) is convergent for $|z| \leq 1$ and $|w| < 1/4$.

If we multiply (95) by $w^n$ and form the sum for $n = 1, 2, \ldots$, we obtain that

$$\Psi(z,w) = 1 + 2\Phi(z,w)\left(\frac{\partial F(z,w)}{\partial w}\right) = 1 + \frac{2w}{1-F(z,w)}\left(\frac{\partial F(z,w)}{\partial w}\right) \tag{101}$$

for $|z| \leq 1$ and $|w| < 1/4$ where $\Phi(z,w)$ is defined by (30) and $F(z,w)$ is defined by (33) and determined by (36).

## 6. THE MOMENTS OF $\rho_n$

The equation (101) makes it possible to determine explicitly the moments of $\rho_n$. Let us define

$$a_r(n) = \sum_{j=0}^{\binom{n}{2}} \binom{j}{r} f_n(n+2j) \tag{102}$$

for $r = 0, 1, 2, \ldots$ and $n \geq 1$. Then the $r$-th binomial moment of $(2n\omega_n - n)/2$ is given by

$$E\left\{\binom{(2n\omega_n - n)/2}{r}\right\} = a_r(n)/a_0(n) \tag{103}$$

for $r = 0, 1, 2, \ldots$, where $a_0(n) = C_n$.

The generating function

$$A_r(w) = \sum_{n=0}^{\infty} a_r(n) w^n \tag{104}$$

can also be expressed as

$$A_r(w) = \frac{1}{r!}\left(\frac{\partial^r \Phi(z,w)}{\partial z^r}\right)_{z=1} = \sum_{n=0}^{\infty} \sum_{j=0}^{\binom{n}{2}} \binom{j}{r} f_n(n+2j) w^n \tag{105}$$

for $r = 0, 1, 2, \ldots$ and $|w| < 1/4$.

By (34) we have

$$\Phi(z,w)[1 - F(z,w)] = 1. \tag{106}$$

If we form the $r$-th derivative of (106) with respect to $z$ at $z = 1$, we obtain that

$$A_r(w) - \sum_{i=0}^{r} A_i(w) B_{r-i}(w) = \delta_{r0} \tag{107}$$

for $r \geq 0$ where $\delta_{00} = 1$ and $\delta_{r0} = 0$ for $r > 0$. From (107), $wA_0(w) = B_0(w)$ and $A_1(w), A_2(w), \ldots$ can be determined step by step if $B_0(w), B_1(w), \ldots$ are already known. From (56) and (107) we can conclude that

$$wA_r(w) = \frac{K_r}{(1-4w)^{(3r-1)/2}} + \ldots \tag{108}$$

where $K_r$ has the same meaning as in (56) and the neglected terms are constant multiples of $R^{-j/2}$ for $-3 \leq j \leq 3r-2$. Here $R = 1 - 4w$. To prove (108) we can use either (107) or the obvious relation

$$b_r(n) = \sum_{j=0}^{r} \binom{n}{j} a_{r-j}(n) \tag{109}$$

for $r \geq 0$ and $n \geq 0$. By (108)

$$a_r(n) = K_r \binom{n+1+3(r-1)/2}{n+1} 4^{n+1} + \ldots \tag{110}$$

where the neglected terms are of smaller order than the displayed term as $n \to \infty$. Accordingly,

$$a_r(n) \sim b_r(n) \tag{111}$$

for $r = 0, 1, 2, \ldots$ as $n \to \infty$. This follows also from (109).

The moments of $\rho_n$ can be determined in the following way. By (101).

$$H_r(w) = \frac{1}{r!} \left( \frac{\partial^r \Psi(z, w)}{\partial z^r} \right)_{z=1} = \delta_{r0} + 2w \sum_{j=0}^{r} A_j(w) B'_{r-j}(w) \tag{112}$$

for $r \geq 0$ and $|w| < 1/4$. On the other hand by (99)

$$H_r(w) = \sum_{n=0}^{\infty} \sum_{j=0}^{\binom{n}{2}} \binom{j}{r} h_n(n+2j) w^n = \sum_{n=0}^{\infty} \binom{2n}{n} E\left\{ \binom{(2n\rho_n - n)/2}{r} \right\} w^n \tag{113}$$

for $|w| < 1/4$. By (48), (107), and (113) we can determine $H_r(w)$ explicitly for every $r = 0, 1, 2, \ldots$. If we use the notation (46), that is, $R = 1 - 4w$, then we obtain that

$$H_0(w) = 1 + 2wA_0(w)B'_0(w) = 1 + 2B_0(w)B'_0(w) = R^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} w^n \tag{114}$$

for $|w| < 1/4$, and

$$\begin{gathered} H_1(w) = 2wA_0(w)B'_1(w) + 2wA_1(w)B'_0(w) = 2w(R^{-2} - R^{-3/2}) = \\ (R^{-2} - R^{-3/2} - R + R^{1/2})/2. \end{gathered} \tag{115}$$

By (115)

$$\binom{2n}{n} E\{2n\rho_n + n\} = n4^n. \tag{116}$$

In a similar way we can determine $H_r(w)$ for $r = 2, 3, \ldots$ and if we use the expressions (53), (54) and (55) we can determine

$$E\left\{ \binom{(2n\rho_n - n)/2}{r} \right\} \tag{117}$$

for every $r = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$. The moments of $\rho_n$ are completely determined by (117). For example, if $r = 2$, we obtain that

$$E\{(2n\rho_n + n)^2\} = 2n^2(5n+1)/3. \tag{118}$$

By (56)

$$B_r'(w) = \frac{2(3r-1)K_r}{(1-4w)^{(3r+1)/2}} + \ldots \tag{119}$$

for $r = 0, 1, 2, \ldots$ where the neglected terms are constant multiples of $R^{-j/2}$ for $-1 \leq j \leq 3r$. If $r \geq 1$, then in (112)

$$2wA_0(w)B_r'(w) = 2B_0(w)B_r'(w) = 2(3r-1)K_r R^{-(3r+1)/2} + \ldots \tag{120}$$

and

$$\begin{gathered} 2w\sum_{j=1}^{r} A_j(w)B_{r-j}'(w) = 2\sum_{j=1}^{r} \frac{K_j}{R^{(3j-1)/2}} \cdot \frac{2[3(r-j)-1]K_{r-j}}{R^{[3(r-j)+1]/2}} + \ldots = \\ \frac{4}{R^{3r/2}}\sum_{j=1}^{r} K_j K_{r-j}[3(r-j)-1] + \ldots = \frac{4(3r-1)K_r - (3r-2)(3r-4)K_{r-1}}{2R^{3r/2}} + \ldots \end{gathered} \tag{121}$$

where the neglected terms are of the form $R^{-j/2}$ with $j < 3r/2$. In (121) we used (58). Accordingly,

$$H_r(w) = \frac{2(3r-1)K_r}{R^{(3r+1)/2}} + \ldots \tag{122}$$

where the neglected term is a polynomial of degree $3r$ in the variable $R^{-1/2}$.

Accordingly, if $r \geq 1$

$$\binom{2n}{n} E\left\{\binom{(2n\rho_n - n)/2}{r}\right\} = 2(3r-1)K_r \binom{n + (3r-1)/2}{n} 4^n + \ldots \tag{123}$$

where the neglected term is of smaller order than the displayed term as $n \to \infty$.

If $r = 1, 2, \ldots$, then by (123)

$$E\left\{\binom{(2n\rho_n - n)/2}{r}\right\} = \frac{4K_r\sqrt{\pi}}{\Gamma(\frac{3r-1}{2})} n^{3r/2} + \ldots \tag{124}$$

and

$$E\{(2n\rho_n)^r\} = M_r (2n)^{3r/2} + \ldots \tag{125}$$

where $M_r$ is given by (5) and the neglected term is of smaller order than the displayed one. This proves (17).

## 7. THE ASYMPTOTIC DISTRIBUTION OF $\rho_n$

We have the following result.

***Theorem 6:*** *We have*

$$\lim_{n\to\infty} P\{\rho_n/\sqrt{2n} \leq x\} = W(x) \tag{126}$$

*where $W(x)$ is the distribution function defined in Theorem 4.*

***Proof:*** By (125) we have

$$\lim_{n\to\infty} E\{(\rho_n/\sqrt{2n})^r\} = M_r \tag{127}$$

for $r = 0, 1, 2, \ldots$ where $M_r$ is given by (5). The remainder of the proof is along the same lines as the proof of Theorem 4.

The process $\{\eta_{[2nt]}/\sqrt{2n}, 0 \leq t \leq 1\}$ converges weakly to the Brownian bridge $\{\eta(t), 0 \leq t \leq 1\}$ as $n \to \infty$. If we define

$$\rho = \int_0^1 \eta(t)dt - \min_{0 \leq t \leq 1} \eta(t), \tag{128}$$

then from (126) and (127) we can conclude that

$$P\{\rho \leq x\} = W(x), \tag{129}$$

$E\{\rho^r\} = M_r$ for $r = 0, 1, 2, \ldots$, and

$$E\{e^{-s\rho}\} = \psi(s) \tag{130}$$

for $Re(s) \geq 0$.

Accordingly, the random variables $\omega^+$ and $\rho$, defined by (87) and (128) respectively, have exactly the same distribution function. For a direct proof of

$$P\{\rho \leq x\} = P\{\omega^+ \leq x\} \tag{131}$$

see W. Vervaat [20].

In 1983, D.A. Darling [6] proved that (130) is given by (86). He observed that $\rho$ can also be expressed as

$$\rho = \max_{0 \leq t \leq 1} \zeta(t) \tag{132}$$

where

$$\zeta(t) = \int_0^1 \eta(u)du - \eta(t). \tag{133}$$

In 1961, G.S. Watson [21] found that $\{\zeta(t), 0 \le t \le 1\}$ is a Gaussian process for which $E\{\zeta(t)\} = 0$ for $0 \le t \le 1$ and

$$E\{\zeta(t)\zeta(u)\} = r(t-u) \tag{134}$$

for $0 \le t \le 1$ and $0 \le u \le 1$ where

$$r(t) = \tfrac{1}{2}(\,|\,t\,|\, - \tfrac{1}{2})^2 - \tfrac{1}{24} \tag{135}$$

for $|\,t\,| \le 1$. By using the representation (133) D.A. Darling [6] proved that (130) is equal to the right-hand side of (86).

## 8. THE ASYMPTOTIC DISTRIBUTION OF $\theta_n$

Now we are in a position to find the asymptotic distribution of $\theta_n$ as $n \to \infty$. We shall prove the following result.

***Theorem 7:*** *We have*

$$\lim_{n\to\infty} P\{\theta_n/\sqrt{2n} \le x\} = \lim_{n\to\infty} P\{\rho_n/\sqrt{2n} \le x\} \tag{136}$$

*where the right-hand side is given by* (126).

***Proof:*** If we assume that in (1), $V(x) = x$ for $0 \le x \le 1$, then (1) can also be expressed as

$$\theta_n = \sum_{i=1}^{2n} \eta_i \xi_i + \delta_{2n} \tag{137}$$

where $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ is the tied-down random walk defined in Section 5, $\delta_{2n}$ is defined by (92) and the random variables $\xi_0, \xi_1, \ldots, \xi_{2n}$ are independent of the random walk $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ and are defined in the following way: We choose $2n$ points at random in the interval $(0,1)$. We assume that the $2n$ points are distributed independently and each point has a uniform distribution over the interval $(0,1)$. These $2n$ points divide the interval $(0,1)$ into $2n+1$ subintervals. The random variables $\xi_0, \xi_1, \ldots, \xi_{2n}$ are defined as the lengths of these subintervals.

It is easy to see that

$$E\{\xi_i\} = 1/(2n+1), \tag{138}$$

and

$$E\{\xi_i^2\} = 2/(2n+1)(2n+2) \tag{139}$$

for $i = 0, 1, \ldots, 2n$ and

$$E\{\xi_i \xi_j\} = 1/(2n+1)(2n+2) \tag{140}$$

if $0 \le i < j \le 2n$. Moreover, we have

$$\sum_{i=1}^{2n} E\{\eta_i^2\} = n(2n+1)/3 \tag{141}$$

and

$$\sum_{i \ne j} E\{\eta_i \eta_j\} = (n-1)n(2n+1)/3. \tag{142}$$

By (91) and (137)

$$\theta_n - \rho_n = \sum_{i=1}^{2n} \eta_i(\xi_i - \frac{1}{2n}). \tag{143}$$

By the above formulas

$$E\{\theta_n - \rho_n\} = 0 \tag{144}$$

and

$$E\{(\theta_n - \rho_n)^2\} = 1/12. \tag{145}$$

Accordingly,

$$E\{(\frac{\theta_n - \rho_n}{\sqrt{2n}})^2\} = \frac{1}{24n} \to 0 \tag{146}$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty}(\frac{\theta_n - \rho_n}{\sqrt{2n}}) = 0 \tag{147}$$

in probability. Now, (126) and (147) imply (136) and (7).

## 9. THE MOMENTS OF $\theta_n$

In the representation (137), the joint density function of $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k}$, where $0 \le i_1 < i_2 < \ldots < i_k \le 2n$ and $1 \le k \le 2n$, is

$$f(x_1, x_2, \ldots, x_k) = 2n(2n-1)\ldots(2n+1-k)(1 - x_1 - x_2 - \ldots - x_k)^{2n-k} \tag{148}$$

if $x_1 \ge 0, x_2 \ge 0, \ldots x_k \ge 0, x_1 + x_2 + \ldots + x_k \le 1$, and 0 otherwise. By (148)

$$E\{\xi_{i_1}^{\alpha_1}\xi_{i_2}^{\alpha_2}\ldots\xi_{i_k}^{\alpha_k}\} = \frac{(2n)!\alpha_1!\alpha_2!\ldots\alpha_k!}{(2n+\alpha_1+\alpha_2+\ldots+\alpha_k)!} \tag{149}$$

for $\alpha_i = 0,1,2,\ldots(1 \le i \le k \le 2n)$.

We can write down that

$$\eta_i = \chi_1 + \chi_2 + \ldots + \chi_i \tag{150}$$

for $i = 1,2,\ldots,2n$ where among the random variables $\chi_1, \chi_2, \ldots, \chi_{2n}$, $n$ take the value $+1$ and $n$ take the value $-1$, and all the possible $\binom{2n}{n}$ choices are equally probable. We have

$$P\{\eta_i = 2k - i\} = \binom{i}{k}\binom{2n-i}{n-i}/\binom{2n}{n} \tag{151}$$

for $k = 0,1,\ldots,i$ and by using the reflection principle we obtain that

$$P\{\delta_{2n} \ge k\} = \binom{2n}{n-k}/\binom{2n}{n} \tag{152}$$

for $k = 0,1,\ldots,n$. Moreover, we have

$$E\{\chi_1\chi_2\cdots\chi_k\} = \sum_{j=0}^{k}(-1)^j\binom{n}{j}\binom{n}{k-j}/\binom{2n}{n} = \begin{cases} (-1)^{k/2}\binom{n}{k/2}/\binom{2n}{n} \text{ if } k = \text{even}, \\ 0 \text{ if } k = \text{ odd}. \end{cases} \tag{153}$$

By using the above formulas, we can prove that

$$E\{\theta_n\} = E\{\delta_{2n}\} = \frac{1}{2}\Big[\frac{4^n}{\binom{2n}{n}} - 1\Big] \tag{154}$$

and

$$E\{\theta_n^2\} = \frac{5n}{6} - E\{\theta_n\}. \tag{155}$$

In a similar way, we can also determine, at least in principle, the higher moments of $\theta_n$, but as the order of the moments increases, the actual calculations rapidly become unmanageable. Theorem 7 suggests, and indeed we can prove that $E\{\theta_n^r\}$ and $E\{\rho_n^r\}$ show identical asymptotic behavior for every $r = 0,1,2,\ldots$ as $n \to \infty$. We shall prove the following theorem.

***Theorem 8:*** *We have*

$$\lim_{n\to\infty} E\{(\theta_n/\sqrt{2n})^r\} = \lim_{n\to\infty} E\{(\rho_n/\sqrt{2n})^r\} = M_r \tag{156}$$

*for $r = 0,1,2,\ldots$ where $M_r$ is given by* (5).

***Proof:*** If $r = 0$, then (156) is trivially true. By (154) we obtain that

$$E\{\theta_n\} \sim \sqrt{n\pi}/2 \tag{157}$$

as $n \to \infty$, and by (155)

$$E\{\theta_n^2\} \sim 5n/6 \tag{158}$$

as $n \to \infty$. Thus (156) is true for $r=1$ and $r=2$. Now we shall prove that (156) is true for every $r \geq 1$. If $r \geq 1$, we can write that

$$E\{\theta_n^r\} - E\{\rho_n^r\} = \sum_{j=1}^{r} \binom{r}{j} E\{(\theta_n - \rho_n)^j \rho_n{}^{r-j}\} \tag{159}$$

and by the Schwarz inequality

$$|\,E\{(\theta_n - \rho_n)^j \rho_n{}^{r-j}\}\,| \leq [E\{(\theta_n - \rho_n)^{2j}\} E\{\rho_n{}^{2(r-j)}\}]^{1/2}. \tag{160}$$

Now by (127)

$$E\{\rho_n{}^{2(r-j)}\} \sim (2n)^{r-j} M_{2(r-j)} \tag{161}$$

as $n \to \infty$. We shall prove that

$$\lim_{n\to\infty} E\{(\theta_n - \rho_n)^{2j}\}/n^j = 0 \tag{162}$$

for $j = 1, 2, \ldots$ and this implies that

$$\lim_{n\to\infty} [E\{\theta_n^r\} - E\{\rho_n^r\}]/n^{r/2} = 0, \tag{163}$$

whence (156) follows.

Let $k_1, k_2, \ldots, k_r$ be positive integers and introduce the notations

$$s_{k_1, k_2, \ldots, k_r} = \sum_{(i_1, i_2, \ldots, i_r)} E\{\eta_{i_1}^{k_1} \eta_{i_2}^{k_2} \cdots \eta_{i_r}^{k_r}\} \tag{164}$$

where $(i_1, i_2, \ldots, i_r)$ is a combination of size $r$ of $(1, 2, \ldots, 2n)$ and the sum is taken over all possible terms of the indicated form, and

$$w_{k_1, k_2, \ldots, k_r} = E\{(\xi_1 - \tfrac{1}{2n})^{k_1} (\xi_2 - \tfrac{1}{2n})^{k_2} \ldots (\xi_r - \tfrac{1}{2n})^{k_r}\}. \tag{165}$$

By (143) we can write that

$$E\{(\theta_n - \rho_n)^{2j}\} = \sum_{\substack{k_1 + k_2 + \ldots = 2j \\ k_1 \geq 1, k_2 \geq 1, \ldots}} \frac{(2j)!}{k_1! k_2! \ldots} s_{k_1, k_2, \ldots} w_{k_1, k_2, \ldots}. \tag{166}$$

By (149) we obtain that if $k_1 + k_2 + \ldots = 2j$, then

$$w_{k_1,k_2,\ldots} = 0(1/n^{2j}) \tag{167}$$

and, in particular, if $k_1 = k_2 = \ldots = k_{2j} = 1$, then

$$w_{k_1,k_2,\ldots,k_{2j}} = 0(1/n^{2j+1}) \tag{168}$$

as $n \to \infty$. Furthermore, by (150) and (153) we obtain that

$$s_{k_1,k_2,\ldots} = 0(n^{j+\alpha}) \tag{169}$$

as $n \to \infty$, where $\alpha$ is the number of $k_i$'s $\geq 1$. Always $\alpha < 2j$, except when $k_1 = k_2 = \ldots = k_{2j} = 1$ in which case $\alpha = 2j$. Accordingly, by (166)

$$E\{(\theta_n - \rho_n)^{2j}\} = 0(n^{j-1}) \tag{170}$$

as $n \to \infty$. This proves (162). This also completes the proof of Theorem 8. Theorem 1 follows from (156).

## 10. AFTERWORD

In this paper we have demonstrated the appearance of the distribution function $W(x)$ in the theories of random walk, order statistics and Brownian excursion. In addition, $W(x)$ appears also in the solutions of problems connected with round-robin tournaments, queuing processes, branching processes and random graphs. Let us mention briefly the case of random trees. A tree is a connected undirected graph which has no cycles, loops or multiple edges. A rooted tree has a vertex, the root, distinguished from the other vertices. The height of a vertex in a rooted tree is the distance from the vertex to the root, that is, the number of edges in the path from the vertex to the root. The total height of a rooted tree is the sum of the heights of its vertices. Let $T_n$ be the set of rooted trees with $n$ vertices and possessing a certain characteristic property, such as labeled vertices, unlabeled vertices, oriented branches, or trivalent branching. Let us choose a tree at random in the set $T_n$ assuming that all the possible choices are equally probable. Denote by $\tau(n)$ the total height of the tree chosen at random. It turns out that for various different models $E\{\tau(n)\}/n^{3/2}$ has a finite positive limit as $n \to \infty$ and

$$\lim_{n\to\infty} P\{\frac{\sqrt{2\pi}\tau(n)}{4E\{\tau(n)\}} \leq x\} = W(x) \tag{171}$$

where $W(x)$ is given in Theorem 2. The various different models of trees require entirely different mathematical analysis, but surprisingly the random variable $\tau(n)$ has the same asymptotic behavior for each model.

Tables and graphs for $W(x)$ and $W'(x)$ can easily be produced by using formulas (9) and (10) and the remarkable program MATHEMATICA by S. Wolfram [22]. The forthcoming paper [19] contains some tables and graphs for $W(x)$ and $W'(x)$.

## REFERENCES

[1] Abramowitz, M. and Stegun, I.A., "*Handbook of Mathematical Functions*", Dover, New York, (1965).

[2] Biane, Ph. and Yor, M., "Valeurs principales associées aux temps locaux browniens", *Bulletin des Sciences Mathématiques* 111, (1987), 23-101.

[3] Carleman, T., "Sur le problème des moments", *Comptes Rendus Acad. Sci.*, Paris, 174, (1922), 1680-1682.

[4] Carleman, T., "*Les Fonctions Quasi-Analytiques*", Gauthier-Villars, Paris, (1926).

[5] Cohen, J.W. and Hooghiemstra, G., "Brownian excursion, the $M/M/1$ queue and their occupation times", *Mathematics of Operations Research* 6, (1981), 608-629.

[6] Darling, D.A., "On the supremum of a certain Gaussian process", *The Annals of Probability* 11, (1983), 803-806.

[7] Fréchet, M. and Shohat, J., "A proof of the generalized second limit theorem in the theory of probability", *Transactions of the American Mathematical Society* 33, (1931), 533-543.

[8] Getoor, R.K. and Sharpe, M.J., "Excursions of Brownian motion and Bessel processes, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 47, (1979), 83-106.

[9] Groeneboom, P, "Brownian motion with parabolic drift", *Probability Theory and Related Fields* 81, (1989), 79-109.

[10] Hardy, G.H. and Wright, E.M., "*An Introduction to the Theory of Numbers*", Third Edition, Oxford University Press, (1956).

[11] Louchard, G., "Kac's formula, Lévy's local time and Brownian excursion", *Journal of Applied Probability* 26, (1984), 479-499.

[12] Louchard, G., "The Brownian excursion area: a numerical analysis", *Computers and Mathematics with Applications* 10, (1984), 413-417. [Erratum: Ibid A 12 (1986), 375.]

[13] Miller, J.C.P., "*The Airy Integral*", Cambridge University Press, (1946).

[14] Ramanujan, S., "Proof of certain identities in combinatorial analysis", *Proceedings of the Cambridge Philosophical Society* 19, (1919), 214-216.

[15] Sarkadi, K., "Mozdonyok várakozási idejéről, (On the waiting time of locomotives)", *A Magyar Tudományos Akadémia Alkalmazott Matematikai Intézetének Közleményei* 3, (1954), 191-194.

[16] Slater, L.J., "*Confluent Hypergeometric Functions*", Cambridge University Press, (1960).

[17] Szekeres, G., "A combinatorial interpretation of Ramanujan's continued fraction", *Canadian Mathematical Bulletin* 11, (1968), 405-408.

[18] Takács, L., "Egy közlekedéssel kapcsolatos valószinűségszámitási problémáról, (On a probability problem connected with railway traffic)", *Publications of the Mathematical Institute of the Hungarian Academy of Sciences* 1, (1956), 99-107.

[19] Takács, L., "A Bernoulli excursion and its various applications", *Advances in Applied Probability* 23, (1991), to appear.

[20] Vervaat, W., "A relation between Brownian bridge and Brownian excursion", *The Annals of Probability* 7, (1979), 143-149.

[21] Watson, G.S., "Goodness-of-fit test on a circle", *Biometrika* 48, (1961), 109-114.

[22] Wolfram, S., "*Mathematica. A System for Doing Mathematics by Computers*", Addison-Wesley, Redwood City, California, (1988).