AN EXISTENCE THEOREM FOR A VOLTERRA INTEGRAL EQUATION WITH DEVIATING ARGUMENTS*

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ABSTRACT

An existence theorem is proved for a nonlinear Volterra integral equation with deviating arguments.

1. INTRODUCTION

The theory of nonlinear Volterra integral equations with deviating arguments and functional integral equations have been studied by many authors [1,2,5,6]. Banas [4] has proved an existence theorem for functional integral equations and Balachandran [1] has proved an existence theorem for a nonlinear Volterra integral equation with deviating argument. In this paper we shall derive a set of sufficient conditions for the existence of a solution of nonlinear Volterra integral equations with deviating arguments. This result is a generalization of the results in [1,4].

2. BASIC ASSUMPTIONS

Let $p(t)$ be a given continuous function defined on the interval $[0,\infty)$ and taking real positive values. Denote $C([0,\infty), p(t):R^n)$ by C_p, the set of all continuous functions from $[0,\infty)$ into R^n such that

$$\sup \{ |x(t)|p(t): \ t \geq 0 \ \} < \infty.$$

It has been proved [7] that C_p forms a real Banach space with regard to the norm

$$\|x\| = \sup \{ \ |x(t)|p(t): \ t \geq 0 \}.$$

If $x \in C_p$ then we will denote by $\omega^T(x,\varepsilon)$ the usual modulus of continuity of x on the interval $[0,T]$, that is,

$$\omega^T(x,\varepsilon) = \sup \{ |x(t) - x(s)| \colon |t - s| \le \varepsilon, \; t,s \in [0,T] \}.$$

Our existence theorem is based on the following lemma.

Lemma (See [3].): *Let E be a bounded set in the space C_p. If all functions belonging to E are equicontinuous on each interval $[0,T]$ and*

$$\lim_{T \to \infty} \{ |x(t)| p(t) \colon t \ge T \} = 0$$

uniformly with respect to E, then E is relatively compact in C_p.

Consider the nonlinear Volterra integral equation with deviating arguments

(1) $x(t) = G(x)g(t,x(h_1(t)), x(h_2(t)),...,x(h_n(t))) + \int_0^t K(t,s,x(H_1(s)),...,x(H_m(s)))ds$

where x, H and K are n-vectors and G is a real-valued function. Assume the following conditions.

(*i*) Let $\Delta = \{(t,s) \colon 0 \le s \le t < \infty \}$. The kernel $K \colon \Delta \times R^{nm} \to R^n$ is continuous and there exist continuous functions $m \colon \Delta \to [0,\infty)$, $a \colon [0,\infty) \to (0,\infty)$, and $b \colon [0,\infty) \to [0,\infty)$ such that

$$|K(t,s,x_1,x_2,...,x_m)| \le m(t,s) + a(t)b(s) \sum_{i=1}^m |x_i|$$

for all $(t,s) \in \Delta$ and $(x_1,x_2,...,x_m) \in R^{nm}$.

In order to formulate other assumptions let us define $L(t) = \int_0^t a(s)b(s)ds$, $t \ge 0$; furthermore, let us take an arbitrary number $M > 0$ and consider the space C_p with $p(t) = [a(t)e^{ML(t)+t}]^{-1}$.

(*ii*) There exists a constant $A > 0$ such that for any $t \in [0,\infty)$ the following inequality holds.

$$\int_0^t m(t,s)ds \le Aa(t)e^{ML(t)}$$

(*iii*) For $i = 1, 2,...,n$ the functions $h_i \colon [0,\infty) \to [0,\infty)$ are continuous, $h_i(0) = 0$, $h_i(t) \le t$ for $t \ge 0$ and there exists a positive real number B_i such that $a(h_i(t)) \le B_i a(t)$.

(*iv*) $G \colon C_p \to [0,\infty)$ is continuous and bounded. Assume $|G(x)| \le k_1$ where k_1 is a positive constant.

(*v*) The function $g \colon [0,\infty) \times R^{n^2} \to R^n$ is continuous and satisfies the conditions

$$|g(t,x_1,x_2,...,x_n) - g(t,y_1,y_2,...,y_n)| \le \sum_{i=1}^n \alpha_i(t)|x_i - y_i|$$

where $\alpha_i(t)$ is continuous such that

$\alpha_i(t) \le e^{M[L(t) - L(h_i(t))]}$ for $t \ge 0$, for $i = 1,2,...,n$ and $|g(t,0,...,0)| \le a(t)e^{ML(t)}$.

(vi) For $i = 1,2,...,m$, the functions H_i: $[0,\infty) \to [0,\infty)$ are continuous and satisfy the following conditions:

$$L(H_i(t)) - L(t) \le N_i$$

where N_i is a positive constant, and

$$a(H_i(t))/a(t) \le (M/m)[1 - k_1(1 + B) - A]e^{-MN_i}$$

where $B = \sum\limits_{i=1}^{n} B_i$ and we assume $k_1(1 + B) + A < 1$.

3. EXISTENCE THEOREM

Theorem: *Assume that the hypotheses* (i) *through* (vi) *hold; then equation* (1) *has at least one solution x in the space* C_p *such that* $|x(t)| \le a(t)e^{ML(t)}$ *for any* $t \ge 0$.

Proof: Define a transformation F in the space C_p by

(2)
$$(Fx)(t) = G(x)g(t,x(h_1(t)), x(h_2(t)),...,x(h_n(t)))$$
$$+ \int\limits_0^t K(t,s,x(H_1(s)),...,x(H_m(s)))ds.$$

From our assumptions we observe that $(Fx)(t)$ is continuous on the interval $[0,\infty)$. Define the set E in C_p by

$$E = \{x \in C_p: |x(t)| \le a(t)e^{ML(t)}\}.$$

Clearly E is nonempty, bounded, convex, and closed in C_p. Now we prove that F maps the set E into itself. Take $x \in E$. Then from our assumptions we have

$$|(Fx)(t)| \le |G(x)| \, |g(t,x(h_1(t)),...,x(h_n(t)))| + \int\limits_0^t |K(t,s,x(H_1(s)),...,x(H_m(s)))|ds.$$

$$\le k_1|g(t,x(h_1(t)),...,x(h_n(t))) - g(t,0,...,0)| + k_1|g(t,0,...,0)| + \int\limits_0^t m(t,s)ds$$

$$+ a(t) \int\limits_0^t b(s)\sum\nolimits_{i=1}^{m}|x(H_i(s))|ds$$

$$\le k_1 \sum\limits_{i=1}^{n} \alpha_i(t)|x(h_i(t))| + k_1 a(t)e^{ML(t)} + Aa(t)e^{ML(t)}$$

$$+ a(t) \int\limits_0^t b(s)\sum\nolimits_{i=1}^{m}a(H_i(s))e^{ML(H_i(s))}ds$$

$$\le k_1 \sum\limits_{i=1}^{n} e^{M[L(t) - L(h_i(t))]}a(h_i(t))e^{ML(H_i(t))} + k_1 a(t)e^{ML(t)} + Aa(t)e^{ML(t)}$$

$$+ a(t)([1 - k_1(1+ B) - A]/m) \sum_{i=1}^{m} e^{-MN_i} \int_0^t Mb(s)a(s)e^{MN_i} e^{ML(s)}ds$$

$$\leq k_1 Ba(t)e^{ML(t)} + k_1 a(t)e^{ML(t)} + Aa(t)e^{ML(t)} + a(t)[1 - k_1(1 + B) - A]e^{ML(t)}$$

$$= a(t)e^{ML(t)},$$

which proves that FE is a subset of E.

Now we want to prove that F is continuous on the set E. In order to do this we take $F = F_1 + F_2$, where

$$(F_1 x)(t) = G(x)g(t,x(h_1(t)),...,x(h_n(t)))$$

and $$(F_2 x)(t) = \int_0^t K(t,s,x(H_1(s)),...,x(H_m(s)))ds.$$

We shall prove the continuity of F_1 and F_2 separately. Let us fix $\varepsilon > 0$ and take $x,y \in E$ such that $\|x - y\| \leq \varepsilon$. We have

$$\|F_1 x - F_1 y\| \leq k_1 \sup\{|g(t,x(h_1(t)),...,x(h_n(t)))$$

$$- g(t,y(h_1(t)),...,y(h_n(t)))|\cdot[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$

$$+ |G(x) - G(y)|\sup\{|g(t,y(h_1(t)),...,y(h_n(t)))|\cdot[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$

$$\leq k_1 \sum_{i=1}^{n}\sup\{\alpha_i(t)|x(h_i(t)) - y(h_i(t))|[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$

$$+ |G(x) -G(y)| \sup\{|g(t,y(h_1(t)),...,y(h_n(t))) -g(t,0,...,0)|\cdot[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$

$$+ |G(x) -G(y)| \sup\{|g(t,0,...,0)\cdot[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$

$$\leq k_1 \sum_{i=1}^{n}B_i\sup\{|x(h_i(t)) - y(h_i(t))|[a(h_i(t))]^{-1}e^{M[L(t)-L(h_i(t))]}e^{-ML(t)-t}: t \geq 0\}$$

$$+|G(x) - G(y)| \sum_{i=1}^{n} \sup\{\alpha_i(t)|y(h_i(t))| [a(t) e^{ML(t)} + t]^{-1} : t \geq 0\}$$

$$+ |G(x) - G(y)| \sup\{e^{-t}: t \geq 0\}$$

$$\leq k_1 \sum_{i=1}^{n}B_i\sup\{|x(h_i(t)) - y(h_i(t))|[a(h_i(t)) e^{ML(h_i(t))} + h_i(t)]^{-1} : t \geq 0\}$$

$$+ |G(x) - G(y)| \sum_{i=1}^{n} \sup\{a(h_i(t))e^{ML(t)}[a(t) e^{ML(t)+t}]^{-1}: t \geq 0\}+ |G(x)-G(y)|$$

$$\leq k_1 B \| x-y\| + B |G(x) - G(y)| + |G(x) - G(y)|.$$

This implies that F_1 is continuous in view of *(iv)*.

Now we prove that F_2 is continuous on the set E. For this let us fix $\varepsilon > 0$ and $x, y \in E$ such that $\| x - y \| \leq \varepsilon$. Further, let us take an arbitrary fixed $T > 0$. In view of *(i)* and *(vi)* the function $K(t, s, x_1, \ldots, x_m)$ is uniformly continuous on

$$[0, T] \times [0, T] \times [-r(H_1(T)), r(H_2(T))] \times ... \times [-r(H_m(T)), r(H_m(T))]$$

where $r(T) = max\{a(s)e^{ML(s)}_t : s \in [0, T]$. Thus, we have for $t \in [0, T]$

(3) $\quad |(F_2 x) (t) - (F_2 y) (t)| \leq \int_0^t | K(t, s, x(H_1(s)),..., x(H_m(s))$

$$-K(t, s, y (H_1(s)),..., y(H_m(s)) | \, ds$$

$$\leq \beta(\varepsilon)$$

where $\beta(\varepsilon)$ is some continuous function such that $\displaystyle\lim_{\varepsilon \to 0} \beta(\varepsilon) = 0$. Further, let us take $t \geq T$. Then we have

$$| (F_2 x) (t) - (F_2 y) (t) | \leq | (F_2 x) (t) | + | (F_2 y) (t) |$$

$$\leq 2a (t) e^{ML(t)}$$

and $\quad | (F_2 x) (t) - (F_2 y) (t)| / p(t) \leq 2e^{-t}$.

Hence for sufficiently large T we have

(4) $$| (F_2 x) (t) - (F_2 y) (t) | / p(t) \leq \varepsilon .$$

By (3) and (4) we get that F_2 is continuous on the set E. Hence $F = F_1 + F_2$ is continuous on E.

Now we prove that FE is relatively compact. For every $x \in E$ we have $Fx \in E$ which gives $|(Fx)(t)|p(t) \leq e^{-t}$. Hence $\displaystyle\lim_{T \to \infty} \sup\{|(Fx)(t)|p(t):t \geq T \} = 0$ uniformly with respect to $x \in E$.

Furthermore, let us fix $\varepsilon > 0$ and $T > 0$; and let $t, s \in [0, T]$ such that $|t-s| \leq \varepsilon$. Then for $x \in E$, we have

$$| (Fx) (t) - (Fx)(s) | \leq |G(x) | \, | g(t, x(h_1(t)),..., x(h_n(t)))$$

$$- g(s, x(h_1(s)),..., x(h_n(s)) |$$

$$+ | \int_0^t K(t, u, x(H_1(u)),..., x(H_m(u))du$$

$$- \int_0^s K(s, u, y(H_1(u)),..., y(H_m(u))du |$$

$$\leq k_1 \, \omega^T \, (g, \, \varepsilon) + / \int_0^t K(t, \, u, \, x(H_1(u)),..., \, x(H_m(u)) \, du$$

$$- \int_0^s K(t, \, u, \, x(H_1(u)),..., \, x(H_m(u))) \, du/$$

$$+ / \int_0^s K(t, \, u, \, x(H_1(\,)),..., \, x(H_m(u))) \, du$$

$$- \int_0^s K(s, \, u, \, x(H_1(u)),..., \, x(H_m(u))) \, du \, /$$

$$\leq k_1 \, \omega^T(g, \, \varepsilon) + \int_s^t \, / \, K(t, \, u, \, x(H_1(u)),..., \, x(H_m(u))) \, / \, du$$

$$+ \int_0^s \, / \, K(t, \, u, \, x(H_1(u)),..., \, x(H_m(u)))$$

$$- K(s, \, , \, x(H_1(u)),..., \, x(H_m(u))) \, / \, du$$

$$\leq k_1 \, \omega^T \, (g, \, \varepsilon + \varepsilon \, max\{ \, m(t, \, u) + a(t)b(u) \, \sum_{i=1}^m \, / \, x(H_i(u) \, / \, : \, o \leq u \leq t \leq T\}$$

$$+ T \omega^T(K, \, \varepsilon).$$

This tends to zero as $\varepsilon \rightarrow 0$. Thus FE is equicontinuous on $[0, \, T]$.

Therefore from the lemma FE is relatively compact. Thus the Schauder fixed point theorem guarantees that F has a fixed point $x \in E$ such that $(Fx) \, (t) = x(t)$. Hence the theorem holds.

EXAMPLE: Consider the following nonlinear integral equation:

(5) $$x(t) = (1/8) \, tsinx \, (t/2) + \int_0^t [ts + (t^3 + (1/4)) \, x(s/3) \,] \, ds.$$

This is clearly of the form (1) and satisfies all the conditions (i) to (vi) with

$$m = n = 1, G(x) = 1/8, \; g(t, x) = \; t \sin x,$$

$$K(t, s, x) = ts + (t^3 + (1/4)) \, x,$$

$$m(t, s) = ts, a(t) = t^3 + (1/4), b(s) = 1,$$

$$\alpha_1(t) = t, h_1(t) = t/2, H_1(t) = t/3,$$

$$M = 16, A = 1/2, B = 1, k_1 = 1/8, N_1 = 1/16.$$

Therefore from our existence theorem the equation (5) has at least one solution x in the space C_p such that

$$| \, x(t) \, | \le (t^3 + 1/4)e^{4(t^4 + t)} \; \text{for any } t \ge 0.$$

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