Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2009, Article ID 254720, 18 pages
doi:10.1155/2009/254720

Research Article

Implicit Difference Inequalities Corresponding to First-Order Partial Differential Functional Equations

Z. Kamont and K. Kropielnicka

Institute of Mathematics, University of Gdańsk, Wit Stwosz Street 57, 80-952 Gdańsk, Poland

Correspondence should be addressed to Z. Kamont, zkamont@math.univ.gda.pl

Received 19 August 2008; Accepted 5 January 2009

Recommended by Donal O'Regan

We give a theorem on implicit difference functional inequalities generated by mixed problems for nonlinear systems of first-order partial differential functional equations. We apply this result in the investigations of the stability of difference methods. Classical solutions of mixed problems are approximated in the paper by solutions of suitable implicit difference schemes. The proof of the convergence of difference method is based on comparison technique, and the result on difference functional inequalities is used. Numerical examples are presented.

1. Introduction

The papers [1, 2] initiated the theory of difference inequalities generated by first-order partial differential equations. The results and the methods presented in [1, 2] were extended in [3, 4] on functional differential problems, and they were generalized in [5–8] on parabolic differential and differential functional equations. Explicit difference schemes were considered in the above papers.

Our purpose is to give a result on implicit difference inequalities corresponding to initial boundary value problems for first-order functional differential equations.

We prove also that that there are implicit difference methods which are convergent. The proof of the convergence is based on a theorem on difference functional inequalities.

We formulate our functional differential problems. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E=[0,a]\times(-b,b),\qquad D=\left[-d_0,0\right]\times[-d,d],\tag{1.1}$$

where $a > 0$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, $b_i > 0$ for $1 \le i \le n$ and $d = (d_1, \dots, d_n) \in \mathbb{R}_+^n$, $d_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$. Let $c = b + d$ and

$$\begin{gathered} E_0 = \big[-d_0, 0\big] \times [-c, c], \\ \partial_0 E = [0, a] \times ([-c, c] \setminus (-b, b)), \\ \Omega = E \cup E_0 \cup \partial_0 E. \end{gathered} \tag{1.2}$$

For a function $z : \Omega \longrightarrow \mathbb{R}^k$, $z = (z_1, \dots, z_k)$, and for a point $(t, x) \in \overline{E}$ where $\overline{E}$ is the closure of E, we define a function $z_{(t,x)} : D \longrightarrow \mathbb{R}^k$ by $z_{(t,x)}(\tau, y) = z(t + \tau, x + y)$, $(\tau, y) \in D$. Then $z_{(t,x)}$ is the restriction of z to the set $[t - d_0, t] \times [x - d, x + d]$ and this restriction is shifted to the set D. Write $\Sigma = E \times C(D, \mathbb{R}^k) \times \mathbb{R}^n$ and suppose that $f = (f_1, \dots, f_k) : \Sigma \longrightarrow \mathbb{R}^k$ and $\varphi : E_0 \cup \partial_0 E \longrightarrow \mathbb{R}^k$, $\varphi = (\varphi_1, \dots, \varphi_k)$, are given functions. Let us denote by $z = (z_1, \dots, z_k)$ an unknown function of the variables (t, x), $x = (x_1, \dots, x_n)$. Write

$$\mathbb{F}[z](t, x) = \big(f_1\big(t, x, z_{(t,x)}, \partial_x z_1(t, x)\big), \dots, f_k\big(t, x, z_{(t,x)}, \partial_x z_k(t, x)\big)\big) \tag{1.3}$$

and $\partial_x z_i = (\partial_{x_1} z_i, \dots, \partial_{x_n} z_i)$, $1 \le i \le k$. We consider the system of functional differential equations

$$\partial_t z(t, x) = \mathbb{F}[z](t, x) \tag{1.4}$$

with the initial boundary condition

$$z(t, x) = \varphi(t, x) \quad \text{on } E_0 \cup \partial_0 E. \tag{1.5}$$

In the paper we consider classical solutions of (1.4), (1.5).

We give examples of equations which can be obtained from (1.4) by specializing the operator f.

Example 1.1. Suppose that the function $\alpha : E \longrightarrow \mathbb{R}^{1+n}$ satisfies the condition: $\alpha(t, x) - (t, x) \in D$ for $(t, x) \in E$. For a given $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_k) : E \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^k$ we put

$$f(t, x, w, q) = \tilde{f}\big(t, x, w(0, \theta), w\big(\alpha(t, x) - (t, x)\big), q\big) \quad \text{on } \Sigma, \tag{1.6}$$

where $\theta = (0, \dots, 0) \in \mathbb{R}^n$. Then (1.4) is reduced to the system of differential equations with deviated variables

$$\partial_t z_i(t, x) = \tilde{f}_i\big(t, x, z(t, x), z\big(\alpha(t, x)\big), \partial_x z_i(t, x)\big), \quad i = 1, \dots, k. \tag{1.7}$$

Example 1.2. For the above $\tilde{f}$ we define

$$f(t, x, w, q) = \tilde{f}\left(t, x, w(0, \theta), \int_D w(\tau, y) dy\, d\tau, q\right) \quad \text{on } \Sigma. \tag{1.8}$$

Then (1.4) is equivalent to the system of differential integral equations

$$\partial_t z_i(t,x) = \tilde{f}_i\left(t, x, z(t,x), \int_D z(t+\tau, x+y)\,dy\,d\tau, \partial_x z_i(t,x)\right), \quad i = 1,\ldots,k. \tag{1.9}$$

It is clear that more complicated differential systems with deviated variables and differential integral problems can be obtained from (1.4) by a suitable definition of f. Sufficient conditions for the existence and uniqueness of classical or generalized solutions of (1.4), (1.5) can be found in [9, 10].

Our motivations for investigations of implicit difference functional inequalities and for the construction of implicit difference schemes are the following. Two types of assumptions are needed in theorems on the stability of difference functional equations generated by (1.4), (1.5). The first type conditions concern regularity of f. It is assumed that

(i) the function f of the variables (t, x, w, q), $q = (q_1, \ldots, q_n)$, is of class C^1 with respect to q and the functions $\partial_q f_i = (\partial_{q_1} f_i, \ldots, \partial_{q_n} f_i)$, $1 \le i \le k$, are bounded,

(ii) f satisfies the Perron type estimates with respect to the functional variable w.

The second type conditions concern the mesh. It is required that difference schemes generated by (1.4), (1.5) satisfy the condition

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \left|\partial_{q_j} f_i(t, x, w, q)\right| \ge 0 \quad \text{on } \Sigma \text{ for } i = 1,\ldots,k, \tag{1.10}$$

where h_0 and $h' = (h_1, \ldots, h_n)$ are steps of the mesh with respect to t and $(x_1, \ldots, x_n)$ respectively. The above assumption is known as a generalized Courant-Friedrichs-Levy (CFL) condition for (1.4), (1.5) (see [11, Chapter 3] and [10, Chapter 5]). It is clear that strong assumptions on relations between h_0 and h' are required in (1.10). It is important in our considerations that assumption (1.10) is omitted in a theorem on difference inequalities and in a theorem on the convergence of difference schemes.

We show that there are implicit difference methods for (1.4), (1.5) which are convergent while the corresponding explicit difference schemes are not convergent. We give suitable numerical examples.

The paper is organized as follows. A theorem on implicit difference functional inequalities with unknown function of several variables is proved in Section 2. We propose in Section 3 implicit difference schemes for the numerical solving of functional differential equations. Convergence results and error estimates are presented. A theorem on difference inequalities is used in the investigation of the stability of implicit difference methods. Numerical examples are given in the last part of the paper.

We use in the paper general ideas for finite difference equations which were introduced in [12–14]. For further bibliographic informations concerning differential and functional differential inequalities and applications see the survey paper [15] and the monographs [16, 17].

2. Functional Difference Inequalities

For any two sets U and W we denote by $F(U,W)$ the class of all functions defined on U and taking values in W. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. For $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $p = (p_1,\ldots,p_k) \in \mathbb{R}^k$ we put

$$\|x\| = |x_1| + \cdots + |x_n|, \qquad \|p\|_\infty = \max\{|p_i| : 1 \le i \le k\}. \tag{2.1}$$

We define a mesh on Ω in the following way. Suppose that (h_0, h'), $h' = (h_1,\ldots,h_n)$, stand for steps of the mesh. For $(r,m) \in \mathbb{Z}^{1+n}$ where $m = (m_1,\ldots,m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \qquad x^{(m)} = \left(x_1^{(m_1)},\ldots,x_n^{(m_n)}\right) = (m_1h_1,\ldots,m_nh_n). \tag{2.2}$$

Let us denote by H the set of all $h = (h_0,h')$ such that there are $K_0 \in \mathbb{Z}$ and $K = (K_1,\ldots,K_n) \in \mathbb{Z}^n$ satisfying the conditions: $K_0h_0 = d_0$ and $(K_1h_1,\ldots,K_nh_n) = d$. Set

$$\begin{gathered}\mathbb{R}_h^{1+n} = \left\{\left(t^{(r)},x^{(m)}\right) : (r,m) \in \mathbb{Z}^{1+n}\right\},\\ D_h = D \cap \mathbb{R}_h^{1+n}, \qquad E_h = E \cap \mathbb{R}_h^{1+n}, \qquad E_{0.h} = E_0 \cap \mathbb{R}_h^{1+n},\\ \partial_0 E_h = \partial_0 E \cap \mathbb{R}_h^{1+n}, \qquad \Omega_h = E_h \cup E_{0.h} \cup \partial_0 E_h.\end{gathered} \tag{2.3}$$

Let $N_0 \in \mathbb{N}$ be defined by the relations: $N_0h_0 \le a < (N_0+1)h_0$ and

$$E'_h = \left\{\left(t^{(r)},x^{(m)}\right) \in E_h : 0 \le r \le N_0 - 1\right\}. \tag{2.4}$$

For functions $w : D_h \to \mathbb{R}^k$ and $z : \Omega_h \to \mathbb{R}^k$ we write $w^{(r,m)} = w(t^{(r)},x^{(m)})$ on D_h and $z^{(r,m)} = z(t^{(r)},x^{(m)})$ on Ω_h. We need a discrete version of the operator $(t,x) \to z_{(t,x)}$. For a function $z : \Omega_h \to \mathbb{R}^k$ and for a point $(t^{(r)},x^{(m)}) \in E_h$ we define a function $z_{[r,m]} : D_h \to \mathbb{R}^k$ by

$$z_{[r,m]}(\tau,y) = z\left(t^{(r)}+\tau, x^{(m)}+y\right), \quad (\tau,y) \in D_h. \tag{2.5}$$

Solutions of difference equations corresponding to (1.4), (1.5) are functions defined on the mesh. On the other hand (1.4) contains the functional variable $z_{(t,x)}$ which is an element of the space $C(D,\mathbb{R}^k)$. Then we need an interpolating operator $T_h : F(D_h,\mathbb{R}^k) \to C(D,\mathbb{R}^k)$. We define T_h in the following way. Let us denote by $(\vartheta_1,\ldots,\vartheta_n)$ the family of sets defined by

$$\vartheta_i = \{0,1\} \quad \text{if } d_i > 0, \quad \vartheta_i = \{0\} \quad \text{if } d_i = 0,\ 1 \le i \le n. \tag{2.6}$$

Set $\upsilon = (\upsilon_1,\ldots,\upsilon_n) \in \mathbb{Z}^n$ and $\upsilon_i = 0$ if $d_i = 0$, $\upsilon_i = 1$ if $d_i > 0$ where $1 \le i \le n$. Write

$$\Delta_+ = \left\{\lambda = (\lambda_1,\ldots,\lambda_n) :\ \lambda_i \in \vartheta_i \text{ for } 1 \le i \le n\right\}. \tag{2.7}$$

Set $e_i = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^n$ with 1 standing on the ith place.

Let $w \in F(D_h, \mathbb{R}^k)$ and $(t,x) \in D$. Suppose that $d_0 > 0$. There exists $(t^{(r)}, x^{(m)}) \in D_h$ such that $(t^{(r+1)}, x^{(m+v)}) \in D_h$ and $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+v)}$. Write

$$\begin{aligned} T_h[w](t,x) = {}& \left(1 - \frac{t-t^{(r)}}{h_0}\right) \sum_{\lambda \in \Delta_+} w^{(r,m+\lambda)} \left(\frac{x-x^{(m)}}{h'}\right)^{\lambda} \left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\lambda} \\ & + \frac{t-t^{(r)}}{h_0} \sum_{\lambda \in \Delta_+} w^{(r+1,m+\lambda)} \left(\frac{x-x^{(m)}}{h'}\right)^{\lambda} \left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\lambda}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \left(\frac{x-x^{(m)}}{h'}\right)^{\lambda} &= \prod_{i=1}^{n} \left(\frac{x_i - x_i^{(m_i)}}{h_i}\right)^{\lambda_i}, \\ \left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\lambda} &= \prod_{i=1}^{n} \left(1 - \frac{x_i - x_i^{(m_i)}}{h_i}\right)^{1-\lambda_i} \end{aligned} \tag{2.9}$$

and we take $0^0 = 1$ in the above formulas. If $d_0 = 0$ then we put

$$T_h[w](t,x) = \sum_{\lambda \in \Delta_+} w^{(r,m+\lambda)} \left(\frac{x-x^{(m)}}{h'}\right)^{\lambda} \left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\lambda}. \tag{2.10}$$

Then we have defined $T_h[w]$ on D. It is easy to see that $T_h[w] \in C(D, \mathbb{R}^k)$. The above interpolating operator has been first proposed in [10, Chapter 5].

For $w, \overline{w} \in F(D_h, \mathbb{R}^k)$ we write $w \le \overline{w}$ if $w^{(r,m)} \le \overline{w}^{(r,m)}$ where $(t^{(r)}, x^{(m)}) \in D_h$. In a similar way we define the relation $w \le \overline{w}$ for $w, \overline{w} \in C(D, \mathbb{R}^k)$ and the relation $z \le \overline{z}$ for $z, \overline{z} \in F(\Omega_h, \mathbb{R}^k)$ and for $z, \overline{z} \in C(\Omega, \mathbb{R}^k)$.

We formulate an implicit difference scheme for (1.4), (1.5). For $x, y \in \mathbb{R}^n$ we write $x \diamond y = (x_1 y_1, \ldots, x_n y_n) \in \mathbb{R}^n$.

Assumption $(H[f])$. The function $f = (f_1, \ldots, f_k) : \Sigma \to \mathbb{R}^k$ of the variables (t, x, w, q), $q = (q_1, \ldots, q_n)$, is continuous and

(1) the partial derivatives $(\partial_{q_1} f_i, \ldots, \partial_{q_n} f_i) = \partial_q f_i$, $i = 1, \ldots, k$, exist on Σ and the functions $\partial_q f_i$, $i = 1, \ldots, k$, are continuous and bounded on Σ,

(2) there is $\tilde{x} \in (-b, b)$, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, such that

$$(x - \tilde{x}) \diamond \partial_q f_i(t, x, w, q) \ge \theta \quad \text{on } \Sigma \text{ for } i = 1, \ldots, k, \tag{2.11}$$

(3) there is $\varepsilon_0 > 0$ such that for $0 < h_0 < \varepsilon_0$ and $w, \overline{w} \in C(D, \mathbb{R}^k)$, $w \le \overline{w}$, we have

$$w(0,\theta) + h_0 f(t, x, w, q) \le \overline{w}(0,\theta) + h_0 f(t, x, \overline{w}, q), \quad (t, x, q) \in E \times \mathbb{R}^n. \tag{2.12}$$

Remark 2.1. The existence theory of classical or generalized solutions to (1.4), (1.5) is based on a method of bicharacteristics. Suppose that $z \in C(\Omega, \mathbb{R}^k)$, $u \in C(\Omega, \mathbb{R}^n)$. Let us denote by

$$g_i[z,u](\cdot,t,x) = \big(g_{i.1}[z,u](\cdot,t,x), \dots, g_{i.n}[z,u](\cdot,t,x)\big) \tag{2.13}$$

the ith bicharacteristic of (1.4) corresponding to (z,u). Then $g_i[z,u](\cdot,t,x)$ is a solution of the Cauchy problem

$$y'(\tau) = -\partial_q f_i\big(\tau, y(\tau), z_{(\tau,y(\tau))}, u(\tau, y(\tau))\big), \qquad y(t) = x. \tag{2.14}$$

Assumption (2.11) states that the bicharacteristics satisfy the following monotonicity conditions: If $x_j - \tilde{x}_j \geq 0$ the function $g_{ij}[z,u](\cdot,t,x)$ is non increasing. If $x_j - \tilde{x}_j < 0$ then $g_{ij}[z,u](\cdot,t,x)$ is nondecreasing.

The same property of bicharacteristics is needed in a theorem on the existence and uniqueness of solutions to (1.4), (1.5) see [9]. It is important that our theory of difference methods is consistent with known theorems on the existence of solutions to (1.4), (1.5).

Remark 2.2. Given the function $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_k) : E \times \mathbb{R} \times C(D, \mathbb{R}^k) \times \mathbb{R}^n \to \mathbb{R}^k$ of the variables (t,x,p,w,q). Write $f_i(t,x,w,q) = \tilde{f}_i(t,x,w_i(0,\theta),w,q)$, $i = 1,\dots,k$, on Σ. Then system (1.4) is equivalent to

$$\partial_t z_i(t,x) = \tilde{f}_i\big(t,x,z_i(t,x), z_{(t,x)}, \partial_x z_i(t,x)\big), \quad i = 1,\dots,k. \tag{2.15}$$

Note that the dependence of $\tilde{f}$ on the classical variable $z(t,x)$ is distinguished in (2.15). Suppose that

(1) $\tilde{f}$ is nondecreasing with respect to the functional variable,

(2) there exists the derivative $\partial_p \tilde{f} = (\partial_p \tilde{f}_1, \dots, \partial_p \tilde{f}_k)$ and $\partial_p \tilde{f}_i(t,x,p,w,q) \geq L$ for $i = 1,\dots,k$ and $1 + Lh_0 \geq 0$.

Then the monotonicity condition (3) of Assumption $(H[f])$ is satisfied.

Let us denote by $H^\star$ the set of all $h = (h_0, h') \in H$ such that

$$h_i < \min\{b_i - \tilde{x}_i,\ \tilde{x}_i + b_i\}, \quad i = 1,\dots,n. \tag{2.16}$$

Suppose that $\omega : \Omega_h \to \mathbb{R}$. We apply difference operators $\delta = (\delta_1, \dots, \delta_n)$ given by

$$\begin{aligned} &\text{if } \tilde{x}_j \leq x_j^{(m_j)} < b_j \text{ then } \delta_j \omega^{(r,m)} = \frac{1}{h_j}\big[\omega^{(r,m+e_j)} - \omega^{(r,m)}\big], \\ &\text{if } -b_j < x_j^{(m_j)} < \tilde{x}_j \text{ then } \delta_j \omega^{(r,m)} = \frac{1}{h_j}\big[\omega^{(r,m)} - \omega^{(r,m-e_j)}\big], \end{aligned} \tag{2.17}$$

and we put $j = 1, \dots, n$ in (2.17). Let δ_0 be defined by

$$\delta_0 \omega^{(r,m)} = \frac{1}{h_0}\left[\omega^{(r+1,m)} - \omega^{(r,m)}\right] \tag{2.18}$$

and $\delta_0 z = (\delta_0 z_1, \dots, \delta_0 z_k)$. Write

$$\mathbb{F}_h[z]^{(r,m)} = \left(f_1\left(t^{(r)}, x^{(m)}, T_h z_{[r,m]}, \delta z_1^{(r+1,m)}\right), \dots, f_k\left(t^{(r)}, x^{(m)}, T_h z_{[r,m]}, \delta z_k^{(r+1,m)}\right)\right). \tag{2.19}$$

Given $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^k$, we consider the functional difference equation

$$\delta_0 z^{(r,m)} = \mathbb{F}_h[z]^{(r,m)} \tag{2.20}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h. \tag{2.21}$$

The above problem is considered as an implicit difference method for (1.4), (1.5). It is important that the difference expressions $(\delta_1 z_i, \dots, \delta_n z_i)$, $1 \le i \le k$, are calculated at the point $(t^{(r+1)}, x^{(m)})$ and the functional variable $T_h z_{[r,m]}$ appears in a classical sense.

We prove a theorem on implicit difference inequalities corresponding to (2.20), (2.21). Note that results on implicit difference methods presented in [18] are not applicable to (2.20), (2.21).

Theorem 2.3. *Suppose that Assumption $(H[f])$ is satisfied and*

(1) $h \in H^\star$, $h_0 < \varepsilon_0$ *and the functions* $u, v : \Omega_h \to \mathbb{R}^k$ *satisfy the difference functional inequality*

$$\delta_0 u^{(r,m)} - \mathbb{F}_h[u]^{(r,m)} \le \delta_0 v^{(r,m)} - \mathbb{F}_h[v]^{(r,m)} \quad \text{on } E_h', \tag{2.22}$$

(2) *the initial boundary estimate* $u^{(r,m)} \le v^{(r,m)}$ *holds on* $E_{0.h} \cup \partial_0 E_h$.

Then

$$u^{(r,m)} \le v^{(r,m)} \quad \text{on } E_h. \tag{2.23}$$

Proof. We prove (2.23) by induction on r. It follows from assumption (2) that estimate (2.23) is satisfied for $r = 0$ and $(t^{(0)}, x^{(m)}) \in E_h$. Assume that $u^{(j,m)} \le v^{(j,m)}$ for $(t^{(j)}, x^{(m)}) \in E_h \cap ([0, t^{(r)}] \times \mathbb{R}^n)$. We prove that $u^{(r+1,m)} \le v^{(r+1,m)}$ for $(t^{(r+1,m)}, x^{(m)}) \in E_h$. Write

$$\begin{aligned} U_i^{(r,m)} = {} & u_i^{(r,m)} + h_0 f_i\left(t^{(r)}, x^{(m)}, T_h u_{[r,m]}, \delta u_i^{(r+1,m)}\right) \\ & - v_i^{(r,m)} - h_0 f_i\left(t^{(r)}, x^{(m)}, T_h v_{[r,m]}, \delta u_i^{(r+1,m)}\right), \quad i = 1, \dots, k. \end{aligned} \tag{2.24}$$

It follows from (2.22) that

$$\begin{aligned}(u_i - v_i)^{(r+1,m)} \le U_i^{(r,m)} + h_0\big[& f_i\big(t^{(r)}, x^{(m)}, T_h v_{[r,m]}, \delta u_i^{(r+1,m)}\big)\\ & - f_i\big(t^{(r)}, x^{(m)}, T_h v_{[r,m]}, \delta v_i^{(r+1,m)}\big)\big],\end{aligned} \tag{2.25}$$

where $i = 1, \ldots, k$. The monotonicity condition (3) of Assumption $(H[f])$ implies the inequalities $U_i^{(r,m)} \le 0$ for $(t^{(r)}, x^{(m)}) \in E_h$, $i = 1, \ldots, k$. Then we have

$$(u_i - v_i)^{(r+1,m)} \le h_0 \sum_{j=1}^{n} \int_0^1 \partial_{q_j} f_i\big(Q_i^{(r,m)}[v,\tau]\big) d\tau\, \delta_j (u_i - v_i)^{(r+1,m)}, \tag{2.26}$$

where $i = 1, \ldots, k$ and

$$Q_i^{(r,m)}[v,\tau] = \big(t^{(r)}, x^{(m)}, T_h v_{[r,m]}, \delta v_i^{(r+1,m)} + \tau\delta(u_i - v_i)^{(r+1,m)}\big). \tag{2.27}$$

Write

$$\Gamma_+^{(m)} = \big\{ j \in \{1, \ldots, n\} : x_j^{(m_j)} \in [\tilde{x}_j, b_j)\big\}, \qquad \Gamma_-^{(m)} = \{1, \ldots, n\} \setminus \Gamma_+^{(m)}. \tag{2.28}$$

It follows from (2.11), (2.17) that

$$\begin{aligned}(u_i - v_i)^{(r+1,m)} & \left[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} \int_0^1 \big|\partial_{q_j} f_i\big(Q_i^{(r,m)}[v,\tau]\big)\big| d\tau\right]\\ & \le h_0 \sum_{j \in \Gamma_+^{(m)}} \frac{1}{h_j} \int_0^1 \partial_{q_j} f_i\big(Q_i^{(r,m)}[v,\tau]\big) d\tau (u_i - v_i)^{(r+1,m+e_j)}\\ & \quad - h_0 \sum_{j \in \Gamma_-^{(m)}} \frac{1}{h_j} \int_0^1 \partial_{q_j} f_i\big(Q_i^{(r,m)}[v,\tau]\big) d\tau (u_i - v_i)^{(r+1,m-e_j)}, \quad i = 1, \ldots, k.\end{aligned} \tag{2.29}$$

We define $\tilde{m} \in \mathbb{Z}^n$ and $\mu \in \mathbb{N}$, $1 \le \mu \le k$, as follows:

$$(u_\mu - v_\mu)^{(r+1,\tilde{m})} = \max_{1 \le i \le k} \max\big\{ (u_i - v_i)^{(r+1,m)} : \big(t^{(r+1)}, x^{(m)}\big) \in \Omega_h \big\}. \tag{2.30}$$

If $(t^{(r+1)}, x^{(\tilde{m})}) \in \partial_0 E_h$ then assumption (2) implies that $(u_\mu - v_\mu)^{(r+1,\tilde{m})} \le 0$. Let us consider the case when $(t^{(r+1)}, x^{(\tilde{m})}) \in E_h$. Then we have from (2.29) that

$$\begin{aligned}(u_\mu - v_\mu)^{(r+1,\tilde{m})} &\left[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} \int_0^1 \left|\partial_{q_j} f_i(Q_i^{(r,\tilde{m})}[v,\tau])\right| d\tau\right] \\ &\le h_0 (u_\mu - v_\mu)^{(r+1,\tilde{m})} \left[\sum_{j\in\Gamma_+^{(\tilde{m})}} \frac{1}{h_j} \int_0^1 \partial_{q_j} f_i(Q_i^{(r,\tilde{m})}[v,\tau]) d\tau \right. \\ &\qquad \left. - \sum_{j\in\Gamma_-^{(\tilde{m})}} \frac{1}{h_j} \int_0^1 \partial_{q_j} f_i(Q_i^{(r,\tilde{m})}[v,\tau]) d\tau\right].\end{aligned} \tag{2.31}$$

It follows that $(u_\mu - v_\mu)^{(r+1,\tilde{m})} \le 0$. The the proof of (2.23) is completed by induction. □

3. Implicit Difference Schemes

We define $N = (N_1, \ldots, N_n) \in N^n$ by the relations:

$$(N_1 h_1, \ldots, N_n h_n) < (b_1, \ldots, b_n) \le ((N_1 + 1)h_1, \ldots, (N_n + 1)h_n) \tag{3.1}$$

and we assume that $(N_i + 1)h_i = b_i$ if $d_i = 0$. For $w \in C(D, \mathbb{R}^k)$ we write

$$\|w\|_D = \max\left\{\|w(t,x)\|_\infty : (t,x) \in D\right\}. \tag{3.2}$$

In a similar way we define the norm in the space $F(D_h, \mathbb{R}^k)$: if $w : D_h \to \mathbb{R}^k$ then

$$\|w\|_{D_h} = \max\left\{\|w^{(r,m)}\|_\infty : (t^{(r)}, x^{(m)}) \in D_h\right\}. \tag{3.3}$$

The following properties of the operator T_h are important in our considerations.

Lemma 3.1. *Suppose that $w : D \to \mathbb{R}^k$ is of class C^1 and w_h is the restriction of w to the set D_h. Let $\widetilde{C}$ be such a constant that $\|\partial_t w\|_D$, $\|\partial_{x_i} w\|_D \le \widetilde{C}$ for $1 \le i \le n$. Then $\|T_h[w_h] - w\|_D \le \widetilde{C}\|h\|$ where $\|h\| = h_0 + h_1 + \cdots + h_n$.*

Lemma 3.2. *Suppose that $w : D \to \mathbb{R}^k$ is of class C^2 and w_h is the restriction of w to the set D_h. Let $\widetilde{C}$ be such a constant that $\|\partial_{tt} w\|_D$, $\|\partial_{tx_i} w\|_D$, $\|\partial_{x_i x_j} w\|_D \le \widetilde{C}$, $i, j = 1, \ldots, n$. Then $\|T_h[w_h] - w\|_D \le \widetilde{C}\|h\|^2$.*

The above lemmas are consequences of [10, Lemma 3.19 and Theorem 5.27].

We first prove a theorem on the existence and uniqueness of solutions to (2.20), (2.21).

Theorem 3.3. *If Assumption $(H[f])$ is satisfied and $\varphi_h \in F(E_{0.h} \cup \partial_0 E_h, \mathbb{R}^k)$ then there exists exactly one solution $u_h = (u_{h.1}, \ldots, u_{h.k}) : \Omega_h \to \mathbb{R}^k$ of difference functional problem* (2.20), (2.21).

Proof. Suppose that $0 \leq r \leq N_0 - 1$ is fixed and that the solution u_h of problem (2.20), (2.21) is given on the set $\Omega_h \cap ([-d_0, t^{(r)}] \times \mathbb{R}^n)$. We prove that the vectors $u_h^{(r+1,m)}$, $-N \leq m \leq N$, exist and that they are unique. It is sufficient to show that there exists exactly one solution of the system of equations

$$\frac{1}{h_0}\left(z_i^{(r+1,m)} - u_{h.i}^{(r,m)}\right) = f_i\left(t^{(r)}, x^{(m)}, T(u_h)_{[r,m]}, \delta z_i^{(r+1,m)}\right), \tag{3.4}$$

where $-N \leq m \leq N$, $i = 1, \ldots, k$, with the initial boundary condition (2.21). There exists $Q_h > 0$ such that

$$Q_h \geq h_0 \left[\sum_{j \in \Gamma_+^{(m)}} \frac{1}{h_j} \partial_{q_j} f_i\left(t^{(r)}, x^{(m)}, T_h(u_h)_{[r,m]}, q\right) - \sum_{j \in \Gamma_-^{(m)}} \frac{1}{h_j} \partial_{q_j} f_i\left(t^{(r)}, x^{(m)}, T_h(u_h)_{[r,m]}, q\right) \right], \tag{3.5}$$

where $-N \leq m \leq N$, $i = 1, \ldots, k$. It is clear that system (3.4) is equivalent to the following one:

$$z_i^{(r+1,m)} = \frac{1}{Q_h + 1}\left[Q_h z_i^{(r+1,m)} + u_{h.i}^{(r,m)} + h_0 f_i\left(t^{(r)}, x^{(m)}, T_h(u_h)_{[r,m]}, \delta z_i^{(r+1,m)}\right)\right], \qquad -N \leq m \leq N, \; i = 1, \ldots, k \tag{3.6}$$

Write $S_h = \{x^{(m)} : x^{(m)} \in [-c, c]\}$. Elements of the space $F(S_h, \mathbb{R}^k)$ are denoted by ξ, $\overline{\xi}$. For $\xi : S_h \to \mathbb{R}^k$, $\xi = (\xi_1, \ldots, \xi_k)$, we write $\xi^{(m)} = \xi(x^{(m)})$ and

$$\begin{aligned} \delta \xi_i^{(m)} &= \left(\delta_1 \xi_i^{(m)}, \ldots, \delta_n \xi_i^{(m)}\right), \quad 1 \leq i \leq k, \\ \delta_j \xi_i^{(m)} &= \frac{1}{h_j}\left[\xi_i^{(m+e_j j)} - \xi_i^{(m)}\right] \quad \text{if } x_j^{(m_j)} \in [\tilde{x}_j, b_j), \\ \delta_j \xi_i^{(m)} &= \frac{1}{h_j}\left[\xi_i^{(m)} - \xi_i^{(m-e_j)}\right] \quad \text{if } x_j^{(m_j)} \in (b_j, \tilde{x}_j), \end{aligned} \tag{3.7}$$

where $j = 1, \ldots, n$. The norm in the space $F(S_h, \mathbb{R}^k)$ is defined by

$$\|\xi\|_\star = \max\left\{ \left\|\xi^{(m)}\right\|_\infty : x^{(m)} \in S_h \right\}. \tag{3.8}$$

Let us consider the set

$$X_h = \left\{ \xi \in F\left(S_h, \mathbb{R}^k\right) : \xi^{(m)} = \varphi^{(r+1,m)} \text{ for } x^{(m)} \in [-c, c] \setminus (-b, b) \right\}. \tag{3.9}$$

We consider the operator $W_h : X_h \to X_h$, $W_h = (W_{h.1}, \ldots, W_{h.n})$ defined by

$$W_{h.i}[\xi]^{(m)} = \frac{1}{Q_h + 1}\left[Q_h \xi_i^{(m)} + u_{h.i}^{(r,m)} + h_0 f_i\left(t^{(r)}, x^{(m)}, T(u_h)_{[r,m]}, \delta \xi_i^{(m)}\right)\right], \tag{3.10}$$

where $-N \le m \le N$, $i = 1, \ldots, k$ and

$$W_h[\xi]^{(m)} = \varphi_h^{(r+1,m)} \quad \text{for } x^{(m)} \in [-c, c] \setminus (-b, b), \tag{3.11}$$

where $\xi = (\xi_1, \ldots, \xi_k) \in F(S_h, \mathbb{R}^k)$. We prove that

$$\left\| W_h[\xi] - W_h[\bar{\xi}] \right\|_\star \le \frac{Q_h}{Q_h + 1} \|\xi - \bar{\xi}\|_\star \quad \text{on } F(S_h, \mathbb{R}^k). \tag{3.12}$$

It follows from (3.10) that we have for $-N \le m \le N$:

$$\begin{aligned}
&W_{h.i}[\xi]^{(m)} - W_{h.i}[\bar{\xi}]^{(m)} \\
&\quad = \frac{1}{Q_h + 1} \Bigg[Q_h \left(\xi_i - \bar{\xi}_i \right)^{(m)} - h_0 \sum_{j \in \Gamma_+^{(m)}} \frac{1}{h_j} \int_0^1 \partial_{q_j} f_i \left(P_i^{(r,m)}[u_h, \tau] \right) d\tau \left(\xi_i - \bar{\xi}_i \right)^{(m)} \\
&\qquad + \sum_{j \in \Gamma_-^{(m)}} \frac{1}{h_j} \int_0^1 \partial_{q_j} f_i \left(P_i^{(r,m)}[u_h, \tau] \right) d\tau \left(\xi_i - \bar{\xi}_i \right)^{(m)} \\
&\qquad + h_0 \sum_{j \in \Gamma_+^{(m)}} \frac{1}{h_j} \int_0^1 \partial_{q_j} f_i \left(P_i^{(r,m)}[u_h, \tau] \right) d\tau \left(\xi_i - \bar{\xi}_i \right)^{(m+e_j)} \\
&\qquad - h_0 \sum_{j \in \Gamma_-^{(m)}} \frac{1}{h_j} \int_0^1 \partial_{q_j} f_i \left(P_i^{(r,m)}[u_h, \tau] \right) d\tau \left(\xi_i - \bar{\xi}_i \right)^{(m-e_j)} \Bigg],
\end{aligned} \tag{3.13}$$

where $i = 1, \ldots, k$ and

$$P_i^{(r,m)}[u_h, \tau] = \left(t^{(r)}, x^{(m)}, T_h(u_h)_{[r,m]}, \delta \bar{\xi}_i^{(m)} + \tau \delta \left(\xi_i - \bar{\xi}_i \right)^{(m)} \right). \tag{3.14}$$

It follows from the above relations and from (3.5) that

$$\left| W_{h.i}[\xi]^{(m)} - W_{h.i}[\bar{\xi}]^{(m)} \right| \le \frac{Q_h}{Q_h + 1} \|\xi - \bar{\xi}\|_\star \quad \text{for } -N \le m \le N, \ i = 1, \ldots, k. \tag{3.15}$$

According to (3.12) we have

$$W_{h.i}[\xi]^{(m)} - W_{h.i}[\bar{\xi}]^{(m)} = 0 \quad \text{for } x^{(m)} \in [-c, c] \setminus (-b, b), \ i = 1, \ldots, k. \tag{3.16}$$

This completes the proof of (3.12).

It follows from the Banach fixed point theorem that there exists exactly one solution $\bar{\xi} : S_h \to \mathbb{R}^k$ of the equation $\xi = W_h[\xi]$ and consequently, there exists exactly one solution of (3.6), (2.21). Then the vectors $u_h^{(r+1,m)}$, $-N \le m \le N$, exist and they are unique. Then the proof is completed by induction with respect to r, $0 \le r \le N_0$. □

Assumption $(H[\sigma])$. The function $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions:

(1) σ is continuous and it is nondecreasing with respect to the both variables,

(2) $\sigma(t,0) = 0$ for $t \in [0,a]$ and the maximal solution of the Cauchy problem

$$\eta'(t) = \sigma(t, \eta(t)), \qquad \eta(0) = 0, \tag{3.17}$$

is $\tilde{\eta}(t) = 0$ for $t \in [0,a]$.

Assumption $(H[f,\sigma])$. There is $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $(H[\sigma])$ is satisfied and for $w, \overline{w} \in \mathbb{C}(D, \mathbb{R}^k)$, $w \geq \overline{w}$, we have

$$f_i(t,x,w,q) - f_i(t,x,\overline{w},q) \leq \sigma\big(t, \|w - \overline{w}\|_D\big), \quad i = 1,\ldots,k, \tag{3.18}$$

where $(t,x,q) \in E \times \mathbb{R}^n$.

Theorem 3.4. *Suppose that Assumptions* $(H[f])$ *and* $(H[f,\sigma])$ *are satisfied and*

(1) $v : \Omega \to \mathbb{R}$ *is a solution of* (1.4), (1.5) *and* v *is of class* C^1 *on* Ω,

(2) $h \in H^*$, $h_0 < \varepsilon$ *and* $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^k$ *and there is* $\alpha_0 : H^* \to \mathbb{R}_+$ *such that*

$$\big\|\varphi^{(r,m)} - \varphi_h^{(r,m)}\big\|_\infty \leq \alpha_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h,\ \lim_{h\to 0} \alpha_0(h) = 0. \tag{3.19}$$

Under these assumptions there is a solution $u_h : \Omega_h \to \mathbb{R}^k$ *of* (2.20), (2.21) *and there is* $\alpha : H^* \to \mathbb{R}_+$ *such that*

$$\big\|(u_h - v_h)^{(r,m)}\big\|_\infty \leq \alpha(h) \quad \text{on } E_h,\ \lim_{h\to 0} \alpha(h) = 0, \tag{3.20}$$

where v_h *is the restriction of* v *to the set* Ω_h.

Proof. The existence of u_h follows from Theorem 3.3. Let $\Gamma_h : E'_h \to \mathbb{R}^k$, $\Gamma_{0.h} : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^k$ be defined by the relations

$$\delta_0 v_h^{(r,m)} = \mathbb{F}_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)} \quad \text{on } E'_h, \tag{3.21}$$

$$v_h^{(r,m)} = \varphi_h^{(r+1,m)} + \Gamma_{0.h}^{(r,m)} \quad \text{for } \big(t^{(r)}, x^{(m)}\big) \in E_{0.h} \cup \partial_0 E_h. \tag{3.22}$$

From Lemma 3.1 and from assumption (1) of the theorem it follows that there are $\gamma, \gamma_0 : H^* \to \mathbb{R}_+$ such that

$$\big\|\Gamma_h^{(r,m)}\big\|_\infty \leq \gamma(h) \quad \text{on } E'_h, \qquad \big\|\Gamma_{0.h}^{(r,m)}\big\|_\infty \leq \gamma_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h \tag{3.23}$$

and $\lim_{h\to 0}\gamma(h) = 0$, $\lim_{h\to 0}\gamma_0(h) = 0$. Write $J = [0,a]$ and $J_h = \{t^{(r)} : 0 \leq r \leq N_0\}$. For $\beta : J_h \to \mathbb{R}$ we put $\beta^{(r)} = \beta(t^{(r)})$. Let $\beta_h : J_h \to \mathbb{R}_+$ be a solution of the difference problem

$$\beta^{(r+1)} = \beta^{(r)} + h_0 \sigma\big(t^{(r)}, \beta^{(r)}\big) + h_0 \gamma(h), \quad 0 \leq r \leq N_0 - 1, \qquad \beta^{(0)} = \alpha_0(h). \tag{3.24}$$

We prove that

$$\left\| (u_h - v_h)^{(r,m)} \right\|_\infty \le \beta_h^{(r)} \quad \text{on } E_h. \tag{3.25}$$

Let $\tilde{v}_h = (\tilde{v}_{h.1}, \dots, \tilde{v}_{h.k}) : \Omega_h \to \mathbb{R}^k$ be defined by

$$\begin{aligned} \tilde{v}_{h.i}^{(r,m)} &= v_{h.i}^{(r,m)} + \beta_h^{(0)} \quad \text{on } E_{0.h}, \\ \tilde{v}_{h.i}^{(r,m)} &= v_{h.i}^{(r,m)} + \beta_h^{(i)} \quad \text{on } E_h \cup \partial_0 E_h, \end{aligned} \tag{3.26}$$

where $i = 1, \dots, k$. We prove that the difference functional inequality

$$\delta_0 \tilde{v}_h \ge \mathbb{F}_h [\tilde{v}_h]^{(r,m)}, \qquad (t^{(r)}, x^{(m)}) \in E'_h, \tag{3.27}$$

is satisfied. It follows from Assumption $(H[f,\sigma])$ and from (3.21) that

$$\begin{aligned} \delta_0 \tilde{v}_{h.i}^{(r,m)} &= \delta_0 v_{h.i}^{(r,m)} + \frac{1}{h_0} \left(\beta_h^{(r+1)} - \beta_h^{(r)} \right) \\ &= f_i \left(t^{(r)}, x^{(m)}, T_h (\tilde{v}_h)_{[r,m]}, \delta \tilde{v}_{h.i}^{(r+1,m)} \right) + \frac{1}{h_0} \left(\beta_h^{(r+1)} - \beta_h^{(r)} \right) \\ &\quad + \left[f_i \left(t^{(r)}, x^{(m)}, T_h (v_h)_{[r,m]}, \delta v_{h.i}^{(r+1,m)} \right) - f_i \left(t^{(r)}, x^{(m)}, T_h (\tilde{v}_h)_{[r,m]}, \delta v_{h.i}^{(r+1,m)} \right) \right] \\ &\ge f_i \left(t^{(r)}, x^{(m)}, T_h (\tilde{v}_h)_{[r,m]}, \delta \tilde{v}_{h.i}^{(r+1,m)} \right) - \sigma \left(t^{(r)}, \beta_h^{(r)} \right) + \frac{1}{h_0} \left(\beta_h^{(r+1)} - \beta_h^{(r)} \right) \\ &= f_i (t^{(r)}, x^{(m)}, T_h (\tilde{v}_h)_{[r,m]}, \delta \tilde{v}_{h.i}^{(r+1,m)}), \quad i = 1, \dots, k. \end{aligned} \tag{3.28}$$

This completes the proof of (3.27).

Since $v_h^{(r,m)} \le \tilde{v}_h^{(r,m)}$ on $E_{0.h} \cup \partial_0 E_h$, it follows from Theorem 2.3 that $u_h^{(r,m)} \le v_h^{(r,m)} + \beta_h^{(r)}$ on E_h. In a similar way we prove that $v_h^{(r,m)} - \beta_h^{(r)} \le u_h^{(r,m)}$ on E_h. The above estimates imply (3.25). Consider the Cauchy problem

$$\eta'(t) = \sigma(t, \eta(t)) + \gamma(h), \qquad \eta(0) = \alpha_0(h). \tag{3.29}$$

It follows from Assumption $(H[\sigma])$ that there is $\tilde{\varepsilon} > 0$ such that for $\|h\| \le \tilde{\varepsilon}$ the maximal solution $\eta(\cdot, h)$ of (3.29) is defined on $[0, a]$ and

$$\lim_{h \to 0} \eta(t, h) = 0 \quad \text{uniformly on } [0, a]. \tag{3.30}$$

Since $\eta(\cdot, h)$ is convex function then we have the difference inequality

$$\eta(t^{(r+1)}, h) \ge \eta(t^{(r)}, h) + h_0 \sigma(t^{(r)}, \eta(t^{(r)}, h)) + h_0 \gamma(h), \tag{3.31}$$

where $0 \le r \le N_0 - 1$. Since β_h satisfies (3.24), the above relations imply the estimate

$$\beta_h^{(r)} \le \eta\big(t^{(r)}, h\big) \le \eta(a,h), \quad 0 \le r \le N_0. \tag{3.32}$$

It follows from (3.30) that condition (3.20) is satisfied with $\alpha(h) = \eta(a,h)$. This completes the proof. □

Lemma 3.5. *Suppose that Assumption* $(H[f])$ *is satisfied and*

(1) $v : \Omega \to \mathbb{R}$ *is a solution of* (1.4), (1.5) *and* v *is of class* C^2 *on* Ω,

(2) $h \in H^*$, $h_0 < \varepsilon$ *and* $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^k$ *and there is* $\alpha_0 : H^* \to \mathbb{R}_+$ *such that*

$$\big\|\varphi^{(r,m)} - \varphi_h^{(r,m)}\big\|_\infty \le \alpha_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h, \ \lim_{h\to 0} \alpha_0(h) = 0. \tag{3.33}$$

(3) *there exists* $L \in \mathbb{R}_+$ *such that estimates*

$$f_i(t,x,w,q) - f_i(t,x,\tilde{w},q) \le L\|w - \tilde{w}\|_D, \quad i = 1,\ldots,k, \tag{3.34}$$

are satisfied for $(t,x,q) \in E \times \mathbb{R}^n$, $w, \tilde{w} \in C(D,\mathbb{R}^k)$ *and* $w \ge \tilde{w}$,

(4) *there is* $\overline{C} \in \mathbb{R}_+$ *such that*

$$\big\|\partial_q f_i(t,x,w,q)\big\| \le \overline{C} \quad \text{on } \Sigma \text{ for } i = 1,\ldots,k. \tag{3.35}$$

Under these assumptions there is a solution $u_h : \Omega_h \to \mathbb{R}^k$ *of* (2.20), (2.21), *and*

$$\big\|(u_h - v_h)^{(r,m)}\big\|_\infty \le \tilde{\alpha}(h) \quad \text{on } E_h, \tag{3.36}$$

where

$$\begin{gathered}\tilde{\alpha}(h) = \alpha_0(h)e^{La} + \tilde{\gamma}(h)\frac{e^{La}-1}{L} \quad \text{if } L > 0,\\ \tilde{\alpha}(h) = \alpha_0(h) + a\tilde{\gamma}(h) \quad \text{if } L = 0,\\ \tilde{\gamma}(h) = 0.5\,\tilde{C}h_0(1+\overline{C}) + L\tilde{C}\|h'\|^2 + 0.5\,\overline{C}\tilde{C}\|h\|\end{gathered} \tag{3.37}$$

and $\tilde{C} \in \mathbb{R}_+$ *is such that*

$$\big\|\partial_{tt}v(t,x)\big\|_\infty, \ \big\|\partial_{tx_i}v(t,x)\big\|_\infty, \big\|\partial_{x_ix_j}v(t,x)\big\|_\infty \le \tilde{C} \tag{3.38}$$

on Ω *for* i, $i = 1,\ldots,n$.

Proof. It follows that the solution $\beta_h : J_h \to \mathbb{R}_+$ of the difference problem

$$\begin{gathered}\beta^{(r+1)} = (1 + Lh_0)\beta^{(r)} + h_0\gamma(h), \quad 0 \le r \le N_0 - 1, \\ \beta^{(0)} = \alpha_0(h)\end{gathered} \tag{3.39}$$

satisfies the condition: $\beta_h^{(r)} \le \tilde{\alpha}(h)$ for $0 \le r \le N_0$. Moreover we have

$$\left\|\Gamma_h^{(r,m)}\right\|_\infty \le \tilde{\gamma}(h) \quad \text{on } E'_h, \tag{3.40}$$

where Γ_h is given by (3.21). Then we obtain the assertion from Lemma 3.2 and Theorem 3.4. □

Remark 3.6. In the result on error estimates we need estimates for the derivatives of the solution v of problem (1.4), (1.5). One may obtain them by the method of differential inequalities, see [10, Chapter 5].

4. Numerical Examples

Example 4.1. For $n = 2$ we put

$$E = [0, 0.5] \times [-1, 1] \times [-1, 1], \qquad E_0 = \{0\} \times [-1, 1] \times [-1, 1]. \tag{4.1}$$

Consider the differential integral equation

$$\begin{aligned}\partial_t z(t,x,y) = \arctan&\left[2x\partial_x z(t,x,y) + 2y\partial_y z(t,x,y) - t\left(2x^2y^2 - x^2 - y^2\right)z(t,x,y)\right] \\ &+ t\left(1-y^2\right)\int_{-1}^{x} sz(t,s,y)\,ds + t\left(1-x^2\right)\int_{-1}^{y} sz(t,x,s)\,ds \\ &+ z(t,x,y)\left[4 + 0.25\left(x^2-1\right)\left(y^2-1\right)\right] - 4\end{aligned} \tag{4.2}$$

with the initial boundary condition

$$\begin{gathered}z(0,x,y) = 1, \quad (x,y) \in [-1,1] \times [-1,1], \\ z(t,-1,y) = z(t,1,y) = 1, \quad (t,y) \in [0,0.5] \times [-1,1], \\ z(t,x,-1) = z(t,x,1) = 1, \quad (t,x) \in [0,0.5] \times [-1,1].\end{gathered} \tag{4.3}$$

The function $v(t,x,y) = \exp[0.25\,t(x^2-1)(y^2-1)]$ is the solution of the above problem. Let us denote by z_h an approximate solution which is obtained by using the implicit difference scheme.

Table 1: Table of errors.

$t^{(r)}$	0.25	0.30	0.35	0.40	0.45	0.50
$\varepsilon_h^{(r)}$	0.0006	0.0007	0.0009	0.0010	0.0012	0.0014

The Newton method is used for solving nonlinear systems generated by the implicit difference scheme. Write $m = (m_1, m_2)$ and

$$\varepsilon_h^{(r)} = \frac{1}{(2N_1 - 1)(2N_2 - 1)} \sum_{m\in\Pi} \left| z_h^{(r,m)} - v^{(r,m)} \right|, \quad 0 \le r \le N_0, \tag{4.4}$$

where

$$\Pi = \{ m = (m_1, m_2) : \ \in \mathbb{Z}^2 : \ -N_1 + 1 \le m_1 \le N_1 - 1, \ -N_1 + 1 \le m_2 \le N_2 - 1 \} \tag{4.5}$$

and $N_1 h_1 = 1$, $N_2 h_2 = 1$, $N_0 h_0 = 0.5$. The numbers $\varepsilon_h^{(r)}$ can be called average errors of the difference method for fixed $t^{(r)}$. We put $h_0 = h_1 = h_2 = 0.005$ and we have the values of the above defined errors which are shown in Table 1.

Note that our equation and the steps of the mesh do not satisfy condition (1.10) which is necessary for the explicit difference method to be convergent. In our numerical example the average errors for the explicit difference method exceeded 10^2.

Example 4.2. Let $n = 2$ and

$$E = [0, 0.5] \times [-0.5, 0.5] \times [-0.5, 0.5], \qquad E_0 = \{0\} \times [-0.5, 0.5] \times [-0.5, 0.5]. \tag{4.6}$$

Consider the differential equation with deviated variables

$$\begin{aligned} \partial_t z(t,x,y) = {} & 2x\partial_x z(t,x,y) + 2y\partial_y z(t,x,y) \\ & + \cos\left[2x\partial_x z(t,x,y) - 2y\partial_y z(t,x,y) - t(x^2 - y^2) z(t,x,y) \right] \\ & + \sqrt{z(t^2, x, y)} + f(t,x,y) z(t,x,y) - 1, \end{aligned} \tag{4.7}$$

with the initial boundary conditions

$$\begin{aligned} z(0,x,y) = 1, &\quad (x,y) \in [-0.5, 0.5] \times [-0.5, 0.5] \\ z(t,-0.5,y) = z(t,0.5,y) = 1, &\quad (t,y) \in [0,0.5] \times [-0.5, 0.5], \\ z(t,x,-0.5) = z(t,x,0,5) = 1, &\quad (t,x) \in [0,0.5] \times [-0.5, 0.5], \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} f(t,x,y) = {} & (x^2 - 0.25)(0.25 - y^2) + t\left[8x^2y^2 - x^2 - y^2 \right] \\ & - \exp\left\{ (0.5t^2 - t)(x^2 - 0.25)(0.25 - y^2) \right\}. \end{aligned} \tag{4.9}$$

Table 2: Table of errors.

$t^{(r)}$	0.25	0.30	0.35	0.40	0.45	0.5
$\varepsilon_h^{(r)}$	0.0002	0.0003	0.0004	0.0004	0.0005	0.0006

The function $v(t,x,y) = \exp[[t(x^2 - 0.25)(0.25 - y^2)]$ is the solution of the above problem. Let us denote by z_h an approximate solution which is obtained by using the implicit difference scheme.

The Newton method is used for solving nonlinear systems generated by the implicit difference scheme.

Let ε_h be defined by (4.4) with $N_1h_1 = 0.5$, $N_2h_2 = 0.5$, $N_0h_0 = 0.5$. We put $h_0 = h_1 = h_2 = 0.005$ and we have the values of the above defined errors which are shown in Table 2.

Note that our equation and the steps of the mesh do not satisfy condition (1.10) which is necessary for the explicit difference method to be convergent. In our numerical example the average errors for the explicit difference method exceeded 10^2.

The above examples show that there are implicit difference schemes which are convergent, and the corresponding classical method is not convergent. This is due to the fact that we need assumption (1.10) for explicit difference methods. We do not need this condition in our implicit methods.

Our results show that implicit difference schemes are convergent on all meshes.

References

[1] Z. Kowalski, "A difference method for the non-linear partial differential equation of the first order," *Annales Polonici Mathematici*, vol. 18, pp. 235–242, 1966.

[2] A. Pliś, "On difference inequalities corresponding to partial differential inequalities of the first order," *Annales Polonici Mathematici*, vol. 20, pp. 179–181, 1968.

[3] P. Brandi, Z. Kamont, and A. Salvadori, "Differential and differential-difference inequalities related to mixed problems for first order partial differential-functional equations," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 39, no. 1, pp. 255–276, 1991.

[4] K. Prządka, "Difference methods for non-linear partial differential-functional equations of the first order," *Mathematische Nachrichten*, vol. 138, pp. 105–123, 1988.

[5] M. Malec, C. Mączka, and W. Voigt, "Weak difference-functional inequalities and their application to the difference analogue of non-linear parabolic differential-functional equations," *Beiträge zur Numerischen Mathematik*, no. 11, pp. 69–79, 1983.

[6] M. Malec and M. Rosati, "Weak monotonicity for nonlinear systems of functional-finite difference inequalities of parabolic type," *Rendiconti di Matematica. Serie VII*, vol. 3, no. 1, pp. 157–170, 1983.

[7] M. Malec and M. Rosati, "A convergent scheme for nonlinear systems of differential functional equations of parabolic type," *Rendiconti di Matematica. Serie VII*, vol. 3, no. 2, pp. 211–227, 1983.

[8] M. Malec and A. Schiaffino, "Méthode aux différences finies pour une équation non-linéaire differentielle fonctionnelle du type parabolique avec une condition initiale de Cauchy," *Bollettino della Unione Matemática Italiana. Serie VII. B*, vol. 1, no. 1, pp. 99–109, 1987.

[9] W. Czernous, "Generalized solutions of mixed problems for first-order partial functional differential equations," *Ukraïnskyj Matematychnyj Zhurnal*, vol. 58, no. 6, pp. 803–828, 2006.

[10] Z. Kamont, *Hyperbolic Functional Differential Inequalities and Applications*, vol. 486 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[11] E. Godlewski and P.-A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, vol. 118 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1996.

[12] K. M. Magomedov and A. S. Kholodov, *Mesh Characteristics Numerical Methods*, Nauka, Moscow, Russia, 1988.

[13] A. A. Samarskii, *The Theory of Difference Schemes*, vol. 240 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2001.

[14] A. A. Samarskii, P. P. Matus, and P. N. Vabishchevich, *Difference Schemes with Operator Factors*, vol. 546 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[15] N. N. Luzin, "On the method of approximate integration of academician S. A. Chaplygin," *Uspekhi Matematicheskikh Nauk*, vol. 6, no. 6(46), pp. 3–27, 1951 (Russian).

[16] N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatulina, *Introduction to the Theory of Functional Differential Equations*, vol. 3 of *Advanced Series in Mathematical Science and Engineering*, World Federation, Atlanta, Ga, USA, 1995.

[17] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, vol. 55 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1969.

[18] A. Kępczyńska, "Implicit difference methods for first order partial differential functional equations," *Nonlinear Oscillations*, vol. 8, no. 2, pp. 198–213, 2005.