FIXED POINTS OF CONE COMPRESSION AND EXPANSION MULTIMAPS DEFINED ON FRÉCHET SPACES: THE PROJECTIVE LIMIT APPROACH

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We present a generalization of the cone compression and expansion results due to Krasnoselskii and Petryshyn for multivalued maps defined on a Fréchet space E. The proof relies on fixed point results in Banach spaces and viewing E as the projective limit of a sequence of Banach spaces.

1. Introduction

This paper presents cone compression and expansion fixed point results of Krasnoselskii-Petryshyn type for multimaps between Fréchet spaces. Two approaches have recently been presented in the literature, both of which are based on the fact that a Fréchet space can be viewed as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in\mathbb{N}}$ (here $\mathbb{N}=\{1,2,\ldots\}$). Both approaches are based on constructing maps F_n defined on subsets of E_n whose fixed points converge to a fixed point of the original operator F. In the first approach [6, 7], for each $n\in\mathbb{N}$ a specific map F_n is discussed; whereas in the second approach [2–4], the maps $\{F_n\}_{n\in\mathbb{N}}$ only need to satisfy a closure-type property. Both approaches have advantages and disadvantages over the other [1]. In this paper, we combine the advantages of both approaches to present a very general fixed point result.

Existence in Section 2 is based on the following result of Petryshyn [14, Theorem 3].

Theorem 1.1. *Let E be a Banach space and let $C\subseteq E$ be a closed cone. Let U and V be bounded open subsets in E such that $0\in U\subseteq\overline{U}\subseteq V$ and let $F:\overline{W}\to CK(C)$ be an upper semicontinuous, k-set contractive (countably) map; here $0\le k<1$, $W=V\cap C$, $\overline{W}$ denotes the closure of W in C and $CK(C)$ denotes the family on nonempty, compact, convex subsets of C. Assume that*

(1) *$\|y\|\ge\|x\|$ for all $y\in Fx$ and $x\in\partial\Omega$, and $\|y\|\le\|x\|$ for all $y\in Fx$ and $x\in\partial W$ (here $\Omega=U\cap C$ and ∂W denotes the boundary of W in C)*

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or

(2) $\|y\| \le \|x\|$ *for all* $y \in Fx$ *and* $x \in \partial\Omega$, *and* $\|y\| \ge \|x\|$ *for all* $y \in Fx$ *and* $x \in \partial W$. *Then* F *has a fixed point in* $\overline{W}\backslash\Omega$.

For the rest of this section, we gather some definitions and a known result which will be needed in Section 2. Let (X,d) be a metric space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha:\Omega_X \to [0,\infty]$ defined by (here $A \in \Omega_X$)

$$\alpha(A) = \inf\left\{r > 0 : A \subseteq \bigcup_{i=1}^{n} A_i \text{ and } \operatorname{diam}(A_i) \le r\right\}. \tag{1.1}$$

Let S be a nonempty subset of X. For each $x \in X$, define $d(x,S) = \inf_{y\in S} d(x,y)$. We say a set is countably bounded if it is countable and bounded. Now suppose that $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then $G : S \to 2^X$ is

(i) countably k-set contractive (here $k \ge 0$) if $G(S)$ is bounded and $\alpha(G(W)) \le k\alpha(W)$ for all countably bounded sets W of S,

(ii) countably condensing if $G(S)$ is bounded, G is countably 1-set contractive and $\alpha(G(W)) < \alpha(W)$ for all countably bounded sets W of S with $\alpha(W) \ne 0$,

(iii) hemicompact if each sequence $\{x_n\}_{n\in\mathbb{N}}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

We now recall a result from the literature [1].

Theorem 1.2. *Let* (Y,d) *be a metric space,* D *a nonempty, complete subset of* Y, *and* $G : D \to 2^Y$ *a countably condensing map. Then* G *is hemicompact.*

Now let I be a directed set with order $\le$ and let $\{E_\alpha\}_{\alpha\in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \le \beta$, let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{x = (x_\alpha) \in \prod_{\alpha\in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta)\ \forall \alpha,\beta \in I,\ \alpha \le \beta\right\} \tag{1.2}$$

is a closed subset of $\prod_{\alpha\in I} E_\alpha$, is called the projective limit of $\{E_\alpha\}_{\alpha\in I}$, and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [9, page 439] $\bigcap_{\alpha\in I} E_\alpha$).

2. Fixed point theory in Fréchet spaces

Let $E = (E, \{|\cdot|_n\}_{n\in\mathbb{N}})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in \mathbb{N}\}$. We assume that the family of seminorms satisfies

$$|x|_1 \le |x|_2 \le |x|_3 \le \cdots \qquad \text{for every } x \in E. \tag{2.1}$$

For $r > 0$ and $x \in E$, we let $B(x,r) = \{y \in E : |x-y|_n \le r \text{ for all } n \in \mathbb{N}\}$. A subset X of E is bounded if for every $n \in \mathbb{N}$, there exists $r_n > 0$ such that $|x|_n \le r_n$ for all $x \in X$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff } |x - y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied, the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume that the following condition holds:

$$\begin{gathered}\text{for each } n \in \mathbb{N}, \text{ there exist a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \longrightarrow E_n.\end{gathered} \tag{2.3}$$

Remark 2.1. (i) For convenience, the norm on E_n is denoted by $|\cdot|_n$.

(ii) In our applications, $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in \mathbb{N}$.

(iii) Note that if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$), then $x \in E$. However if $x \in E_n$, then x is not necessarily in E and in fact E_n is easier to use in applications as we will see in Theorem 3.2 (even though E_n is isomorphic to $\mathbf{E}_n$).

For $r > 0$ and $x \in E_n$, we let $B_n(x,r) = \{y \in E_n : |x - y|_n \leq r\}$. Finally we assume that

$$E_1 \supseteq E_2 \supseteq \cdots \text{ and for each } n \in \mathbb{N}, \qquad |x|_n \leq |x|_{n+1} \quad \forall x \in E_{n+1}. \tag{2.4}$$

Let $\lim_{\leftarrow} E_n$ (or $\bigcap_1^\infty E_n$, where $\bigcap_1^\infty$ is the generalized intersection [9]) denote the projective limit of $\{E_n\}_{n\in\mathbb{N}}$ (note that $\pi_{n,m} = j_n\mu_{n,m}j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note that $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in \mathbb{N}$, we set $X_n = j_n\mu_n(X)$, and we let $\overline{X_n}$ and ∂X_n denote, respectively, the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by [6]

$$\text{pseudo-int}(X) = \{x \in X : j_n\mu_n(x) \in \overline{X_n} \backslash \partial X_n \ \text{ for every } n \in \mathbb{N}\}. \tag{2.5}$$

The set X is pseudo-open if $X = \text{pseudo-int}(X)$.

We begin with our main result.

Theorem 2.2. *Let E and E_n be as described above, C a closed cone in E, U and V are bounded pseudo-open subsets of E with $0 \in U \subseteq \overline{U} \subseteq V$, and $F : C \cap \overline{V} \to 2^E$ (here 2^E*

denotes the family of nonempty subsets of E). Suppose the following conditions are satisfied:

$$\overline{W_1} \supseteq \overline{W_2} \supseteq \cdots; \quad \text{here } W_n = \overline{C_n} \cap V_n, \tag{2.6}$$

$$\begin{aligned}&\text{for each } n \in \mathbb{N},\ F_n : \overline{W_n} \longrightarrow CK\left(\overline{C_n}\right) \text{ is an}\\ &\text{upper semicontinuous map } \left(\text{here } \overline{W_n} \text{ denotes}\right.\\ &\left.\text{the closure of } W_n \text{ in } \overline{C_n}\right).\end{aligned} \tag{2.7}$$

Also for each $n \in \mathbb{N}$, assume that either (here $\Omega_n = U_n \cap \overline{C_n}$)

$$\begin{aligned}&|y|_n \geq |x|_n \quad \forall y \in F_n x,\ \forall x \in \partial\Omega_n,\\ &|y|_n \leq |x|_n \quad \forall y \in F_n x,\ \forall x \in \partial W_n,\\ &\left(\text{here } \partial W_n \text{ denotes the boundary of } W_n \text{ in } \overline{C_n}\right)\end{aligned} \tag{2.8}$$

or

$$\begin{aligned}&|y|_n \leq |x|_n \quad \forall y \in F_n x,\ \forall x \in \partial\Omega_n,\\ &|y|_n \geq |x|_n \quad \forall y \in F_n x,\ \forall x \in \partial W_n,\end{aligned} \tag{2.9}$$

hold. Finally suppose the following three conditions hold:

$$\begin{aligned}&\text{for each } n \in \mathbb{N}, \text{ the map } \mathcal{K}_n : \overline{W_n} \longrightarrow 2^{E_n}, \text{ given by}\\ &\mathcal{K}_n(y) = \bigcup_{m=n}^{\infty} F_m(y) \text{ (see Remark 2.3), is k-set}\\ &\text{(countably) contractive (here } 0 \leq k < 1);\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\text{for every } k \in \mathbb{N} \text{ and any subsequence } A \subseteq \{k, k+1, \dots\}\\ &\text{if } x \in \overline{C_n} \text{ is such that } x \in \overline{W_n} \backslash \Omega_n \text{ for some } n \in A,\\ &\text{then there exists a } \gamma > 0 \text{ with } |x|_k \geq \gamma;\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\text{if there exist a } w \in E \text{ and a sequence } \{y_n\}_{n \in \mathbb{N}}\\ &\text{with } y_n \in \overline{W_n} \backslash \Omega_n \text{ and } y_n \in F_n y_n \text{ in } E_n \text{ such that}\\ &\text{for every } k \in \mathbb{N}, \text{ there exists a subsequence}\\ &S \subseteq \{k+1, k+2, \dots\} \text{ of } \mathbb{N} \text{ with } y_n \longrightarrow w \text{ in } E_k\\ &\text{as } n \longrightarrow \infty \text{ in } S, \text{ then } w \in Fw \text{ in } E.\end{aligned} \tag{2.12}$$

Then F has a fixed point in E.

Remark 2.3. The definition of $\mathcal{K}_n$ in (2.10) is as follows. If $y \in \overline{W_n}$ and $y \notin \overline{W_{n+1}}$, then $\mathcal{K}_n(y) = F_n(y)$; whereas if $y \in \overline{W_{n+1}}$ and $y \notin \overline{W_{n+2}}$, then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in \mathbb{N}$. We would like to apply Theorem 1.1. To do so, we need to show that

$$\overline{C_n} \text{ is a cone,} \tag{2.13}$$

$$U_n \text{ and } V_n \text{ are open and bounded with } 0 \in U_n \subseteq \overline{U_n} \subseteq V_n. \tag{2.14}$$

First we check (2.13). To see this, let $\widehat{x}, \widehat{y} \in \mu_n(C)$ and $\lambda \in [0,1]$. Then for every $x \in \mu_n^{-1}(\widehat{x})$ and $y \in \mu_n^{-1}(\widehat{y})$, we have $\lambda x + (1-\lambda)y \in C$ since C is convex and so $\lambda\widehat{x} + (1-\lambda)\widehat{y} = \lambda\mu_n(x) + (1-\lambda)\mu_n(y)$. It is easy to check that $\lambda\mu_n(x) + (1-\lambda)\mu_n(y) = \mu_n(\lambda x + (1-\lambda)y)$, so as a result

$$\lambda\widehat{x} + (1-\lambda)\widehat{y} = \mu_n(\lambda x + (1-\lambda)y) \in \mu_n(C), \tag{2.15}$$

and so $\mu_n(C)$ is convex. Now since j_n is linear, $C_n = j_n(\mu_n(C))$ is convex, and as a result $\overline{C_n}$ is convex. Similarly it is easy to show that $t\widehat{x} \in \mu_n(C)$ for every $t \geq 0$, so $\overline{C_n}$ is a cone. Thus (2.13) holds.

Now since U is pseudo-open and $0 \in U$, then $0 \in \text{pseudo-int}\, U$, and so $0 = j_n\mu_n(0) \in \overline{U_n} \backslash \partial U_n$. Of course

$$\overline{U_n} \backslash \partial U_n = (U_n \cup \partial U_n) \backslash \partial U_n = U_n \backslash \partial U_n, \tag{2.16}$$

so $0 \in U_n \backslash \partial U_n$, and in particular $0 \in U_n$. Next we show that U_n is open. First note that $U_n \subseteq \overline{U_n} \backslash \partial U_n$ since if $y \in U_n$, then there exists $x \in U$ with $y = j_n\mu_n(x)$ and this together with $U = \text{pseudo-int}\, U$ yields $j_n\mu_n(x) \in \overline{U_n} \backslash \partial U_n$, that is, $y \in \overline{U_n} \backslash \partial U_n$. In addition note that,

$$\overline{U_n} \backslash \partial U_n = (\text{int}\, U_n \cup \partial U_n) \backslash \partial U_n = \text{int}\, U_n \backslash \partial U_n = \text{int}\, U_n \tag{2.17}$$

since $\text{int}\, U_n \bigcap \partial U_n = \emptyset$. Consequently

$$U_n \subseteq \overline{U_n} \backslash \partial U_n = \text{int}\, U_n, \quad \text{so } U_n = \text{int}\, U_n. \tag{2.18}$$

As a result U_n is open. Clearly U_n is bounded since U is bounded (note that if $y \in U_n$, then there exists $x \in U$ with $y = j_n\mu_n(x)$). It just remains to show that $U_n \subseteq \overline{U_n} \subseteq V_n$ in (2.14). Of course since $U \subseteq \overline{U} \subseteq V$, we have

$$U_n = j_n\mu_n(U) \subseteq j_n\mu_n(\overline{U}) \subseteq j_n\mu_n(V) = V_n; \tag{2.19}$$

and since $j_n\mu_n$ is continuous, $U_n \subseteq j_n\mu_n(\overline{U}) \subseteq \overline{j_n\mu_n(U)} = \overline{U_n}$. Also we see that $\overline{\mu_n(U)} \subseteq \mu_n(V)$ (note that $\overline{U} \subseteq V$), so since j_n is an isometry,

$$\overline{U_n} = \overline{j_n\mu_n(U)} = j_n\overline{\mu_n(U)} \subseteq j_n\mu_n(V) = V_n. \tag{2.20}$$

Thus (2.14) holds.

Theorem 1.1 guarantees that there exist $y_n \in \overline{W_n}\backslash\Omega_n$ with $y_n \in F_n y_n$ in E_n. Let us look at $\{y_n\}_{n\in\mathbb{N}}$. Note $y_n \in \overline{W_1}$ for each $n \in \mathbb{N}$ from (2.6). Now Theorem 1.2 (with $Y = E_1$, $G = \mathcal{K}_1$, and $D = \overline{W_1}$ and note that $d_1(y_n,\mathcal{K}_1(y_n)) = 0$ for each $n \in \mathbb{N}$ since $|x|_1 \le |x|_n$ for all $x \in E_n$ and $y_n \in F_n y_n$ in E_n; here $d_1(x,Z) = \inf_{y\in Z}|x-y|_1$) guarantees that there exist a subsequence $\mathbb{N}_1^\star$ of $\mathbb{N}$ and a $z_1 \in \overline{W_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $\mathbb{N}_1^\star$. Also $y_n \in \overline{W_n}\backslash\Omega_n$ for $n \in \mathbb{N}$ together with (2.11) yields $|y_n|_1 \ge \gamma$ for $n \in \mathbb{N}$, and so $|z_1|_1 \ge \gamma$. Let $\mathbb{N}_1 = \mathbb{N}_1^\star\backslash\{1\}$ and look at $\{y_n\}_{n\in\mathbb{N}_1}$. Note that $y_n \in \overline{W_2}$ for $n \in \mathbb{N}_1$ from (2.6). Now Theorem 1.2 (with $Y = E_2$, $G = \mathcal{K}_2$ and $D = \overline{W_2}$) guarantees that there exists a subsequence $\mathbb{N}_2^\star$ of $\mathbb{N}_1$ and a $z_2 \in \overline{W_2}$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $\mathbb{N}_2^\star$. Note that $z_2 = z_1$ in E_1 since $\mathbb{N}_2^\star \subseteq \mathbb{N}_1^\star$. Also $y_n \in \overline{W_n}\backslash\Omega_n$ for $n \in \mathbb{N}_1$ together with (2.11) yields $|y_n|_2 \ge \gamma$ for $n \in \mathbb{N}_1$, and so $|z_2|_2 \ge \gamma$. Let $\mathbb{N}_2 = \mathbb{N}_2^\star\backslash\{2\}$. Proceed inductively to obtain subsequences of integers

$$\mathbb{N}_1^\star \supseteq \mathbb{N}_2^\star \supseteq \cdots, \qquad \mathbb{N}_k^\star \subseteq \{k,k+1,\ldots\} \tag{2.21}$$

and $z_k \in \overline{W_k}$ for $k \in \mathbb{N}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $\mathbb{N}_k^\star$. Note that $z_{k+1} = z_k$ in E_k for $k \in \mathbb{N}$ and $|z_k|_k \ge \gamma$ for $k \in \mathbb{N}$. Also let $\mathbb{N}_k = \mathbb{N}_k^\star \setminus \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in E_k. Note that y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n y_n$ in E_n for $n \in \mathbb{N}_k$ and $y_n \to y$ in E_k as $n \to \infty$ in $\mathbb{N}_k$ (since $y = z_k$ in E_k) together with (2.12) implies that $y \in Fy$ in E. □

Of course for the proof, one sees that (2.11) is only needed to guarantee that the fixed point $y \in E$ satisfies $|z_k|_k \ge \gamma$ for $k \in \mathbb{N}$; here $y = z_k$ in E_k.

Theorem 2.4. *Let E and E_n be as described in the beginning of Section 2, C a closed cone in E, U and V are bounded pseudo-open subsets of E with $0 \in U \subseteq \overline{U} \subseteq V$, and $F : C \cap \overline{V} \to 2^E$. Suppose that (2.6) and (2.7) hold and in addition assume that either (2.8) or (2.9) is satisfied. Finally assume that (2.10) and (2.12) hold. Then F has a fixed point in E.*

Of course a special case of Theorem 2.2 occurs if $F_n = F$ (i.e., $F_n = F|_{E_n}$.

Theorem 2.5. *Let E and E_n be as described in the beginning of Section 2, C a closed cone in E, U and V are bounded pseudo-open subsets of E with $0 \in U \subseteq \overline{U} \subseteq V$, and $F : C \cap \overline{V} \to 2^E$. Suppose (with $W_n = \overline{C_n} \cap V_n$ and $\Omega_n = \overline{C_n} \cap U_n$) the following is satisfied:*

$$\begin{gathered}\text{for each } n \in \mathbb{N},\ F : \overline{W_n} \to CK(\overline{C_n}) \text{ is an upper}\\ \text{semicontinuous } k\text{-set (countably) contractive map}\\ (\text{here } 0 \le k < 1).\end{gathered} \tag{2.22}$$

Also for each $n \in \mathbb{N}$, assume that either

$$\begin{aligned}|y|_n &\ge |x|_n \quad \forall y \in Fx,\ \forall x \in \partial\Omega_n,\\ |y|_n &\le |x|_n \quad \forall y \in Fx,\ \forall x \in \partial W_n,\end{aligned} \tag{2.23}$$

or

$$\begin{aligned} |y|_n &\le |x|_n \quad \forall y \in Fx,\ \forall x \in \partial\Omega_n, \\ |y|_n &\ge |x|_n \quad \forall y \in Fx,\ \forall x \in \partial W_n, \end{aligned} \tag{2.24}$$

hold. Finally suppose that (2.11) and the following hold that:

$$\begin{aligned} &\textit{for each } n \in \{2,3,\dots\} \textit{ if } y \in \overline{W_n} \textit{ solves } y \in Fy \\ &\textit{in } E_n, \textit{ then } y \in \overline{W_k} \textit{ for } k \in \{1,\dots,n-1\}. \end{aligned} \tag{2.25}$$

Then F has a fixed point in E.

Remark 2.6. Note again that (2.11) could be removed from the statement of Theorem 2.5. The result in Theorem 2.2 is of course based on Theorem 1.1 which is of course based on (1) and (2). One could replace Theorem 1.1 with the Leggett-Williams theorem (see [2]) or with results in [5, 13], and analogous results can be obtained in the Fréchet space setting. Also multiplicity results could be presented as in [10].

Remark 2.7. The Kakutani maps in Theorem 2.2 could be replaced by maps admissible with respect to Gorniewicz (if one uses results in [8]) or indeed the $\mathcal{U}_c^\kappa$ maps of Park (if one uses the results in [11]).

3. Application

In this section, we apply the results in Section 2 to the integral equation

$$y(t) = \int_0^\infty k(t,s) f(s, y(s))\, ds \quad \text{for } t \in [0,\infty). \tag{3.1}$$

Our result, Theorem 3.2, was established in [10]. However, our goal here is to show how easily and naturally Section 2 (in particular Theorem 2.2) applies when discussing problems of the form (3.1).

Remark 3.1. One could also obtain a result for the inclusion

$$y(t) \in \int_0^\infty k(t,s) F(s, y(s))\, ds \quad \text{for } t \in [0,\infty) \tag{3.2}$$

if one uses the ideas in the proof below with the ideas in [4].

Theorem 3.2. *Let $1 \leq p \leq \infty$ be a constant and q the conjugate to p. Suppose the following conditions are satisfied:*

$$\text{for each } t \in [0,\infty), \text{ the map } s \longmapsto k(t,s) \text{ is measurable,} \tag{3.3}$$

$$\sup_{t\in[0,\infty)} \left(\int_0^\infty |k(t,s)|^q ds\right)^{1/q} < \infty, \tag{3.4}$$

$$\int_0^\infty |k(t',s) - k(t,s)|^q ds \longrightarrow 0 \text{ as } t \longrightarrow t', \quad \text{for each } t' \in [0,\infty), \tag{3.5}$$

$$\begin{gathered} f : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is an } L^p\text{-Carathéodory function: by this,} \\ \text{(a) the map } t \longmapsto f(t,y) \text{ is measurable, } \forall y \in \mathbb{R}, \\ \text{(b) the map } y \longmapsto f(t,y) \text{ is continuous for a.e. } t \in [0,\infty) \\ \text{(c) for each } r > 0 \text{ there exists } h_r \in L^p[0,\infty) \text{ such that } |y| \leq r \\ \Longrightarrow |f(t,y)| \leq h_r(t) \text{ for a.e. } t \in [0,\infty), \end{gathered} \tag{3.6}$$

$$\begin{gathered} \text{for each } t \in [0,T],\ k(t,s) \geq 0, \text{ for a.e. } s \in [0,t], \\ f : [0,\infty) \times \mathbb{R} \longrightarrow [0,\infty) \text{ with } f(s,u) > 0, \text{ for } (s,u) \in [0,\infty) \times (0,\infty), \end{gathered} \tag{3.7}$$

$$\begin{gathered} \exists g : [0,\infty) \longrightarrow (0,\infty) \text{ with } g \in L^q[0,\infty), \\ \text{and with } k(t,s) \leq g(s) \text{ for } t \in [0,\infty), \end{gathered} \tag{3.8}$$

$$\begin{gathered} \exists a,b \in [0,1],\ a < b,\ M,\ 0 < M < 1 \\ \text{with } k(t,s) \geq M g(s) \text{ for } t \in [a,b], \text{ a.e. } s \in [0,\infty), \end{gathered} \tag{3.9}$$

$$\begin{gathered} \text{there exists a continuous nondecreasing function} \\ w : [0,\infty) \longrightarrow [0,\infty), \text{ a } \phi \in L^p[0,\infty) \text{ with} \\ f(s,u) \leq \phi(s) w(u) \text{ for a.e. } s \in [0,\infty) \\ \text{and all } u \in [0,\infty), \end{gathered} \tag{3.10}$$

$$\exists r > 0 \text{ with } r > w(r) \sup_{t\in[0,\infty)} \int_0^\infty \phi(s) k(t,s) ds, \tag{3.11}$$

$$\begin{gathered} \text{there exists } \tau \in L^p[a,b] \text{ with } f(s,u) \geq \tau(s) w(u) \\ \text{for a.e. } s \in [a,b] \text{ and all } u \in [0,\infty), \end{gathered} \tag{3.12}$$

$$\exists R > r \text{ with } R < w(MR) \sup_{t\in[0,1]} \int_a^b \tau(s) k(t,s) ds. \tag{3.13}$$

Then (3.1) has at least one solution in $C[0,\infty)$.

Remark 3.3. In (3.9), we picked $b \in [0,1]$ for convenience. Also if there exists a σ, $0 \le \sigma < \infty$ with

$$\sup_{t\in[0,\infty)} \int_a^b \tau(s)k(t,s)ds = \int_a^b \tau(s)k(\sigma,s)ds, \tag{3.14}$$

then one could replace (3.13) by

$$R < w(MR) \sup_{t\in[0,\infty)} \int_a^b \tau(s)k(t,s)ds. \tag{3.15}$$

Proof. Here $E = C[0,\infty)$, $\mathbf{E}^k$ consists of the class of functions in E which coincide on the interval $[0,k]$, and $E_k = C[0,k]$. We will apply Theorem 2.2 with $\overline{U} = B(0,r)$, $\overline{V} = B(0,R)$,

$$\begin{gathered} C = \{y \in E : y(t) \ge 0 \text{ on } [0,\infty) \textit{ and } y(t) \ge M|y|_n,\ \forall t \in [a,b],\ \forall n \in \mathbb{N}\}, \\ Fy(t) = \int_0^\infty k(t,s)f(s,y(s))ds; \end{gathered} \tag{3.16}$$

here $|y|_n = \sup_{t\in[0,n]}|y(t)|$. Fix $n \in \mathbb{N}$. Note that

$$\overline{C_n} = C_n = \{y \in E_n : y(t) \ge 0 \text{ on } [0,n] \text{ and } y(t) \ge M|y|_n,\ \forall t \in [a,b]\}, \tag{3.17}$$

with $\overline{U_n} = B_n(0,r)$ and $\overline{V_n} = B_n(0,R)$. Also we let

$$F_n y(t) = \int_0^n k(t,s)f(s,y(s))ds. \tag{3.18}$$

Clearly (2.6) and (2.7) hold. A standard argument in the literature [12] guarantees that (here $W_n = \overline{C_n} \cap V_n$)

$$F : \overline{W_n} \longrightarrow E_n \text{ is continuous and compact.} \tag{3.19}$$

In addition for any $y \in \overline{W_n}$, note that

$$|F_n y(t)| \le \int_0^n g(s)f(s,y(s))ds, \quad \text{for } t \in [0,n], \tag{3.20}$$

from (3.8), and

$$F_n y(t) \ge M \int_0^n g(s)f(s,y(s))ds, \quad \text{for } t \in [a,b], \tag{3.21}$$

from (3.9), and these two inequalities yield

$$F_n y(t) \ge M|F_n y|_n \quad \text{for } t \in [a,b], \tag{3.22}$$

so (2.7) holds.

Next we show that (2.9) is satisfied (here $\Omega_n = \overline{C_n} \cap U_n$). Let $y \in \partial\Omega_n = \partial U_n \cap \overline{C_n}$. Then $|y|_n = r$ and this together with (3.10) yields

$$|F_n y(t)| \leq w(|y|_n) \int_0^n k(t,s)\phi(s)ds \leq w(r) \sup_{t\in[0,\infty)} \int_0^\infty k(t,s)\phi(s)ds \tag{3.23}$$

for $t \in [0,n]$, and so (3.11) yields

$$|F_n y|_n \leq w(r) \sup_{t\in[0,\infty)} \int_0^\infty k(t,s)\phi(s)ds < r = |y|_n. \tag{3.24}$$

Now let $y \in \partial W_n = \partial V_n \cap \overline{C_n}$. Then $|y|_n = R$ and $y(t) \geq M|y|_n = MR$ for $t \in [a,b]$ (in particular $y(t) \in [MR,R]$ for $t \in [a,b]$). Now (3.12) implies that

$$\begin{aligned} |F_n y(t)| &= \int_0^n k(t,s)f(s,y(s))ds \geq \int_a^b k(t,s)f(s,y(s))ds \\ &\geq w(MR)\int_a^b k(t,s)\tau(s)ds, \end{aligned} \tag{3.25}$$

so (3.13) yields

$$\begin{aligned} |F_n y|_n &\geq w(MR) \sup_{t\in[0,n]} \int_a^b k(t,s)\tau(s)\,ds \\ &\geq w(MR) \sup_{t\in[0,1]} \int_a^b k(t,s)\tau(s)ds > R = |y|_n. \end{aligned} \tag{3.26}$$

Thus (2.9) holds.

To show (2.10), fix $n \in \mathbb{N}$. Let $y \in \overline{W_n}$. Without loss of generality, assume that there exists $l \in \{0,1,2,\dots\}$ with $y \in \overline{W_{n+l}}$ and $y \notin \overline{W_{n+l+1}}$. Then by definition, $\mathcal{K}_n(y) = \bigcup_{m=n}^{n+l} F_m(y)$. Now since $y \in \overline{W_{n+l}}$ we have from (3.6) that there exists an $h_R \in L^p[0,\infty)$ with $|f(s,y(s))| \leq h_R(s)$ for a.e. $s \in [0,n+l]$. Fix $j \in \{0,1,\dots,l\}$ and so we have for $t \in [0,n]$ that

$$\begin{aligned} |F_{n+j}y(t)| &\leq \int_0^{n+j} h_R(s)k(t,s)ds \\ &\leq \left(\int_0^\infty [h_R(s)]^p ds\right)^{1/p} \sup_{t\in[0,\infty)} \left(\int_0^\infty [k(t,s)]^q ds\right)^{1/q}, \end{aligned} \tag{3.27}$$

so

$$|F_{n+j}y|_n \leq \left(\int_0^\infty [h_R(s)]^p ds\right)^{1/p} \sup_{t\in[0,\infty)} \left(\int_0^\infty [k(t,s)]^q ds\right)^{1/q}, \tag{3.28}$$

and as a result

$$|\mathcal{K}_n y|_n \leq \left(\int_0^\infty [h_R(s)]^p ds\right)^{1/p} \sup_{t\in[0,\infty)} \left(\int_0^\infty [k(t,s)]^q ds\right)^{1/q}, \tag{3.29}$$

that is,

$$|u|_n \leq \left(\int_0^\infty [h_R(s)]^p ds\right)^{1/p} \sup_{t\in[0,\infty)} \left(\int_0^\infty [k(t,s)]^q ds\right)^{1/q} \quad \forall u \in \mathcal{K}_n y. \tag{3.30}$$

Also for $t_1,t_2 \in [0,n]$ and $j \in \{0,1,\ldots,l\}$, we have

$$\begin{aligned} |F_{n+j}y(t_1) - F_{n+j}y(t_2)| &\leq \int_0^{n+j} h_R(s)|k(t_1,s) - k(t_2,s)|ds \\ &\leq \left(\int_0^\infty [h_R(s)]^p ds\right)^{1/p} \left(\int_0^\infty |k(t_1,s) - k(t_2,s)|^q ds\right)^{1/q}, \end{aligned} \tag{3.31}$$

and so

$$|\mathcal{K}_n y(t_1) - \mathcal{K}_n y(t_1)| \leq \left(\int_0^\infty [h_R(s)]^p ds\right)^{1/p} \left(\int_0^\infty |k(t_1,s) - k(t_2,s)|^q ds\right)^{1/q}, \tag{3.32}$$

that is,

$$|u(t_1) - u(t_2)| \leq \left(\int_0^\infty [h_R(s)]^p ds\right)^{1/p} \left(\int_0^\infty |k(t_1,s) - k(t_2,s)|^q ds\right)^{1/q} \tag{3.33}$$

for all $u \in \mathcal{K}_n y$. Thus $\{\mathcal{K}_n y : y \in \overline{W_n}\}$ is uniformly bounded and equicontinuous on $[0,n]$. The Arzela-Ascoli theorem implies that $\mathcal{K}_n : \overline{W_n} \to 2^{E_n}$ is compact, so (2.10) holds.

Next we show (2.11) is satisfied with $\gamma = Mr$. Fix $k \in \mathbb{N}$ and take a subsequence $A \subseteq \{k,k+1,\ldots\}$. Let $x \in \overline{C_n}$ be such that $x \in \overline{W_n}\backslash\Omega_n$ (i.e., $R \geq |x|_n \geq r$) for some $n \in A$. Then $\min_{t\in[a,b]} x(t) \geq M|x|_n \geq Mr = \gamma$, so as a result $|x|_k = \max_{t\in[0,k]} |x(t)| \geq \gamma$.

Finally, we show (2.12). Suppose that there exist a $w \in C[0,\infty)$ and a sequence $\{w_n\}_{n\in\mathbb{N}}$ with $y_n \in \overline{W_n}\backslash\Omega_n$ (i.e., $R \geq |w_n|_n \geq r$) and $w_n = F_n w_n$ in $C[0,n]$ such that for every $k \in \mathbb{N}$, there exists a subsequence $S \subseteq \{k+1,k+2,\ldots\}$ of $\mathbb{N}$ with $w_n \to w$ in $C[0,k]$ as $n \to \infty$ in S. If we show that

$$w(t) = \int_0^\infty k(t,s)f(s,w(s))ds \quad \text{for } t \in [0,\infty), \tag{3.34}$$

then (2.12) holds. To see (3.34), fix $t \in [0,\infty)$. Consider $k \geq t$ and $n \in S$ (as described above). Then $w_n = F_n w_n$ for $n \in S$, so

$$w_n(t) - \int_0^k k(t,s)f(s,w_n(s))\,ds = \int_k^n k(t,s)f(s,w_n(s))\,ds, \tag{3.35}$$

and so

$$\left| w_n(t) - \int_0^k k(t,s)f(s,w_n(s))\,ds \right| \leq \int_k^n k(t,s)h_R(s)\,ds \tag{3.36}$$

(here (3.6) guarantees that there exists $h_R \in L^p[0,\infty)$ with $|f(s,w_n(s))| \leq h_R(s)$ for a.e. $s \in [0,\infty)$). Let $n \to \infty$ through S and use the Lebesgue dominated convergence theorem to obtain

$$\left| w(t) - \int_0^k K(t,s)f(s,w(s))\,ds \right| \leq \int_k^\infty k(t,s)h_R(s)\,ds \tag{3.37}$$

since $w_n \to w$ in $C[0,k]$. Finally, let $k \to \infty$ (note (3.5)) to obtain

$$w(t) - \int_0^\infty k(t,s)f(s,w(s))\,ds = 0. \tag{3.38}$$

Thus (2.12) holds. Our result now follows from Theorem 2.2, that is, there exists a solution $y \in C[0,\infty)$ to (3.1). Note in fact that $y \leq |y|_n \leq R$ for each $n \in \mathbb{N}$. □

References

[1] R. P. Agarwal, M. Frigon, and D. O'Regan, *A survey of recent fixed point theory in Fréchet spaces*, Nonlinear Analysis and Applications: to V. Lakshmikantham on His 80th Birthday. Vol. 1, Kluwer Academic, Dordrecht, 2003, pp. 75–88.

[2] R. P. Agarwal and D. O'Regan, *Cone compression and expansion fixed point theorems in Fréchet spaces with applications*, Journal of Differential Equations **171** (2001), no. 2, 412–429.

[3] ______, *Fixed point theory for self maps between Fréchet spaces*, Journal of Mathematical Analysis and Applications **256** (2001), no. 2, 498–512.

[4] ______, *Multivalued nonlinear equations on the half line: a fixed point approach*, The Korean Journal of Computational & Applied Mathematics **9** (2002), no. 2, 509–524.

[5] Y. Q. Chen, Y. J. Cho, and D. O'Regan, *On positive fixed points of countably condensing mappings*, Dynamics of Continuous, Discrete & Impulsive Systems. Series A. Mathematical Analysis **12** (2005), no. 3-4, 519–527.

[6] M. Frigon, *Fixed point results for compact maps on closed subsets of Fréchet spaces and applications to differential and integral equations*, Bulletin of the Belgian Mathematical Society. Simon Stevin **9** (2002), no. 1, 23–37.

[7] M. Frigon and D. O'Regan, *Fixed points of cone-compressing and cone-extending operators in Fréchet spaces*, The Bulletin of the London Mathematical Society **35** (2003), no. 5, 672–680.

[8] M. Izydorek and Z. Kucharski, *The Krasnosielski theorem for permissible multivalued maps*, Bulletin of the Polish Academy of Sciences. Mathematics **37** (1989), no. 1–6, 145–149 (1990).

[9] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1964.

[10] D. O'Regan, *A multiplicity fixed point theorem in Fréchet spaces*, Zeitschrift für Analysis und ihre Anwendungen **19** (2000), no. 3, 853–862.

[11] ———, *A Krasnoselskii cone compression theorem for $\mathcal{U}_c^k$ maps*, Mathematical Proceedings of the Royal Irish Academy **103A** (2003), no. 1, 55–59.

[12] D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integrodifferential Equations*, Mathematics and Its Applications, vol. 445, Kluwer Academic, Dordrecht, 1998.

[13] D. O'Regan and M. Zima, *Leggett-Williams norm type fixed point theorems for multivalued mappings*, to appear.

[14] W. V. Petryshyn, *Existence of fixed points of positive k-set-contractive maps as consequences of suitable boundary conditions*, Journal of the London Mathematical Society. Second Series **38** (1988), no. 3, 503–512.

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