STABILIZATION OF VOLTERRA EQUATIONS BY NOISE

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The paper studies the stability of an autonomous convolution Itô-Volterra equation where the linear diffusion term depends on the current value of the state only, and the memory of the past fades exponentially fast. It is shown that the presence of noise can stabilize an equilibrium solution which is unstable in the absence of this noise.

1. Introduction

This paper aims to contribute to research on the question of the stabilization or destabilization of a deterministic dynamical system (differential equation, partial differential equation, or functional differential equation) by a noise perturbation, and in particular, perturbations which transform the differential equation (or FDE) to one of Itô-type. Mao has written an interesting paper [9] devoted to the study of stabilization and destabilization of nonlinear finite-dimensional differential equations, which has been extended to examine the stabilization of partial differential equations by Caraballo et al. [6]. The asymptotic behaviour of linear functional differential equations with bounded delay has been studied by Mohammed and Scheutzow [11], wherein it is shown that time delays in the diffusion coefficient can destablilize a linear functional differential equation. The stabilization of nonlinear finite-dimensional functional differential equations with (sufficiently small) bounded delay has been covered by the author in [3], and the destabilization of even-dimensional equations in [2]. In the latter paper, however, the delay can be unbounded, so Volterra equations can be destabilized, as a special case. For scalar, linear, convolution Volterra equations with positive and integrable kernel, Appleby has shown in [1] that the corresponding family of Itô-Volterra equations with a diffusion term of the form σx is almost surely asymptotically stable, provided the deterministic problem is uniformly asymptotically stable, so the addition of noise is not destabilizing. To the authors' knowledge, the issue of stabilization of a Volterra equation by noise perturbations of Itô-type has not, to date, been studied. The question is of interest in applications in

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the context of the dynamic stability of viscoelastic members subject to stochastic perturbations.

In this paper, we first consider a very simple type of nonlinear scalar convolution Volterra equation:

$$x'(t) = -af(x(t)) + \int_0^t \lambda e^{-\mu(t-s)} g(x(s))\,ds, \tag{1.1}$$

and its Itô perturbation

$$dX(t) = \left(- af(X(t)) + \int_0^t \lambda e^{-\mu(t-s)} g(X(s))\,ds \right) dt + \sigma X(t)\,dB(t). \tag{1.2}$$

The presence of an exponential kernel is central to our initial analysis: it means that (1.1) (resp., (1.2)) can be written as a differential equation (resp., SDE) in $\mathbb{R}^2$. There are two important benefits of this alternative representation. First, a finite-dimensional stochastic differential equation is easier to analyze than an infinite-dimensional Volterra equation. Second, it is true that solutions of stochastic functional differential equations can oscillate about their zero equilibrium position in the sense that there is no last zero of $X(t)$, almost surely. Therefore we cannot, in general, use Itô's rule to obtain the semimartingale decomposition of $|X(t)|^p$ for $0 < p < 1$ (or $\log|X(t)|$), which is very helpful in obtaining good information on the asymptotic stability of stochastic differential equations. For linear equations, Arnold has shown in [4] (under appropriate Lie algebra conditions on the matrices A and B) that the linear stochastic system

$$dX(t) = AX(t)\,dt + BX(t)\,dW(t) \tag{1.3}$$

has almost sure top Lyapunov exponent which is the limit as $p \downarrow 0$ of the top Lyapunov exponent of the pth mean $\mathbb{E}[\|X(t)\|^p]$. However, the system of stochastic differential equations created from (1.2) is nonzero for all time, a.s., so, with $Z(t) = (X(t), Y(t))$, where

$$Y(t) = \int_0^t e^{-\mu(t-s)} X(s)\,ds, \tag{1.4}$$

we can consider the semimartingale $\|Z(t)\|^p$ for small $p > 0$ and use a nonlinear analogue of Arnold's result above (which is due to Mao [10]) to extract an upper bound on the top Lyapunov exponent of Z.

Once results on stabilization have been obtained for the linear case of (1.2), it is then possible to use a type of comparison principle argument to deal with the more general equation

$$dX(t) = \left(- af(X(t)) + \int_0^t k(t-s) g(X(s))\,ds \right) dt + \sigma X(t)\,dB(t) \tag{1.5}$$

when $|k(t)| \le \lambda e^{-\mu t}$, for all $t \ge 0$, and some positive μ and λ. We also extend these results to nonconvolution equations and general finite-dimensional equations.

The paper is organized as follows: the details of the problem to be studied, notation, and supporting results are presented in Section 2. An upper bound on the pth mean and a.s. top Lyapunov exponent of (1.2) is obtained in Section 3. The bound can be written in terms of an optimization problem which is parameterized by the model parameters a, μ, λ and the functions f, g. In Section 4, we obtain sufficient conditions on the model parameters to ensure that the a.s. top Lyapunov exponent of solutions of (1.2) is negative. In Section 5, these results are formally stated, and we show how the region in (μ,a,λ) parameter space for which global a.s. exponential asymptotic stability can be enlarged under certain hypotheses on the sign of f. Indeed, it is established that deterministic problems of the form (1.1) which have unstable zero solutions can be stabilized for all initial conditions, almost surely, whenever σ lies in a nonempty a, μ, λ-dependent interval. Results on nondestabilization for linear problems are also established in Section 5. Section 6 has the same concerns as Section 5, but instead covers the more general problem (1.5). In Section 7, our results are applied to some specific problems. Further generalizations are presented in Section 8.

2. The scalar problem

We will use the following standard notation in this paper. Let $\mathbb{R} = (-\infty,\infty)$, and $\mathbb{R}^+ = [0,\infty)$. Denote the minimum of x, y in $\mathbb{R}$ by $x \wedge y$, their maximum by $x \vee y$, and the signum function by $\operatorname{sgn}(x) = 1$ for $x > 0$, and $\operatorname{sgn}(x) = -1$ for $x \le 0$. If I, J are open sets in a Banach space, we denote the class of continuous functions taking I onto J by $C(I;J)$. Let $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$, and let $\{\mathbf{e}_1,\mathbf{e}_2\}$ be the standard basis vectors in $\mathbb{R}^2$. If $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$, we denote the standard innerproduct of $\mathbf{x}$ and $\mathbf{y}$ by $\langle \mathbf{x},\mathbf{y}\rangle$. The standard Euclidean norm of $\mathbf{x} \in \mathbb{R}^2$ is given by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x},\mathbf{x}\rangle}$. We denote the space of 2×2 matrices with real entries by $M_{2,2}(\mathbb{R})$, so the space of continuous functions taking $\mathbb{R}^2$ onto $\mathbb{M}_{2,2}(\mathbb{R})$ is given by $C(\mathbb{R}^2;M_{2,2}(\mathbb{R}))$. We take the standard operator norm as norm on $M_{2,2}(\mathbb{R})$; for $A \in M_{2,2}(\mathbb{R})$, the operator norm of A is given by

$$\|A\| = \sup\{\mathbf{x}\in\mathbb{R}^2, \|\mathbf{x}\| = 1 : \|A\mathbf{x}\|\}. \tag{2.1}$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. We study the scalar Itô-Volterra equation

$$dX(t) = \Big(-af(X(t)) + \int_0^t k(t-s)g(X(s))\,ds\Big)dt + \sigma X(t)\,dB(t), \tag{2.2}$$

where $a \in \mathbb{R}$, $\sigma \neq 0$, and f, g, k are continuous functions, whose properties we later specify in more detail. Here, $B = \{B(t),\mathcal{F}_t^B;\ 0 \le t < \infty\}$ is a standard one-dimensional Brownian motion on the probability space, with natural filtration $(\mathcal{F}_t^B)_{t\ge 0}$. We suppose that (2.2) is an initial value problem, with $X(0) = \xi$, where ξ is a square-integrable random variable which is independent of B. In the usual way, the filtration $(\mathcal{F}_t^B)_{t\ge 0}$ can be extended to $(\mathcal{F}_t)_{t\ge 0}$ in such a way that $B = \{B(t),\mathcal{F}_t;\ 0 \le t < \infty\}$ is a standard Brownian motion.

In this paper, we assume that $k \in C(\mathbb{R};\mathbb{R})$ and that $\sup_{t\ge 0}|k(t)|e^{\mu t}$ is finite for some $\mu > 0$. In particular, we first study (2.2), where $k(t)$ is a negative exponential, namely, $k(t) = \lambda e^{-\mu t}$ for $t \ge 0$, so

$$dX(t) = \Big(-af(X(t)) + \int_0^t \lambda e^{-\mu(t-s)}g(X(s))\,ds\Big)dt + \sigma X(t)\,dB(t), \tag{2.3}$$

where $\lambda \neq 0$ and $\mu > 0$. We remark in advance that all results relating to existence and uniqueness of solutions, and to the zero solution of (2.3) apply equally to the more general problem (2.2), under the hypotheses which we impose below on the functions f, g.

Suppose that $f, g \in C(\mathbb{R};\mathbb{R})$ and suppose that

$$f(0) = 0, \qquad g(0) = 0. \tag{2.4}$$

Let f, g satisfy local Lipschitz conditions, and the following global linear bounds:

$$|f(x)| \leq \overline{f}|x|, \qquad |g(x)| \leq \overline{g}|x|, \tag{2.5}$$

where $\overline{f}, \overline{g}$ are finite positive constants. We also suppose without loss of generality that $\overline{f}$, $\overline{g}$ are optimal in (2.5), so that

$$\sup_{x\in\mathbb{R}\setminus\{0\}} \frac{|f(x)|}{|x|} = \overline{f}, \qquad \sup_{x\in\mathbb{R}\setminus\{0\}} \frac{|g(x)|}{|x|} = \overline{g}. \tag{2.6}$$

Under conditions (2.4), (2.5), and the local Lipschitz continuity of f, g, (2.3) has a unique solution on any compact interval $[0,T]$ in the space of Itô processes (which is consequently continuous). In addition, the solution satisfies

$$\mathbb{E}\Big[\max_{0\leq T} |X(t)|^2\Big] < \infty. \tag{2.7}$$

These results are proven by Berger and Mizel in [5]. To emphasize dependence on the initial condition, we denote the value of the solution of (2.3) at time $t \geq 0$ with initial condition ξ by $X(t;\xi)$. Moreover, by dint of (2.4), $X(t;0) = 0$ for all $t \geq 0$, a.s. This is called the zero solution of (2.3). It is the almost sure asymptotic stability of this solution which is the main topic of this paper. We say that all solutions are almost surely exponentially stable, or that the zero solution of (2.3) is globally almost surely exponentially stable, if there exists a positive nonrandom constant δ_0 such that for all ξ, we have

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t,\xi)| \leq -\delta_0, \quad \text{a.s.} \tag{2.8}$$

In order to compare the stability of the solution of (2.3) with that of the deterministic analogue of (2.3)

$$x'(t) = -af(x(t)) + \int_0^t \lambda e^{-\mu(t-s)} g(X(s))\,ds, \quad t \geq 0, \tag{2.9}$$

(namely, (2.3) with $\sigma = 0$), we will wish to consider the linearization of (2.9) at $x = 0$. We therefore assume that f and g are continuously differentiable in an open interval around 0, and without loss of generality, we set

$$f'(0) = 1, \qquad g'(0) = 1. \tag{2.10}$$

Note that by writing

$$y(t) = \int_0^t e^{-\mu(t-s)} g(x(s))\,ds, \tag{2.11}$$

we can reexpress (2.9) as a first-order system of differential equations: if $z(t) = (x(t), y(t))$, then $z'(t) = F(z(t))$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$F(x,y) = (-af(x) + \lambda y,\ g(x) - \mu y). \tag{2.12}$$

By considering the linearization of this planar system at $x = 0$, we can use the Hartman-Grobman theorem to conclude that there exist η, $\delta > 0$ such that for all $|x(0)| < \eta$ we have

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| = -\delta, \tag{2.13}$$

whenever

$$a > \frac{\lambda}{\mu}, \quad a + \mu > 0. \tag{2.14}$$

Moreover, the zero solution of (2.9) is unstable if the sense of either (or both) of the inequalities in (2.14) is reversed.

In this paper, we wish to obtain conditions on σ, and the parameters a, μ, λ such that the solution of (2.3) is globally a.s. exponentially asymptotically stable (namely, that (2.8) is satisfied for all initial conditions ξ and some $\delta_0 > 0$). In particular, we would like to establish (2.8) when (2.14) is false, establishing the asymptotic stability of (2.3) when the zero solution of (2.9) is unstable.

For much of the paper, we impose the following additional conditions on f:

$$\begin{gathered} xf(x) > 0, \quad \forall x \neq 0, \\ 0 < \underline{f} = \inf_{x\in\mathbb{R}\setminus\{0\}} \frac{|f(x)|}{|x|} \leq \sup_{x\in\mathbb{R}\setminus\{0\}} \frac{|f(x)|}{|x|} = \overline{f} < \infty. \end{gathered} \tag{2.15}$$

Under (2.15), more precise results on stabilization are possible. We note that the linear case, in which $f(x) = x$, satisfies these conditions, and therefore use (2.15) to obtain sharp results in this case. Moreover, once the analysis has been conducted under these conditions, it is easy to see how to proceed when they do not hold.

In studying problem (2.2) where $|k(t)| \leq \lambda e^{-\mu t}$ for some λ, $\mu > 0$, we will invoke the above hypotheses on f and g. We defer any further specific comments relating to this problem until we study it in Section 6, save to say that we try, as far as is possible, to implement the program outlined above for (2.3) for the more general equation (2.2) also.

3. Exponential asymptotic stability

In this section, we establish a sufficient algebraic condition under which solutions of (2.3) satisfy (2.8). The analysis of this algebraic condition is the subject of Section 4.

Introduce the functions

$$\tilde{f}(x) = \begin{cases} \dfrac{f(x)}{x} & x \neq 0, \\ 1 & x = 0, \end{cases} \tag{3.1}$$

$$\tilde{g}(x) = \begin{cases} \dfrac{g(x)}{x} & x \neq 0, \\ 1 & x = 0. \end{cases} \tag{3.2}$$

Then by (2.4), (2.10), $\tilde{f}, \tilde{g}$ are continuous, and by virtue of (2.5), are bounded on $\mathbb{R}$, with $\sup_{y\in\mathbb{R}} |\tilde{g}(y)| = \bar{g}$. Next, introduce the process

$$Y(t) = \int_0^t e^{-\mu(t-s)} g(X(s))\,ds, \tag{3.3}$$

and define the matrices

$$A(y) = \begin{pmatrix} -a\tilde{f}(y) & \lambda \\ \tilde{g}(y) & -\mu \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4}$$

Then, setting

$$Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \tag{3.5}$$

we see that (3.1), (3.2), (3.3), and (3.4) imply that

$$dZ(t) = A(X(t))Z(t)dt + \Sigma Z(t)dB(t), \tag{3.6}$$

with $Z(0) = (\xi, 0)$. Next, suppose that P is an invertible matrix in $M_{2,2}(\mathbb{R})$, and define, for $z \in \mathbb{R}^2$,

$$\tilde{A}(z) = PA(\langle P^{-1}z, \mathbf{e}_1\rangle)P^{-1}, \qquad \tilde{\Sigma} = P\Sigma P^{-1}, \tag{3.7}$$

and $\tilde{Z}(t) = PZ(t)$ so that $\tilde{Z}$ obeys the stochastic differential equation

$$d\tilde{Z}(t) = \tilde{A}(\tilde{Z}(t))\tilde{Z}(t)dt + \tilde{\Sigma}\tilde{Z}(t)dB(t), \tag{3.8}$$

and $\tilde{Z}(0) = PZ(0)$ is nonzero provided that $\xi \neq 0$. Noticing from (3.4), (3.7) that $\tilde{A} \in$

$C(\mathbb{R}^2;M_{2,2}(\mathbb{R}))$ is a bounded function, we can use [8, Proposition 2.1] to establish for $\xi \neq 0$ that

$$\widetilde{Z}(t) \neq 0 \quad \forall t \geq 0, \quad \text{a.s.} \tag{3.9}$$

Next, the boundedness of the entries of $\widetilde{A}$ ensures the existence of a finite δ_0 such that

$$-\delta_0 = \sup_{y\in\mathbb{R}^2} \max_{\|x\|=1} \langle x, \widetilde{A}(y)x\rangle + \frac{1}{2}\langle \widetilde{\Sigma}x, \widetilde{\Sigma}x\rangle - \langle x, \widetilde{\Sigma}x\rangle^2. \tag{3.10}$$

Let us temporarily assume that it has been shown that

$$\delta_0 > 0. \tag{3.11}$$

Thus, for every $\delta_1 \in (0,\delta_0)$, there exists $p \in (0,1)$ satisfying

$$p \sup_{\|x\|=1} \langle \widetilde{\Sigma}x, x\rangle^2 \leq 2(\delta_0 - \delta_1). \tag{3.12}$$

Define

$$p^* = \frac{2\delta_0}{\sup_{\|x\|=1}\langle \widetilde{\Sigma}x, x\rangle^2}, \tag{3.13}$$

so that $p < p^*$ satisfies (3.12) for some $\delta_1 = \delta_1(p) > 0$.

By (3.10), (3.12), it follows for all $y \in \mathbb{R}^2$, $\|x\| = 1$ that

$$\langle x, \widetilde{A}(y)x\rangle + \frac{1}{2}\langle \widetilde{\Sigma}x, \widetilde{\Sigma}x\rangle + \frac{p-2}{2}\langle x, \widetilde{\Sigma}x\rangle^2 \leq -\delta_1. \tag{3.14}$$

Now, by (3.8), Itô's rule furnishes us with the semimartingale decomposition of $\|\widetilde{Z}\|^2 = \{\|\widetilde{Z}(t)\|^2; \mathcal{F}_t;\ 0 \leq t < \infty\}$, and so, by (3.9), we may use Itô's rule again to obtain the semimartingale decomposition of $\|\widetilde{Z}\|^p = (\|\widetilde{Z}\|^2)^{p/2}$, which is

$$\begin{aligned}\|\widetilde{Z}(t)\|^p = \|\widetilde{Z}(0)\|^p + \int_0^t p\|\widetilde{Z}(s)\|^p \Bigg(&\frac{\langle \widetilde{Z}(s), \widetilde{A}(\widetilde{Z}(s))\widetilde{Z}(s)\rangle}{\|\widetilde{Z}(s)\|^2} + \frac{1}{2}\frac{\langle \widetilde{\Sigma}\widetilde{Z}(s), \widetilde{\Sigma}\widetilde{Z}(s)\rangle}{\|\widetilde{Z}(s)\|^2} \\ &+ \frac{p-2}{2}\left(\frac{\langle \widetilde{Z}(s), \widetilde{\Sigma}\widetilde{Z}(s)\rangle}{\|\widetilde{Z}(s)\|^2}\right)^2\Bigg) ds \\ &+ \int_0^t p\|\widetilde{Z}(s)\|^{p-2}\langle \widetilde{Z}(s), \widetilde{\Sigma}\widetilde{Z}(s)\rangle\, dB(s).\end{aligned} \tag{3.15}$$

Therefore, if t, $t+h \geq 0$, (3.14) and (3.15) yield

$$\|\widetilde{Z}(t+h)\|^p - \|\widetilde{Z}(t)\|^p \leq \int_t^{t+h} -\delta_1 p\|\widetilde{Z}(s)\|^p ds + \int_t^{t+h} p\|\widetilde{Z}(s)\|^{p-2}\langle \widetilde{Z}(s), \widetilde{\Sigma}\widetilde{Z}(s)\rangle\, dB(s). \tag{3.16}$$

Using the Cauchy-Schwarz inequality, and then Lyapunov's inequality (as $p < 1$), we get

$$\mathbb{E}\left[\int_0^t \|\widetilde{Z}(s)\|^{2p-4}\langle \widetilde{\Sigma}\widetilde{Z}(s), \widetilde{Z}(s)\rangle^2 ds\right] \leq \|\widetilde{\Sigma}\|^2 \int_0^t \mathbb{E}\left[\|\widetilde{Z}(s)\|^2\right]^{1/p} ds, \tag{3.17}$$

which is finite for all finite $t \geq 0$, as $\mathbb{E}[\|\widetilde{Z}(t)\|^2]$ is finite for all $0 \leq t < \infty$. Therefore,

$$\mathbb{E}\left[\int_0^t p\|\widetilde{Z}(s)\|^{p-2}\langle \widetilde{Z}(s), \widetilde{\Sigma}\widetilde{Z}(s)\rangle dB(s)\right] = 0, \tag{3.18}$$

so defining $V(t) = \mathbb{E}[\|\widetilde{Z}(t)\|^p]$ and taking the expectation on both sides of (3.16) yields

$$V(t+h) - V(t) \leq -\delta_1 p \int_t^{t+h} V(s)ds. \tag{3.19}$$

Obviously, $V(t)$ is nonnegative and finite for $0 \leq t < \infty$. In Lemma 3.4, we show that $t \mapsto V(t)$ is continuous. Therefore, taking the limsup as $h \to 0$ in (3.19), we arrive at

$$D_+V(t) \leq -\delta_1 p V(t), \quad t \geq 0, \tag{3.20}$$

where

$$D_+V(t) = \limsup_{h \downarrow 0} \frac{V(t+h) - V(t)}{h}. \tag{3.21}$$

Therefore, we get

$$\mathbb{E}[\|\widetilde{Z}(t)\|^p] \leq \mathbb{E}[\|\widetilde{Z}(0)\|^p]e^{-\delta_1 p t}. \tag{3.22}$$

Using (3.5), the invertibility of P, and the definition of $\widetilde{Z}$ in terms of Z, there exists a P-dependent constant $K(P)$, such that

$$\mathbb{E}[|X(t)|^p] \leq K(P)\mathbb{E}[|\xi|^p]e^{-\delta_1 p t}, \quad t \geq 0. \tag{3.23}$$

This gives us our first result.

Theorem 3.1. *Suppose that there exists an invertible matrix $P \in M_{2,2}(\mathbb{R})$ such that δ_0 defined in (3.10) is positive. If p^* is defined by (3.13), then for $p < p^* \wedge 1$, there exists $\delta_1 = \delta_1(p) > 0$ such that (3.23) holds.*

To establish a.s. global exponential asymptotic stability, observe that the stochastic differential equation (3.8) satisfies all the hypotheses of [10, Theorem 4.2], so using this result, we get

$$\limsup_{t \to \infty} \frac{1}{t} \log\|\widetilde{Z}(t)\| \leq \frac{-\delta_1 p}{p} = -\delta_1, \quad \text{a.s.} \tag{3.24}$$

Letting $\delta_1 \uparrow \delta_0$ through the rationals yields

$$\limsup_{t \to \infty} \frac{1}{t} \log\|\widetilde{Z}(t)\| \leq -\delta_0, \quad \text{a.s.,} \tag{3.25}$$

and so, using $\widetilde{Z}(t) = PZ(t)$ and (3.5) now gives

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \le -\delta_0, \quad \text{a.s.} \tag{3.26}$$

We therefore have the following theorem.

Theorem 3.2. *Suppose that there exists an invertible matrix $P \in M_{2,2}(\mathbb{R})$ such that δ_0 defined by (3.10) is positive. Then the solution of (2.3) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t;\xi)| \le -\delta_0, \quad \textit{a.s.} \tag{3.27}$$

Hence, if δ_0 defined by (3.10) can be shown to be positive, Theorems 3.1 and 3.2 show that all solutions of (2.3) are pth mean exponentially stable for some $p \in (0,1)$, and also are a.s. globally exponentially asymptotically stable.

Remark 3.3. We could have proved the almost sure exponential stability under the condition that δ_0 defined by (3.10) is more directly positive by obtaining a semimartingale decomposition of $\log(X(t)^2 + Y(t)^2)$. This is essentially Mao's method of proving stabilization of ordinary differential equations by noise in [9, Theorem 3.1]. However, as we prefer to also establish the pth mean exponential asymptotic stability for small p, we proceed as in Theorems 3.1, 3.2.

We now return to the deferred proof of Lemma 3.4.

Lemma 3.4. *If $\widetilde{Z}$ satisfies (3.8), then $t \mapsto \mathbb{E}[\|\widetilde{Z}(t)\|^p]$ is continuous for any $p \in (0,1)$.*

Proof. Since $\widetilde{A}$ is a bounded function, [7, Problem 5.3.15] implies that $\widetilde{Z}$ satisfies

$$\mathbb{E}\left[||\widetilde{Z}(t) - \widetilde{Z}(t_0)||^p\right] \le C(T)\left(1 + \mathbb{E}\left[||\widetilde{Z}(0)||^2\right]\right) |t - t_0| \tag{3.28}$$

for all $0 \le t_0$, $t \le T$, and any $T > 0$. The constant $C(T)$ is positive and finite. Note that $p \in (0,1)$, so

$$\begin{aligned} &|\mathbb{E}||Z(t)||^p - \mathbb{E}||Z(t_0)||^p| \\ &\quad = |\mathbb{E}[||Z(t)||^p - ||Z(t_0)||^p]| \le \mathbb{E}|\,||Z(t)||^p - ||Z(t_0)||^p| \\ &\quad \le \mathbb{E}[\,|\,||Z(t)|| - ||Z(t_0)||\,|^p] \le \mathbb{E}[\,|\,||Z(t)|| - ||Z(t_0)||\,|^2]^{p/2}, \end{aligned} \tag{3.29}$$

where we use the inequality $|x^p - y^p| \le |x - y|^p$, $x, y \ge 0$, at the penultimate step, and Lyapunov's inequality at the last step. Applying the inequality $|\|x\| - \|y\|| \le \|x - y\|$ for $x, y \in \mathbb{R}^2$, and then (3.28) therefore gives

$$|\mathbb{E}||\widetilde{Z}(t)||^p - \mathbb{E}||\widetilde{Z}(t_0)||^p| \le \left(C(T)\left(1 + \mathbb{E}\left[||\widetilde{Z}(0)||^2\right]\right)|t - t_0|\right)^{p/2}, \tag{3.30}$$

from which continuity is immediate. □

4. Sufficient conditions for $\delta_0 > 0$

In this section, we obtain some explicit conditions in terms of the parameters in (2.3) for which δ_0 given by (3.10) can be shown to be positive. In most cases, we do not try to calculate δ_0 directly, but instead obtain bounds on the underlying quantity

$$F(x,y) = \langle x, \widetilde{A}(y)x\rangle + \frac{1}{2}\langle \widetilde{\Sigma}x, \widetilde{\Sigma}x\rangle - \langle x, \widetilde{\Sigma}x\rangle^2 \tag{4.1}$$

which is to be maximized. To do this, we consider a simple class of diagonal matrices

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \tag{4.2}$$

for $\beta > 0$ in the construction of the matrices $\widetilde{A}$, $\widetilde{\Sigma}$ in (3.7). We will try to choose β optimally. Defining $x = (x_1, x_2)$, $w = \langle P^{-1}y, \mathbf{e}_1\rangle$, and using (3.4), (3.7), (4.2), we get

$$F(x,y) = -a\widetilde{f}(w)x_1^2 + \left(\frac{\lambda}{\beta} + \beta\widetilde{g}(w)\beta\right)x_1x_2 - \mu x_2^2 - \sigma^2x_1^4 + \frac{1}{2}\sigma^2x_1^2 =: \widetilde{F}(x,w). \tag{4.3}$$

Next define

$$\Lambda(\beta) = \sup_{w\in\mathbb{R}} \left|\frac{\lambda}{\beta} + \beta\widetilde{g}(w)\beta\right|. \tag{4.4}$$

Then, there exists $\beta^* > 0$ such that

$$\Lambda^* := \Lambda(\beta^*) = \inf_{\beta>0} \Lambda(\beta). \tag{4.5}$$

We will give explicit formulae for such a Λ^* presently. Returning to (4.3) with $\beta = \beta^*$, and by using the inequality

$$x_1x_2 \le \frac{1}{2}\left(\alpha x_1^2 + \frac{1}{\alpha}x_2^2\right), \quad \alpha > 0, \tag{4.6}$$

we obtain

$$\widetilde{F}(x,w) \le -a\widetilde{f}(w)x_1^2 + \frac{1}{2}\Lambda^*\left(\alpha x_1^2 + \frac{1}{\alpha}x_2^2\right) - \mu x_2^2 - \sigma^2x_1^4 + \frac{1}{2}\sigma^2x_1^2. \tag{4.7}$$

We reimpose conditions (2.15). By defining

$$\widetilde{f} = \begin{cases} \inf_{w\in\mathbb{R}} \widetilde{f}(w) = \underline{f} & \text{for } a \ge 0, \\ \sup_{w\in\mathbb{R}} \widetilde{f}(w) = \overline{f} & \text{for } a < 0, \end{cases} \tag{4.8}$$

we see from (3.10), (4.7), and (4.8) that

$$-\delta_0 \le \max_{x\in[0,1]} H(x), \tag{4.9}$$

where

$$H(x) = -\sigma^2 x^2 + x\Big(-a\tilde{f} + \frac{1}{2}\Lambda^*(\alpha - \alpha^{-1}) + \mu + \frac{1}{2}\sigma^2\Big) + \frac{\Lambda^*}{2\alpha} - \mu. \tag{4.10}$$

Note that σ enters (4.10) only through terms involving σ^2, so that without loss of generality, we consider $\sigma > 0$ hereinafter.

The effect of the above argument has been to bound the complicated nonlinear function F with a one-parameter family of quadratic functions. Therefore, in order to show that $\delta_0 > 0$ is possible for some interval of values of σ for a particular triple of parameters (μ, a, Λ^*) determined by the deterministic problem (2.9), it is sufficient to prove that there is a value of $\alpha > 0$—which can depend on (μ, a, Λ^*)—such that for that choice of α, there is an interval of values of $|\sigma|$ such that

$$\max_{x\in[0,1]} H(x) < 0. \tag{4.11}$$

Remark 4.1. We can separate out the dependence of Λ^* on λ by observing (for $\lambda > 0$) that

$$\Lambda^* = \inf_{\beta>0} \Lambda(\sqrt{\lambda}\beta) = \sqrt{\lambda} \inf_{\beta>0} \sup_{w\in\mathbb{R}} |\beta^{-1} + \beta\tilde{g}(w)|, \tag{4.12}$$

and similarly for $\lambda < 0$ that

$$\Lambda^* = \inf_{\beta>0} \Lambda(\sqrt{-\lambda}\beta) = \sqrt{-\lambda} \inf_{\beta>0} \sup_{w\in\mathbb{R}} |-\beta^{-1} + \beta\tilde{g}(w)|, \tag{4.13}$$

so $\Lambda^*(\lambda) = \sqrt{|\lambda|}G^*(\lambda)$, where $G^*(\lambda) = \inf_{\beta>0}\sup_{w\in\mathbb{R}} |\mathrm{sgn}(\lambda)\beta^{-1} + \beta\tilde{g}(w)|$. In fact, it is possible to explicitly compute $\Lambda^*(\lambda)$, given that $g_0 = \inf_{w\in\mathbb{R}}\tilde{g}(w)$ and $g^0 = \sup_{w\in\mathbb{R}}\tilde{g}(w)$. For $\lambda > 0$, we have

$$\Lambda^*(\lambda) = \begin{cases} 2\sqrt{g^0}\sqrt{\lambda} & \text{if } g_0 > 0 \quad \text{or} \quad 0 < -\dfrac{g_0}{3} < g^0, \\ \sqrt{\lambda}\dfrac{1}{\sqrt{2}}\dfrac{-g_0 + g^0}{\sqrt{-g_0 - g^0}} & \text{if } -\dfrac{g_0}{3} > g^0, \end{cases} \tag{4.14}$$

while for $\lambda < 0$, Λ^* is given by

$$\Lambda^*(\lambda) = \begin{cases} \sqrt{|\lambda|}\dfrac{1}{\sqrt{2}}\dfrac{g^0 - g_0}{\sqrt{g^0 + g_0}} & \text{if } g_0 > 0 \quad \text{or} \quad -g_0 > \dfrac{g^0}{3} > 0, \\ 2\sqrt{|\lambda|}\sqrt{-g_0} & \text{if } -g_0 > \dfrac{g^0}{3}. \end{cases} \tag{4.15}$$

By a careful choice of α, the following can be proven.

Proposition 4.2. *Suppose (2.15) is true, and define*

$$\nu_\lambda(\mu) = \begin{cases} \dfrac{\Lambda^*(\lambda)^2}{4\underline{f}}\dfrac{1}{\mu} & \text{if } 0 < \mu \le \dfrac{\Lambda^*(\lambda)}{2}, \\ \dfrac{\Lambda^*(\lambda) - \mu}{\underline{f}} & \text{if } \dfrac{\Lambda^*(\lambda)}{2} \le \mu \le \Lambda^*(\lambda), \\ \dfrac{\Lambda^*(\lambda) - \mu}{\overline{f}} & \text{if } \mu > \Lambda^*(\lambda), \end{cases} \tag{4.16}$$

where Λ^* *is given by (4.5). If*

$$a > \nu_\lambda(\mu), \tag{4.17}$$

there exists a nonempty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ *such that for all* $|\sigma| \in I_{a,\mu,\lambda}$, *there exists* $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \lambda) > 0$ *so that*

$$\delta_0^* < \delta_0, \tag{4.18}$$

where δ_0 *is given by (3.10).*

Proof. In the proof, we suppress the dependence of Λ^* on λ. Define

$$\mu_1 = \frac{\Lambda^*}{2\alpha}, \qquad \mu_2 = \frac{1}{2}\Lambda^*(\alpha + \alpha^{-1}), \qquad \mu_3 = \Lambda^*\left(\alpha + \frac{1}{2\alpha}\right), \tag{4.19}$$

and the functions

$$f_1(\mu) = \frac{\mu + (1/2)\Lambda^*(\alpha - \alpha^{-1})}{\underline{f}}, \quad \mu \ge \mu_1, \tag{4.20}$$

$$f_2(\mu) = \begin{cases} \dfrac{\Lambda^*(2\alpha + \alpha^{-1}) - 2\mu}{4\underline{f}}, & \mu_1 \le \mu \le \mu_2, \\ \dfrac{\Lambda^*(2\alpha + \alpha^{-1}) - 2\mu}{4\overline{f}}, & \mu_2 \le \mu, \end{cases}$$
$$f_3(\mu) = \begin{cases} \dfrac{(1/2)\Lambda^*(\alpha + \alpha^{-1}) - \mu}{4\underline{f}}, & \mu_2 \le \mu \le \mu_3, \\ \dfrac{(1/2)\Lambda^*(\alpha + \alpha^{-1}) - \mu}{4\overline{f}}, & \mu_3 \le \mu. \end{cases} \tag{4.21}$$

Note that

$$f_1(\mu_1) = f_2(\mu_1) = f_3(\mu_1) = \frac{\alpha\Lambda^*}{2\underline{f}}. \tag{4.22}$$

There are three ways in which (4.11) can be satisfied for H defined in (4.10):

(i) $H(0) < 0$, $H'(0) < 0$ (so $\max_{x\in[0,1]} H(x) = H(0) < 0$);

(ii) $H(1) < 0$, $H'(1) > 0$ (so $\max_{x\in[0,1]} H(x) = H(1) < 0$);
(iii) there exists $x^* \in [0,1]$ with $H'(x^*) = 0$, $H(x^*) < 0$ (so $\max_{x\in[0,1]} H(x) = H(x^*) < 0$).

Consider (for fixed α, λ) the subdivision of the (μ,a)-plane:
(1) $a > f_1(\mu)$, $\mu > \mu_1$,
(1′) $a = f_1(\mu)$, $\mu > \mu_1$,
(2) $a < f_1(\mu)$, $a > f_2(\mu)$, $\mu > \mu_1$,
(3) $a \le f_2(\mu)$, $a > f_3(\mu)$, $\mu > \mu_1$.

It follows after some manipulation that (1) implies (i), (2) implies (ii), (3) implies (iii), and (1′) implies (iii) (with $x^* = 1/4$) for appropriate ranges of σ. For future reference, we note that the ranges of σ are given by

$$\frac{1}{2}\sigma^2 < \overline{f}(a - f_1(\mu)) \quad \text{for (1)}, \tag{4.23}$$

$$\sigma^2 < 16(\mu - \mu_1) \quad \text{for (1}'), \tag{4.24}$$

$$2\left(-a\widetilde{f} + \frac{\Lambda^*\alpha}{2}\right) < \sigma^2 < \frac{2}{3}\left(-a\widetilde{f} + \mu + \frac{\Lambda^*}{2}(\alpha - \alpha^{-1})\right) \quad \text{for (2)}, \tag{4.25}$$

$$\sigma_- < |\sigma| < \sigma_+ \quad \text{for (3)}, \tag{4.26}$$

where

$$\sigma_\pm = 2\sqrt{\mu - \mu_1} \pm \sqrt{2\widetilde{f}(a - f_3(\mu))}. \tag{4.27}$$

Fix $\alpha > 0$, $\Lambda^* > 0$ (by fixing λ). Then with

$$S_\alpha = \left\{(\mu,a) \in \mathbb{R}^+ \times \mathbb{R} : \mu > \frac{\Lambda^*}{2\alpha},\ a > f_3(\mu)\right\}, \tag{4.28}$$

the following is true.

If $(\mu,a) \in S_\alpha$ for some $\alpha > 0$, there exists a nonempty interval $I_{a,\mu,\lambda,\alpha} \subset \mathbb{R}^+$ such that for all $|\sigma| \in I_{a,\mu,\lambda,\alpha}$, we have $H(x) < 0$ for all $x \in [0,1]$. (The intervals $I_{a,\mu,\lambda,\alpha}$ in cases (1), (1′), (2), (3) are determined by (4.23), (4.24), (4.25), (4.26), resp.)

Suppose $\nu_\lambda(\mu)$ is as defined in (4.16), and (4.17) holds. Then for $\mu \ge \Lambda^*$, $(\mu,a) \in S_1$, and for $\Lambda^*/2 \le \mu < \Lambda^*$, $(\mu,a) \in S_1$. For $0 < \mu < \Lambda^*/2$, $a > \nu_\lambda(\mu)$, we get $a > 0$, so there exists $\alpha > 0$ such that

$$\frac{\Lambda^*}{2\mu} < \alpha < \frac{2\underline{f}a}{\Lambda^*}, \tag{4.29}$$

which gives $a > \alpha\Lambda^*/2\underline{f}$, $\mu > \Lambda^*/2\alpha$, so by (4.22), (4.28), we have $(\mu,a) \in S_\alpha$. Fixing the α-dependence in terms of a, μ, λ as above yields the appropriate intervals $I_{a,\mu,\lambda} := I_{a,\mu,\lambda,\alpha(a,\mu,\lambda)}$, where $-\delta_0^* = \max_{x\in[0,1]} H(x)$ satisfies (4.18), by (4.9). □

Remark 4.3. Notice that $\Lambda^* = 0$ if and only if $g(x) \equiv x$, and $\lambda < 0$, in which case, by defining

$$\nu_\lambda(\mu) = -\frac{\mu}{\underline{f}}, \quad \mu > 0, \tag{4.30}$$

the result of Proposition 4.2 is true, where $a > \nu_\lambda(\mu)$. Note that the formula for ν_λ coincides with that in (4.16) when $\Lambda^*(\lambda) = 0$.

5. Exponential kernel

This section contains the first set of principal results of this paper. The first subsection states the main results flowing from the analysis in the previous two sections, the second considers the stabilization of the nonlinear equation (2.9), while the third subsection deals with stabilization and nondestabilizing results relating to the linear version of (2.9).

5.1. Theorems. Combining the results of Theorems 3.1, and 3.2, and of Proposition 4.2, we immediately obtain the first main result of the paper.

Theorem 5.1. *Suppose that f, g are locally Lipschitz continuous functions such that (2.4), (2.5), (2.10), (2.15) hold. Let ν_λ be defined by (4.16) and $a > \nu_\lambda(\mu)$. Then there exists a nonempty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda}$, there exists $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \lambda) > 0$ so that all solutions of (2.3) satisfy*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t;\xi)| \le -\delta_0^*, \quad a.s. \tag{5.1}$$

Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there exists $\delta_p > 0$ such that

$$\mathbb{E}[|X(t;\xi)|^p] \le K\mathbb{E}[|\xi|^p|]e^{-\delta_p t}, \quad t \ge 0, \tag{5.2}$$

where K is a positive constant independent of p and ξ.

The analysis of Proposition 4.2 enables a stabilization result to be proven when conditions (2.15) on f are dropped. When these conditions are dropped, (4.9) still holds, but H is now given by the formula

$$H(x) = |a|\overline{f}x + \frac{1}{2}\Lambda^*(\alpha x + \alpha^{-1}(1-x)) - \mu(1-x) - \sigma^2x^2 + \frac{1}{2}\sigma^2 x, \tag{5.3}$$

because $-a\widetilde{f}(w) \le |a|\overline{f}$, by (2.5), (3.1). Hence, by replacing $-a$ by $|a|$ and $\widetilde{f}$ by $\overline{f}$ in Proposition 4.2, we have the following result.

Theorem 5.2. *Suppose that f, g are locally Lipschitz continuous functions such that (2.4), (2.5), (2.10) hold. Let Λ^* be defined by (4.5), and suppose that*

$$|a| < \frac{\mu - \Lambda^*(\lambda)}{\overline{f}}, \quad \mu > \Lambda^*(\lambda). \tag{5.4}$$

Then there exists a nonempty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for all $|\sigma| \in I_{a,\mu,\lambda}$, there exists $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \lambda) > 0$ so that all solutions of (2.3) satisfy

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t;\xi)| \le -\delta_0^*, \quad a.s. \tag{5.5}$$

Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there exists $\delta_p > 0$ such that

$$\mathbb{E}\left[\,|X(t;\xi)|^p\right] \le K\mathbb{E}[\,|\xi|^p\,|]e^{-\delta_p t}, \quad t \ge 0, \tag{5.6}$$

where K is a positive constant independent of p and ξ.

Remark 5.3. It does not appear that conditions given in [9, Theorem 3.1] can be applied directly to the example given here, owing to the structure of the matrix Σ in (3.4). Moreover, it does not seem that applying a coordinate transform will rectify this.

Remark 5.4. It appears that it is possible to exploit the method of proof of [3, Theorem 4] to obtain results on the stabilization of (2.9) by noise. However, the estimates obtained on the region in (μ,a,λ) parameter space in which we can state with certainty that all solutions of (2.3) tend to zero are less sharp than those obtained in Theorems 5.1, 5.2, under the hypotheses imposed in those theorems.

Remark 5.5. Under the hypotheses (2.15) with $a > 0$, it can be shown that

$$a\underline{f} > \frac{\lambda\overline{g}}{\mu} \tag{5.7}$$

implies that all solutions of (2.3) satisfy (5.1) for some $\delta_0 > 0$, for *all* values of $|\sigma| > 0$. The proof of this result in the linear case is established in [1]; the adaptations required in the nonlinear case are straightforward, and not recorded here.

5.2. Stabilization of the nonlinear equation (2.9). We consider the deterministic nonlinear equation (2.9) and the corresponding family of related Itô-Volterra equations (2.3), under the hypotheses (2.15). We deal with the cases $\lambda > 0$, $\lambda < 0$ separately.

For $\lambda > 0$, the zero solution of (2.9) is locally exponentially asymptotically stable if $(\mu,a,\lambda) \in S_D$, where

$$S_D = \left\{(\mu,a) \in \mathbb{R}^+ \times \mathbb{R}^+ : a > \frac{\lambda}{\mu}\right\}. \tag{5.8}$$

For (2.3), the zero solution is almost surely exponentially asymptotically stable if $(\mu,a,\lambda) \in S_S$,

$$S_S = \{(\mu,a) \in \mathbb{R}^+ \times \mathbb{R}^+ : a > \nu_\lambda(\mu)\} \tag{5.9}$$

if $|\sigma| \in I_{a,\mu,\lambda}$. By defining $\mu^*(\lambda) \ge \Lambda(\lambda)/2$ as the solution of $\lambda/\mu^*(\lambda) = \nu_\lambda(\mu^*(\lambda))$, we see that for all $\mu > \mu^*(\lambda)$ and for all a obeying

$$\frac{\lambda}{\mu} < a < \nu_\lambda(\mu), \tag{5.10}$$

there exists an interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda}$, the deterministic problem (2.9) with parameters (μ,a,λ) is unstable, while problem (2.3) with parameters (μ,a,λ,σ) is globally asymptotically stable, a.s.

For $\lambda < 0$, using the same notation as above, we have

$$S_D = \left\{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R}^+ : a > \frac{\lambda}{\mu},\ a + \mu > 0\right\},$$
$$S_S = \left\{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R}^+ : a > \nu_\lambda(\mu)\right\}. \tag{5.11}$$

Hence, by defining $\mu^*(\lambda) \geq \Lambda(\lambda)$ as above, with $\mu > \mu^*(\lambda)$, and a obeying (5.10), the deterministic problem is unstable, while the stochastic problem can be stabilized for the appropriate choice of σ, as above.

Relaxing the hypotheses (2.15), we see that the unstable deterministic problem with parameters (μ, a, λ) can be stabilized for any $|\sigma| \in I_{a,\mu,\lambda}$, which is a nonempty interval in $\mathbb{R}^+$ provided that

$$a < \frac{\lambda}{\mu}, \quad |a| < \frac{\mu - \Lambda^*(\lambda)}{\overline{f}}, \quad \mu > \Lambda^*(\mu). \tag{5.12}$$

5.3. Stabilization of the linear equation. For a given value of λ, we explicitly recognize the λ-dependence of S_D and S_S by writing $S_D = S_D(\lambda)$, $S_S = S_S(\lambda)$. Then for fixed λ, it is the case that $\overline{S_S(\lambda)} \cap S_D(\lambda)$ and $S_S(\lambda) \cap \overline{S_D(\lambda)}$ are, in general, nonempty. This does not mean, however, that noise can also have a destabilizing effect, as $(\mu, a, \lambda) \notin S_S$ does not imply that there is destabilization. For the linear version of (2.3), (2.9) however (where $f(x) = g(x) = x$), we show in this section that $\overline{S_S(\lambda)} \cap S_D(\lambda)$ is empty for all $\lambda \neq 0$, so that noise can always be added to a stable deterministic problem in such a way that the stochastic problem remains stable, while $S_S(\lambda) \cap \overline{S_D(\lambda)}$ is nonempty for all $\lambda \neq 0$, so that there exist unstable deterministic problems which can be stabilizied. It is therefore justifiable to talk about the stabilizing effect of noise for linear problems.

To make the discussion concrete, the deterministic linear problem is

$$x'(t) = -ax(t) + \int_0^t \lambda e^{-\mu(t-s)} x(s)\,ds \tag{5.13}$$

with the corresponding family of Itô-Volterra equations

$$dX(t) = \left(-aX(t) + \int_0^t \lambda e^{-\mu(t-s)} X(s)\,ds\right)dt + \sigma X(t)\,dB(t). \tag{5.14}$$

Consider $\lambda > 0$, and note that we have

$$\nu_\lambda(\mu) = \begin{cases} \dfrac{\lambda}{\mu} & \text{if } 0 < \mu \leq \sqrt{\lambda}, \\ 2\sqrt{\lambda} - \mu & \text{if } \sqrt{\lambda} < \mu. \end{cases} \tag{5.15}$$

Therefore, $S_D(\lambda) \subset S_S(\lambda)$ for all $\lambda > 0$, as claimed. For $\lambda < 0$, we get $\nu_\lambda(\mu) = -\mu$ for all $\mu > 0$, and again $S_D(\lambda) \subset S_S(\lambda)$ for all $\lambda < 0$.

The method outlined in this paper for obtaining upper bounds on the a.s. decay rate of Itô-Volterra equations of the form (2.3) is sharp for the linear problem (5.14) when

$\lambda > 0$ (and $\sigma = 0$) in the following sense: the region in (μ, a, λ)-parameter space for which convergence occurs is correctly identified ($a > \lambda/\mu$), and the exponential decay rate of solutions identified as a solution of $\gamma^2 + (a+\mu)\gamma + (a\mu - \lambda) = 0$.

To see this, note for $\sigma = 0, \lambda > 0$ that $H(x) = x(-a + \sqrt{\lambda}(\alpha - \alpha^{-1}) + \mu) + (-\mu + \sqrt{\lambda}\alpha^{-1})$. Then $H(0)$, $H(1) < 0$ if and only if we can choose $\alpha \in (\sqrt{\lambda}/\mu, a/\sqrt{\lambda})$, which requires $a > \lambda/\mu$. Define the bound on the decay rate by $\gamma = \sup_{x\in[0,1]} H(x) = H(0) \vee H(1)$. The best estimate can be obtained by choosing α such that $H(0) = H(1)$, in which case we get the required $\gamma^2 + (a+\mu)\gamma + (a\mu - \lambda) = 0$.

6. General exponential kernel

We now study the general scalar Itô-Volterra equation (2.2), relaxing the hypothesis that the kernel k is exponential, instead assuming that

$$|k(t)| \le \lambda e^{-\mu t}, \quad t \ge 0, \tag{6.1}$$

for some $\lambda, \mu > 0$. In this case, the solution of the Itô-Volterra equation cannot be written in terms of a stochastic differential equation. However, condition (6.1) enables us to be able to obtain an upper bound on solutions of (2.2) in terms of a linear problem of the form (5.14), from which we can obtain the required bound on the negative Lyapunov exponent.

Lemma 6.1. *Suppose that f, g satisfy (2.4), (2.5), (2.10), (2.15), k satisfies (6.1), and X is the solution of (2.2). Let $(Y(t))_{t\ge0}$ be the process given by $Y(0) = |X(0)|$ and*

$$dY(t) = \left(-a\tilde{f}Y(t) + \int_0^t \lambda\bar{g}e^{-\mu(t-s)}Y(s)ds\right)dt + \sigma Y(t)dB(t), \tag{6.2}$$

where $\tilde{f}$ is given by (4.8) and $\bar{g}$ by (2.5). Then $|X(t)| \le Y(t)$ for all $t \ge 0$, a.s.

Proof. Define $(\phi(t))_{t\ge0}$ by $\phi(0) = 1$ and $d\phi(t) = \sigma\phi(t)dB(t)$, and let $X_1(t) = \phi(t)^{-1}X(t)$. Using integration by parts, we find that

$$X_1(t) = X(0) + \int_0^t \phi(s)^{-1}R(s)ds, \tag{6.3}$$

where $R(t) = -af(\phi(t)X_1(t)) + \int_0^t k(t-s)g(\phi(s)X_1(s))ds$. Since R and ϕ have continuous paths, we see that X_1 is C^1, and

$$X_1'(t) = -af(\phi(t)X_1(t))\phi(t)^{-1} + \int_0^t k(t-s)g(\phi(s)X_1(s))\phi(t)^{-1}ds. \tag{6.4}$$

We now analyze this smooth system (6.4) on a pathwise basis, so we assume that $\omega \in \Omega$ is fixed. Considering the case $a > 0$, and invoking (2.15), we have

$$D_-|X_1(t)| \le -a\underline{f}|X_1(t)| + \left|\int_0^t k(t-s)g(\phi(s)X_1(s))\phi(t)^{-1}ds\right|, \tag{6.5}$$

so with $a > 0$, and using (2.5), (4.8), (6.1) gives

$$D_-|X_1(t)| \le -a\tilde{f}|X_1(t)| + \int_0^t \lambda e^{-\mu(t-s)}\overline{g}|X_1(s)|\phi(s)\phi(t)^{-1}ds. \tag{6.6}$$

The same analysis for $a \le 0$ also gives (6.6), recalling that $\tilde{f}$ is defined by (4.8). Introduce the process $Y_1(t) = \phi(t)^{-1}Y(t)$, so $Y_1(0) = |X(0)|$, and (6.2) gives

$$Y_1'(t) = -a\tilde{f}Y(t) + \int_0^t \lambda e^{-\mu(t-s)}\overline{g}Y_1(s)\phi(s)\phi(t)^{-1}ds. \tag{6.7}$$

Applying the deterministic comparison principle on a pathwise basis now gives $|X_1(t)| \le Y_1(t)$ for all $t \ge 0$, and almost all paths $\omega \in \Omega$. The result follows by the definition of X_1, Y_1. □

Using Theorem 5.1, we can determine when the top Lyapunov exponent of Y defined by (6.2) is negative, and hence obtain a stabilization result for (2.2).

Theorem 6.2. *Let X be the solution of (2.2). Suppose that f, g satisfy (2.4), (2.5), (2.10), (2.15), and k satisfies (6.1) for some λ, $\mu > 0$. Define*

$$\nu_\lambda(\mu) = \begin{cases} \dfrac{\lambda\overline{g}}{\mu\underline{f}} & \text{if } 0 < \mu \le \sqrt{\lambda\overline{g}}, \\[2ex] \dfrac{2\sqrt{\lambda\overline{g}} - \mu}{\underline{f}} & \text{if } \sqrt{\lambda\overline{g}} \le \mu \le 2\sqrt{\lambda\overline{g}}, \\[2ex] \dfrac{2\sqrt{\lambda\overline{g}} - \mu}{\overline{f}} & \text{if } \mu > 2\sqrt{\lambda\overline{g}}. \end{cases} \tag{6.8}$$

If $a > \nu_\lambda(\mu)$, there exists a nonempty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda}$, there exists $\delta_0^* = \delta_0^*(a,\mu,\lambda,|\sigma|) > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t;\xi)| \le -\delta_0^*, \quad \text{a.s.} \tag{6.9}$$

Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there exists $\delta_p > 0$ such that

$$\mathbb{E}[|X(t;\xi)|^p] \le K\mathbb{E}[|\xi|^p|]e^{-\delta_p t}, \quad t \ge 0, \tag{6.10}$$

where K is a positive constant independent of p and ξ.

Removing the hypotheses (2.15), we obtain a similar result to Lemma 6.1 using a nearly identical argument.

Lemma 6.3. *Suppose that f, g satisfy (2.4), (2.5), (2.10), k satisfies (6.1), and X is the solution of (2.2). Let $(Y(t))_{t\ge 0}$ be the process given by $Y(0) = |X(0)|$ and*

$$dY(t) = \left(|a|\overline{f}Y(t) + \int_0^t \lambda\overline{g}e^{-\mu(t-s)}Y(s)ds\right)dt + \sigma Y(t)dB(t), \tag{6.11}$$

where $\overline{f}$, $\overline{g}$ are given by (2.5). Then $|X(t)| \le Y(t)$ for all $t \ge 0$, a.s.

Equation (6.11) is in the form of (5.14) with $-|a|\overline{f} < 0$ in the role of a and $\lambda\overline{g}$ in the role of λ, so the stabilization region in (μ, a)-parameter space for fixed λ is given by

$$-|a|\overline{f} > 2\sqrt{\lambda\overline{g}} - \mu, \quad \mu > 2\sqrt{\lambda\overline{g}}. \tag{6.12}$$

Therefore, we have the following theorem.

Theorem 6.4. *Let X be the solution of (2.2). Suppose that f, g satisfy (2.4), (2.5), (2.10), and k satisfies (6.1) for some λ, $\mu > 0$. If*

$$|a| < \frac{2\sqrt{\lambda\overline{g}} - \mu}{\overline{f}}, \quad \mu > 2\sqrt{\lambda\overline{g}}, \tag{6.13}$$

there exists a nonempty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda}$, there exists $\delta_0^* = \delta_0^*(a, \mu,\lambda,|\sigma|) > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t;\xi)| \le -\delta_0^*, \quad a.s. \tag{6.14}$$

Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there exists $\delta_p > 0$ such that

$$\mathbb{E}[|X(t;\xi)|^p] \le K\mathbb{E}[|\xi|^p|]e^{-\delta_p t}, \quad t \ge 0, \tag{6.15}$$

where K is a positive constant independent of p and ξ.

7. Examples

To consider the stabilization of solutions of the deterministic equation

$$x'(t) = -af(x(t)) + \int_0^t k(t-s)g(x(s))\,ds, \tag{7.1}$$

(where k satisfies (6.1)), we observe by (2.9) that the zero solution of (7.1) is unstable if

$$a < \int_0^\infty k(s)\,ds, \tag{7.2}$$

as this guarantees that the linearization of (7.1) is unstable. As an example, suppose that $k(t) > 0$ for all $t \ge 0$, so that for $a < 0$, (7.1) is unstable. If, in addition $a > (2\sqrt{\lambda\overline{g}} - \mu)/\mu$ and $\mu > 2\sqrt{\lambda\overline{g}}$, there exists an interval in $\mathbb{R}^+$ such that for all $|\sigma| \in I_{a,\mu,\lambda}$, (2.2) is a.s. globally exponentially asymptotically stable, so stabilization is again possible.

We now give an example under which Theorem 6.2 applies.

Example 7.1. Consider the deterministic problem

$$x'(t) = -ax(t)(2 - \cos(x(t))) + \int_0^t \frac{1}{1+(t-s)^2} e^{-\mu(t-s)} \left(x(s) + \frac{x(s)^3}{1+x(s)^2} \right) ds \tag{7.3}$$

with the corresponding family of stochastic Volterra equations

$$dX(t) = \left(-aX(t)(2-\cos(X(t))) + \int_0^t \frac{1}{1+(t-s)^2} e^{-\mu(t-s)} \left(X(s) + \frac{X(s)^3}{1+X(s)^2} \right) ds \right) dt \\ + \sigma X(t) dB(t). \tag{7.4}$$

Therefore, we identify $f(x) = x(2-\cos x)$, $g(x) = x + x^3/(1+x^2)$, and note that f, g satisfy (2.4), (2.5), (2.10), (2.15), and are both locally Lipschitz. We have $\overline{g} = 2$ and

$$\widetilde{f} = \begin{cases} 1 & \text{if } a > 0, \\ 3 & \text{if } a \le 0. \end{cases} \tag{7.5}$$

Note that $k(t) = (1+(t^2+1)^{-1})e^{-\mu t}$ satisfies condition (6.1) with $\lambda = 2$. Since

$$\int_0^\infty k(s)ds > \frac{1}{\mu}, \tag{7.6}$$

we see that (7.4) is unstable for $a < 1/\mu$. Define

$$\nu(\mu) = \begin{cases} \dfrac{4}{\mu} & \text{if } 0 < \mu \le 2, \\ 4-\mu & \text{if } 2 < \mu \le 4, \\ \dfrac{1}{3}(4-\mu) & \text{if } \mu > 4. \end{cases} \tag{7.7}$$

Hence for $\mu > 1+\sqrt{3}$, $\nu(\mu) < a < 1/\mu$, Theorem 6.2 shows that all solutions of (7.4) tend to zero exponentially fast, almost surely, provided that $|\sigma|$ is contained in an open interval contained in $\mathbb{R}^+$ whose endpoints depend on a, μ. At the same time, the zero solution of the deterministic problem (7.3) is unstable.

We finally show how the estimates on the ranges of admissible σ for which (2.3) is stable in terms of the free parameter α can be established, using a specific example to highlight the construction.

Example 7.2. Consider the nonlinear scalar deterministic Volterra equation

$$x'(t) = x(t) - \frac{1}{2}\frac{x(t)^3}{1+x(t)^2} + \int_0^t 2e^{-10(t-s)}(x(s) + \sin x(s))ds, \tag{7.8}$$

and the corresponding family of Itô-Volterra equations

$$dX(t) = \left(X(t) - \frac{1}{2}\frac{X(t)^3}{1+X(t)^2} + \int_0^t 2e^{-10(t-s)}(X(s) + \sin X(s))ds \right) dt + \sigma X(t) dB(t). \tag{7.9}$$

Note that (7.9) is of the form (2.3) and satisfies the restrictions (2.4), (2.5), (2.10), (2.15), where

$$f(x) = x - \frac{1}{2}\frac{x^3}{1+x^2}, \qquad g(x) = \frac{1}{2}(x + \sin x), \tag{7.10}$$

and $\lambda = 4$, $a = -1$, $\mu = 10$. By (2.14), the nonlinear problem (7.8) is unstable. Observe that Λ^* defined by (4.5) is given by $\Lambda^* = 4$, and $\widetilde{f}$ defined by (4.8) by

$$\widetilde{f} = \begin{cases} 1 & \text{if } a \le 0, \\ \frac{1}{2} & \text{if } a > 0. \end{cases} \tag{7.11}$$

Therefore, with ν defined by (4.16), we have $a > \nu_\lambda(\mu)$, so by Theorem 5.1, we can choose an interval of values of $|\sigma|$ such that all solutions of (7.9) converge to zero exponentially fast, almost surely. Recall the subdivision of the (μ, a)-plane given in (1)–(3) in Proposition 4.2. We find ranges of the parameter $\alpha > 0$ such that $a > f_2(\mu)$, or $f_3(\mu) < a < f_2(\mu)$ (where f_2, f_3 are given by (4.21)), and hence use (4.25), (4.26) to determine the maximal interval of $|\sigma|$ for which solutions of (7.9) are stable.

By rearranging the inequality $a > f_2(\mu)$, we obtain a quadratic in α. Hence for

$$1 - \frac{\sqrt{2}}{2} < \alpha < 1 + \frac{\sqrt{2}}{2}, \tag{7.12}$$

the point $(10, -1)$ (in the (μ, a) plane) satisfies $-1 > f_2(10)$. Rearranging the inequality $a > f_3(\mu)$ yields

$$\alpha > \frac{9 - \sqrt{65}}{4} \quad \text{or} \quad \alpha < \frac{9 + \sqrt{65}}{4}, \tag{7.13}$$

and solving for $a < f_2(\mu)$ gives $\alpha < 1 - \sqrt{2}/2$ or $\alpha > 1 + \sqrt{2}/2$. Thus, $f_3(\mu) < a < f_2(\mu)$ if and only if

$$\frac{9 - \sqrt{65}}{4} < \alpha < 1 - \frac{\sqrt{2}}{2} \quad \text{or} \quad 1 + \frac{\sqrt{2}}{2} < \alpha < \frac{9 + \sqrt{65}}{4}. \tag{7.14}$$

Appealing to (4.25), we see that the appropriate choice of $\sigma^2 > 0$ to stabilize (7.9) is given by

$$2(1 + 2\alpha) < \sigma^2 < \frac{2}{3}\left(11 - 2\left(\alpha - \alpha^{-1}\right)\right) \tag{7.15}$$

for any α satisfying (7.12). This can be achieved for $\sigma^2 \in (3.17157, 8.82843)$.

Similarly, by (4.26), if we can choose α to satisfy (7.14) so that $\sigma^2 > 0$ is given by

$$2\sqrt{10 - 2\alpha^{-1}} - \sqrt{18 - 4\left(\alpha + \alpha^{-1}\right)} < |\sigma| < 2\sqrt{10 - 2\alpha^{-1}} + \sqrt{18 - 4\left(\alpha + \alpha^{-1}\right)}, \tag{7.16}$$

then (7.9) can be stabilized. By (7.14), it follows that this can be achieved for $|\sigma| \in (1.7684, 8.9138)$.

Combining the analysis on the regions $a > f_2(\mu)$, $f_3(\mu) < a < f_2(\mu)$, we see that for any value of $\sigma^2 \in (3.127, 79.456)$ that all solutions of (7.9) satisfy

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t; \xi)| < 0, \quad \text{a.s.}, \tag{7.17}$$

while no solution of (7.8) converges to zero.

8. Finite-dimensional problem

In this section, we consider how the analysis used to tackle the scalar problem can be adapted to deal with the stabilization of the finite-dimensional Itô-Volterra equation.

Suppose $f \in C(\mathbb{R}^d;\mathbb{R}^d)$ is Lipschitz continuous with a global linear bound, and with $f(0) = 0$. Suppose further that there exists $a > 0$ such that

$$\langle f(x),x\rangle \leq -a\|x\|^2, \quad x \in \mathbb{R}^d. \tag{8.1}$$

Let $g \in C(\mathbb{R}^d;\mathbb{R}^r)$ be locally Lipschitz continuous and obeying a global linear bound

$$\|g(x)\| \leq \overline{g}\|x\|, \quad x \in \mathbb{R}^d. \tag{8.2}$$

Finally, suppose that there exist $\lambda > 0$, $\mu > 0$ such that $K \in C(\mathbb{R}^+;\mathbb{R}^d \times \mathbb{R}^r)$ satisfies

$$\|K(t)\| \leq \lambda e^{-\mu t}, \quad t \geq 0. \tag{8.3}$$

In the above, $\langle \cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^d$, $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^d$ (or $\mathbb{R}^r$, as appropriate), and the norm in (8.3) is the standard operator norm on $\mathbb{R}^d \times \mathbb{R}^r$. Also, denote I_d as the $d\times d$ identity matrix.

We study stability of the following Itô-Volterra equation:

$$dX(t) = \Big(f(X(t)) + \int_0^t K(t-s)g(X(s))\,ds\Big)dt + \Sigma X(t)\,dB(t), \tag{8.4}$$

where $\Sigma = \sigma I_d$ and $\sigma \in \mathbb{R}$, and $X(0) = \xi \in \mathbb{R}^d$. Here, as before, B is standard one-dimensional Brownian motion, ξ is independent of B, with $\mathbb{E}[\|\xi\|^2] < \infty$. Under these conditions, there is a unique strong solution of (8.4). Moreover, if $X(0) = 0$, then $X(t) = 0$ for all $t \geq 0$ a.s.

Remark 8.1. We note for further reference that the statement of the stability theorem we obtain for (8.4) is identical for that we achieve for the equation

$$dX(t) = \Big(f(t,X(t)) + \int_0^t K(s,t)g(s,X(s))\,ds\Big)dt + \Sigma X(t)\,dB(t), \tag{8.5}$$

under the following restrictions on f, g, K: $\|K(s,t)\| \leq \lambda e^{-\mu(t-s)}$, $\|g(t,x)\| \leq \overline{g}\|x\|$, and $\langle x, f(t,x)\rangle \leq -\alpha\|x\|^2$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and some λ, μ, $\overline{g}$, $\alpha > 0$.

To prove the stability result for (8.4), first define the process $Y(t) = \|X(t)\|^2$ and let $\widetilde{\varphi}$ be given by $\widetilde{\varphi}(0) = 1$ and

$$d\widetilde{\varphi}(t) = 2\sigma\widetilde{\varphi}(t)\,dB(t). \tag{8.6}$$

Also set $\widetilde{Y}(t) = \widetilde{\varphi}(t)^{-1}Y(t)$, $t \geq 0$. As Y has a semimartingale decomposition given by

$$dY(t) = \Big(2\langle X(t), f(X(t))\rangle + 2\Big\langle X(t), \int_0^t K(t-s)g(X(s))\,ds\Big\rangle + \sigma^2 Y(t)\Big)dt + 2\sigma Y(t)\,dB(t), \tag{8.7}$$

by using (8.6) in conjunction with (8.7), we see that $(\widetilde{Y}(t))_{t\geq 0}$ is a continuously differentiable process obeying

$$\widetilde{Y}'(t) = \sigma^2\widetilde{Y}(t) - 2\langle X(t), f(X(t))\rangle\widetilde{\varphi}(t)^{-1} + \widetilde{\varphi}(t)^{-1}2\left\langle X(t), \int_0^t K(t-s)g(X(s))\,ds\right\rangle. \quad (8.8)$$

Employing (8.1) gives

$$2\langle X(t), f(X(t))\rangle\widetilde{\varphi}(t)^{-1} \leq -2a\widetilde{Y}(t). \quad (8.9)$$

Using the Cauchy-Schwarz inequality (8.2), (8.3), and the inequality

$$2xy \leq \beta x^2 + \frac{1}{\beta}y^2 \quad (\beta > 0,\ x,\ y \geq 0), \quad (8.10)$$

we get

$$\begin{aligned}\left|2\langle X(t), \int_0^t K(t-s)g(X(s))\,ds\rangle\right| &\leq 2\int_0^t ||X(t)||\,||K(t-s)||\overline{g}||X(s)||\,ds\\ &\leq \lambda\overline{g}\int_0^t e^{-\mu(t-s)}\left(\beta||X(t)||^2 + \frac{1}{\beta}||X(s)||^2\right)ds \qquad (8.11)\\ &\leq \frac{\lambda\overline{g}}{\mu}\beta Y(t) + \frac{\lambda\overline{g}}{\beta}\int_0^t e^{-\mu(t-s)}Y(s)\,ds.\end{aligned}$$

Inserting (8.9) and (8.11) into (8.8) yields

$$\widetilde{Y}'(t) \leq \left(\sigma^2 - 2a + \frac{\lambda\overline{g}}{\mu}\beta\right)\widetilde{Y}(t) + \int_0^t \frac{\lambda\overline{g}}{\beta}e^{-\mu(t-s)}\widetilde{\varphi}(t)^{-1}\widetilde{\varphi}(s)\widetilde{Y}(s)\,ds. \quad (8.12)$$

Next, define $(\widetilde{Z}(t))_{t\geq 0}$ according to $\widetilde{Z}(0) = \widetilde{Y}(0)$ and

$$\widetilde{Z}'(t) = \left(\sigma^2 - 2a + \frac{\lambda\overline{g}}{\mu}\beta\right)\widetilde{Z}(t) + \int_0^t \frac{\lambda\overline{g}}{\beta}e^{-\mu(t-s)}\widetilde{\varphi}(t)^{-1}\widetilde{\varphi}(s)\widetilde{Y}(s)\,ds. \quad (8.13)$$

By applying the comparison principle pathwise, we see that $\widetilde{Y}(t) \leq \widetilde{Z}(t)$ for $t \geq 0$, a.s. Now let $Z(t) = \widetilde{\varphi}(t)\widetilde{Z}(t)$; then Z obeys

$$dZ(t) = \left[\left(\sigma^2 - 2a + \frac{\lambda\overline{g}}{\mu}\beta\right)Z(t) + \int_0^t \frac{\lambda\overline{g}}{\beta}e^{-\mu(t-s)}Z(s)\,ds\right]dt + 2\sigma Z(t)\,dB(t), \quad (8.14)$$

and so we obtain

$$||X(t)||^2 = Y(t) = \widetilde{\varphi}(t)\widetilde{Y}(t) \leq \widetilde{\varphi}(t)\widetilde{Z}(t) = Z(t). \quad (8.15)$$

Therefore, if we can show that there exists $\delta > 0$ such that

$$\limsup_{t\to\infty}\frac{1}{t}\log Z(t) \leq -\delta_0, \quad \text{a.s.} \quad (8.16)$$

it is automatically true that

$$\limsup_{t\to\infty} \frac{1}{t}\log||X(t)|| \le -\frac{\delta_0}{2}, \quad \text{a.s.} \tag{8.17}$$

Note further that $Z(0) = \|\xi\|^2$ and for any $p > 0$, note that $\mathbb{E}[\|X(t)\|^p] \le \mathbb{E}[Z(t)^{p/2}]$, so we may equally obtain stability results in pth mean (for sufficiently small p) as before.

The above argument supplies the proof of the following Theorem.

Theorem 8.2. *Suppose f, g are locally Lipschitz continuous and globally linearly bounded with $f(0) = 0$, $g(0) = 0$, and satisfy (8.1), (8.2). If K is continuous and satisfies (8.3), then the solution of (8.4) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\log||X(t;\xi)|| \le -\frac{\delta_0}{2}, \quad \textit{a.s.}, \tag{8.18}$$

for some $\delta_0 > 0$, provided that the solution of (8.14) satisfies (8.16). Moreover, under these conditions, there exists $p^* > 0$ such that for all $p < p^*$, there exists $\delta_p > 0$ such that

$$\mathbb{E}[||X(t;\xi)||^p] \le C\mathbb{E}[\|\xi\|^p]e^{-\delta_p t}, \quad t \ge 0, \tag{8.19}$$

where C is a positive constant independent of p and ξ.

Showing that the solution of the Itô-Volterra equation (8.14) satisfies (8.16) for some $\delta_0 > 0$ is the subject of an earlier section of the paper, and, as before, our interest lies in the stabilizing effect of a noise perturbation. In (8.14), however, unlike the earlier problem studied, the intensity of the noise perturbation arises in both drift and diffusion terms of the auxiliary Itô-Volterra equation (8.14), so although the method of proof is identical in spirit to previous results, the calculation must be done afresh.

Lemma 8.3. *Suppose $\overline{g}$, μ, $\lambda > 0$, and define*

$$\nu(\lambda,\overline{g},\mu) = \begin{cases} \dfrac{\overline{g}\lambda}{\mu} & \textit{for } \mu \in \left(0, \sqrt{3\overline{g}\lambda}\right), \\ \dfrac{3}{2}\left(\dfrac{(\overline{g}\lambda)^2}{3\mu}\right)^{1/3} - \dfrac{1}{6}\mu & \textit{for } \mu \ge \sqrt{3\overline{g}\lambda}. \end{cases} \tag{8.20}$$

If $a > \nu(\lambda,\overline{g},\mu)$, there exists a nonempty interval $I_{a,\mu,\overline{g},\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\overline{g},\lambda}$, there exists $\delta_0^* = \delta_0^*(|\sigma|,a,\mu,\overline{g},\lambda) > 0$ such that $\delta_0^* < \delta_0$, where δ_0 is given by (8.16).

Proof. To obtain sufficient conditions under which $\delta_0 > 0$, note that

$$-\delta_0 \le \max_{x\in[0,1]} H(x), \tag{8.21}$$

where

$$H(x) = -4\sigma^2x^2 + x\left(3\sigma^2 - 2a + \frac{\overline{g}\beta\lambda}{\mu} + \frac{\overline{g}\lambda}{\beta}(\alpha - \alpha^{-1}) + \mu\right) + \frac{1}{\alpha}\sqrt{\frac{\overline{g}\lambda}{\beta}} - \mu, \tag{8.22}$$

and α, β are positive constants that are chosen so as to obtain the best sufficient condition on the parameters under which stability is guaranteed. As before, we note that $\delta_0^* > 0$ if one of the following holds:

(i) $H(0) < 0, H'(0) < 0$;

(ii) $H(1) < 0, H'(1) > 0$;

(iii) there exists $x^* \in [0,1]$ such that $H'(x^*) = 0$ and $H(x^*) < 0$.

Now, define $\mu^* = \alpha^{-1}\sqrt{\overline{g}\lambda/\beta}$, and the functions

$$f_1(\mu,\alpha,\beta) = \frac{1}{2}\left(\overline{g}\beta\frac{\lambda}{\mu} + \sqrt{\frac{\overline{g}\lambda}{\beta}}(\alpha - \alpha^{-1} + \mu)\right),$$

$$f_2(\mu,\alpha,\beta) = \frac{1}{2}\left(\overline{g}\beta\frac{\lambda}{\mu} + \sqrt{\frac{\overline{g}\lambda}{\beta}}\left(\alpha + \frac{1}{4}\alpha^{-1} - \frac{\mu}{4}\right)\right), \tag{8.23}$$

$$f_3(\mu,\alpha,\beta) = \frac{1}{2}\left(\overline{g}\beta\frac{\lambda}{\mu} + \sqrt{\frac{\overline{g}\lambda}{\beta}}\left(\alpha + \frac{1}{3}\alpha^{-1} - \frac{\mu}{3}\right)\right),$$

for $\mu > \mu^*$. Note that $f_1(\mu,\alpha,\beta) > f_2(\mu,\alpha,\beta) > f_3(\mu,\alpha,\beta)$ for fixed $\alpha, \beta > 0$ and $\mu > \mu^*$, while

$$f_j(\mu^*) = \frac{1}{2}\alpha\sqrt{\overline{g}\lambda}(\beta^{-1/2} + \beta^{3/2}), \quad j = 1,2,3. \tag{8.24}$$

Now, consider the parameter regions

(1) $a > f_1(\mu,\alpha,\beta)$, $\mu > \mu^*$;

(1′) $a = f_1(\mu,\alpha,\beta)$, $\mu > \mu^*$;

(2) $f_2(\mu,\alpha,\beta) < a < f_1(\mu,\alpha,\beta)$, $\mu > \mu^*$;

(3) $f_3(\mu,\alpha,\beta) < a \leq f_2(\mu,\alpha,\beta)$, $\mu > \mu^*$.

It transpires that for the appropriate ranges of σ that (1) implies (i), (2) implies (ii), (3) implies (iii), while (1′) gives rise to a special case of (iii). We omit the calculations which justify this statement. In order that the results may be used in practice, we give here, without justification, the appropriate ranges of σ:

$$0 < \sigma^2 < \frac{1}{3}\left(2a - \overline{g}\beta\frac{\lambda}{\mu} - \sqrt{\frac{\overline{g}\lambda}{\beta}}(\alpha - \alpha^{-1}) - \mu\right) \quad \text{for (1)}, \tag{8.25}$$

$$0 < \sigma^2 < \frac{16}{9}\left(\mu - \frac{1}{\alpha}\sqrt{\frac{\overline{g}\lambda}{\beta}}\right) \quad \text{for (1')}, \tag{8.26}$$

$$-2a + \overline{g}\beta\frac{\lambda}{\mu} < \sigma^2 < \frac{1}{5}\left(-2a + \overline{g}\beta\frac{\lambda}{\mu} + \mu + \sqrt{\frac{\overline{g}\lambda}{\beta}}(\alpha - \alpha^{-1})\right) \quad \text{for (2)}, \tag{8.27}$$

$$\sigma_-^2 < \sigma^2 < \sigma_+^2 \quad \text{for (3)}, \tag{8.28}$$

where $0 < \sigma_- < \sigma_+$ are given by

$$\sigma_\pm = \frac{2}{3}\sqrt{\mu - \mu^*} \pm \frac{1}{3}\sqrt{6(a - f_3(\mu,\alpha,\beta))}. \tag{8.29}$$

Now fix α, $\beta > 0$ and $\overline{g}$, $\lambda > 0$, and define

$$S_{\alpha,\beta} = \{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R} : \mu > \mu^*(\alpha,\beta),\ a > f_3(\mu,\alpha,\beta)\}. \tag{8.30}$$

Then the following is true: if $(\mu, a) \in S_{\alpha,\beta}$ for some α, $\beta > 0$, there exists a nonempty interval $I_{a,\mu,\overline{g},\lambda,\alpha,\beta} \subset \mathbb{R}^+$ such that for all $|\sigma| \in I_{a,\mu,\overline{g},\lambda,\alpha,\beta}$, we have $H(x) < 0$ for all $x \in [0,1]$. (The intervals $I_{a,\mu,\overline{g},\lambda,\alpha,\beta}$ in cases (1), (1′), (2), (3) are determined in (8.25), (8.26), (8.27), (8.28), resp.) Now, suppose $0 < \mu \leq \sqrt{3\overline{g}\lambda}$ and $a > \overline{g}\lambda/\mu$. Let $\beta = 1$ and note that there exists $\alpha > 0$ such that

$$\frac{1}{\mu}\sqrt{\overline{g}\lambda} < \alpha < \frac{a}{\sqrt{\overline{g}\lambda}}. \tag{8.31}$$

Hence $(\mu, a) \in S_{\alpha,\beta}$. Next, suppose that $\mu > \sqrt{3\overline{g}\lambda}$ and

$$a > \frac{3}{2}\left(\frac{(\overline{g}\lambda)^2}{3\mu}\right)^{1/3} - \frac{1}{6}\mu. \tag{8.32}$$

Now let $\alpha = 1/\sqrt{3}$, and $\beta^{3/2} = \mu/(3\overline{g}\lambda)^{1/2}$. Then $a > f_3(\mu,\alpha,\beta)$, and so $(\mu, a) \in S_{\alpha,\beta}$. Fixing the $\alpha-$, $\beta-$dependences in terms of a, μ, $\overline{g}$, λ above yields the required intervals $I_{a,\mu,\overline{g},\lambda} = I_{a,\mu,\overline{g},\lambda,\alpha(a,\mu,\overline{g},\lambda),\beta(a,\mu,\overline{g},\lambda)}$, with $-\delta_0^* = \max_{x\in[0,1]} H(x) < 0$. □

This gives the desired stabilization result.

Theorem 8.4. *Suppose that f, g are globally linearly bounded and locally Lipschitz continuous, satisfy $f(0) = 0$, $g(0) = 0$, and obey (8.1), (8.2). Let $K \in C(\mathbb{R}^+; M_{d,r}(\mathbb{R}))$ satisfy (8.3), and $\Sigma = \sigma I_d$. Suppose that $\nu(\mu,\lambda,\overline{g})$ is defined by (8.20) and $a > \nu(\mu,\lambda,\overline{g})$. Then, the following hold.*

(i) *There exists a nonempty interval $I_{a,\mu,\lambda,\overline{g}} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda,\overline{g}}$, there exists $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \lambda, \overline{g}) > 0$ so that all solutions of (8.4) satisfy*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t;\xi)\| \leq -\delta_0^*, \quad a.s. \tag{8.33}$$

(ii) *Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there exists $\delta_p > 0$ such that*

$$\mathbb{E}[\|X(t;\xi)\|^p] \leq C\mathbb{E}[\|\xi\|^p]e^{-\delta_p t}, \quad t \geq 0, \tag{8.34}$$

where C is a positive constant independent of p and ξ.

8.1. Stabilization without a negative definite assumption on f. It is still possible to prove a stabilization result for the finite-dimensional Volterra equation

$$x'(t) = f(x(t)) + \int_0^t K(t-s)g(x(s))\,ds \tag{8.35}$$

even when f does not satisfy the negative definite assumption (8.1) posited earlier in this section. In the sequel, we relax the hypothesis (8.1) on f in favour of requiring that f

obeys a global linear bound of the form

$$||f(x)|| \le \overline{f}\|x\|, \quad x \in \mathbb{R}^d. \tag{8.36}$$

Again, we study the asymptotic behaviour of (8.4) under the assumptions (8.36), (8.2), (8.3), assuming as before, that f, g are locally Lipschitz continuous, and K continuous. To this end, we introduce the $M_{d,d}(\mathbb{R}^+)$-valued process $(\Phi(t))_{t\ge 0}$ determined by

$$d\Phi(t) = \sigma\Phi(t)dB(t), \quad t \ge 0, \tag{8.37}$$

and $\Phi(0) = I_d$. Hence $\Phi(t) = \varphi(t)I_d$, where $\varphi(0) = 1$ and

$$d\varphi(t) = \sigma\varphi(t)dB(t) \tag{8.38}$$

is a scalar geometric Brownian motion. Next, define $Y(t) = \Phi(t)^{-1}X(t)$ for $t \ge 0$; this process is continuously differentiable on $\mathbb{R}^+$ and satisfies

$$Y'(t) = \Phi(t)^{-1}\left(f(\Phi(t)Y(t)) + \int_0^t K(t-s)g(\Phi(s)Y(s))\,ds\right). \tag{8.39}$$

Employing (8.36), (8.2), (8.3), and the fact that $\|\Phi(t)\| = \varphi(t)$, $\|\Phi(t)^{-1}\| = \varphi(t)^{-1}$, we get

$$||Y'(t)|| \le \overline{f}||Y(t)|| + \int_0^t \lambda e^{-\mu(t-s)}\overline{g}\varphi(t)^{-1}\varphi(s)||Y(s)||ds, \tag{8.40}$$

so for $t \ge 0$,

$$D_+||Y(t)|| \le \overline{f}||Y(t)|| + \int_0^t \lambda\overline{g}e^{-\mu(t-s)}\varphi(t)^{-1}\varphi(s)||Y(s)||ds. \tag{8.41}$$

Therefore, if $\widetilde{Z}(0) = \|\xi\|$ and $(\widetilde{Z}(t))_{t\ge 0}$ satisfies

$$\widetilde{Z}'(t) = \overline{f}\widetilde{Z}(t) + \int_0^t \lambda\overline{g}e^{-\mu(t-s)}\varphi(t)^{-1}\varphi(s)\widetilde{Z}(s)ds, \tag{8.42}$$

the comparison principle implies that $\|Y(t)\| \le \widetilde{Z}(t)$ for all $t \ge 0$, a.s. Next, observe that $(Z(t))_{t\ge 0}$ defined by $Z(t) = \varphi(t)\widetilde{Z}(t)$ satisfies

$$dZ(t) = \left(\overline{f}Z(t) + \int_0^t \lambda\overline{g}e^{-\mu(t-s)}Z(s)ds\right)dt + \sigma Z(t)dB(t). \tag{8.43}$$

Since

$$||X(t)|| = ||\Phi(t)Y(t)|| \le \varphi(t)||Y(t)|| \le \varphi(t)\widetilde{Z}(t) = Z(t), \tag{8.44}$$

it follows that there exists $\delta_0 > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\log||X(t;\xi)|| \le -\delta_0 \quad \text{a.s.}, \tag{8.45}$$

providing that Z satisfying (8.43) obeys

$$\limsup_{t\to\infty}\frac{1}{t}\log Z(t) \le -\delta_0, \quad \text{a.s.} \tag{8.46}$$

Applying Theorem 5.2 to (8.43), it is seen that (8.46) holds for some range of σ once

$$\overline{f} \le \mu - 2\sqrt{\lambda\overline{g}}, \quad \mu \ge 2\sqrt{\lambda\overline{g}}. \tag{8.47}$$

Therefore, we have obtained the following result.

Theorem 8.5. *Suppose that f, g satisfy (8.36), (8.2) and are locally Lipschitz continuous. Let $K \in C(\mathbb{R}^+; M_{d,r}(\mathbb{R}))$ satisfy (8.3), and suppose that $\Sigma = \sigma I_d$. If $0 \le \overline{f} \le \mu - 2\sqrt{\lambda\overline{g}}$, then the following hold.*

(i) *There exists a nonempty interval $I_{f,g,K} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{f,g,K}$, there exists $\delta_0^* = \delta_0^*(|\sigma|, f, g, K) > 0$ so that all solutions of (8.4) satisfy*

$$\limsup_{t\to\infty}\frac{1}{t}\log||X(t;\xi)|| \le -\delta_0^*, \quad a.s. \tag{8.48}$$

(ii) *Moreover, under these conditions, there exists $p^* > 0$ such that for all $p < p^*$, there exists $\delta_p > 0$ such that*

$$\mathbb{E}[||X(t;\xi)||^p] \le C\mathbb{E}[\|\xi\|^p]e^{-\delta_p t}, \quad t \ge 0, \tag{8.49}$$

where C is a positive constant independent of p and ξ.

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