A NOTE ON STRONG SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH A DISCONTINUOUS DRIFT COEFFICIENT

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The existence of a mean-square continuous strong solution is established for vector-valued Itô stochastic differential equations with a discontinuous drift coefficient, which is an increasing function, and with a Lipschitz continuous diffusion coefficient. A scalar stochastic differential equation with the Heaviside function as its drift coefficient is considered as an example. Upper and lower solutions are used in the proof.

1. Introduction

Existence theorems [9–12] for Itô stochastic differential equations

$$dX_t = f(t,X_t)\,dt + g(t,X_t)\,dW_t, \quad t \in [0,T], \tag{1.1}$$

usually require that the drift and diffusion coefficients, f and g, be at least continuous (in x) as well as satisfying a growth condition to prevent explosions. An example of Tanaka (e.g., [12, page 71]) with zero drift and a discontinuous diffusion coefficient is known to have no strong solution with zero initial value that is a solution corresponding to a specified Wiener process in contrast to a weak solution where some other Wiener process could be used. Moreover, Barlow [2] shows that a strong solution need not exist when the diffusion process is only continuous. Krylov [7] and Krylov and Liptser [8] (see also the references cited therein) have investigated existence issues for SDE with discontinuous coefficients.

In contrast, here we consider the existence of mean-square continuous strong solutions with a Lipschitz continuous diffusion coefficient but a discontinuous drift coefficient, such as in the scalar SDE

$$dX_t = H(X_t)dt + dW_t, \tag{1.2}$$

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where $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function, which is defined by

$$H(x) := \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases} \tag{1.3}$$

Such equations arise, for example, when one considers the effects of background noise on switching systems or other discontinuous ordinary differential equations [5].

Several possible methods could be used here. Following Krylov, one could approximate the drift coefficient by a sequence of smooth functions (e.g., sigmoidal-shaped functions in the case of the Heaviside function). Alternatively, one could reformulate the equation as a stochastic differential inclusion. However, here we will use a method based on upper and lower solutions of stochastic differential equations.

2. Upper and lower solutions

We consider an Itô stochastic differential equation

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dW_t, \quad t \in [0, T], \tag{2.1}$$

with coefficients $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times k}$, where W_t is a given k-dimensional Wiener process.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\{\mathcal{F}_t\}_{t \geq 0}$ be the smallest filtration generated by the Wiener process W_t.

By a strong solution of the SDE (2.1) on an interval $[0, T]$ we mean a stochastic process X_t which is $\mathcal{F}_t$-measurable for each $t \in [0, T]$ with $\mathbb{E}\|X_t\|^2 < \infty$ for all $t \in [0, T]$ such that

$$X_t = X_0 + \int_0^t f(s, X_s)\,ds + \int_0^t g(s, X_s)\,dW_s, \quad t \in [0, T], \text{ w.p.1.} \tag{2.2}$$

(It is assumed implicitly that the integrals on the right-hand side exist w.p.1.) Such a strong solution is sample-path continuous when the coefficients f and g are sufficiently regular, for example, satisfy a global Lipschitz condition.

Upper and lower solutions of an SDE (2.1) have been considered previously under other names in [1, 9]. Conditions ensuring their existence were given in [1] and they were used in the context of comparison theorems in [9] (see also [3]).

Definition 2.1. A $\mathcal{F}_t$-measurable stochastic process Z_t is an upper solution of the SDE (2.1) on the interval $[0, T]$ if the inequality (interpreted component wise)

$$Z_t \geq Z_0 + \int_0^t f(s, Z_s)\,ds + \int_0^t g(s, Z_s)\,dW_s, \quad t \in [0, T], \tag{2.3}$$

holds with probability 1. If Z_t satisfies the reversed inequality (i.e., with $\leq$), then Z_t is a lower solution.

Upper and lower solutions provide useful bounds on strong solutions of an initial value problem for an SDE (2.1) and are often easier to determine explicitly. In the following theorem we show that a strong solution lying between lower and upper solutions exists in a special case which suffices later to prove the existence of a strong solution of an SDE with a discontinuous drift coefficient. (Analogous definitions of upper, lower, and strong solutions hold if the drift or diffusion coefficient is nonanticipatively random.)

Our first theorem can be considered as a comparison result that we will need in our main result, that is, Theorem 3.1.

THEOREM 2.2. *Suppose that the mapping* $f : [0,T] \times \Omega \to \mathbb{R}^d$ *is measurable and nonanticipative with* $\int_0^T \mathbb{E}\|f(t,\cdot)\|^2 dt < \infty$, *that* $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ *is Lipschitz continuous and satisfies the linear growth bound*

$$||g(t,x)|| \leq K + L\|x\|, \quad t \in [0,T],\ x \in \mathbb{R}^d, \tag{2.4}$$

and that Z_t *and* Y_t *are upper and lower solutions of the SDE*

$$dX_t = f(t,\omega)dt + g(t,X_t)dW_t, \quad t \in [0,T], \tag{2.5}$$

on $[0,T]$ *with* $\mathbb{E}\|Y_t\|^2 < \infty$, $\mathbb{E}\|Z_t\|^2 < \infty$, *and* $Y_t \leq Z_t$ *for* $t \in [0,T]$, *w.p.1. In addition, suppose that* X_0 *is* $\mathcal{F}_0$*-measurable with* $\mathbb{E}\|X_0\|^2 < \infty$ *and* $Y_0 \leq X_0 \leq Z_0$.

Then there exists a unique pathwise continuous strong solution X_t *which satisfies* $Y_t \leq X_t \leq Z_t$ *for* $t \in [0,T]$, *w.p.1.*

Proof. We define the functions $p, r : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ by

$$\begin{aligned} p(t,x,\omega) &:= \max\{Y_t(\omega), \min\{Z_t(\omega), x\}\}, \\ r(t,x,\omega) &:= \frac{p(t,x,\omega) - x}{1 + \|x\|^2}, \end{aligned} \tag{2.6}$$

and consider the stochastic differential equation

$$dX_t = \left(f(t,\omega) + r(t,X_t,\omega)\right)dt + g\left(p(t,X_t,\omega)\right)dW_t, \quad t \in [0,T], \tag{2.7}$$

for the given initial condition X_0. This SDE has nonanticipative random coefficients

$$\begin{aligned} \widetilde{f}(t,x,\omega) &:= f(t,\omega) + r(t,x,\omega), \\ \widetilde{g}(t,x,\omega) &:= g\left(p(t,x,\omega)\right), \end{aligned} \tag{2.8}$$

which are Lipschitz continuous in x and satisfy a growth bound of the form

$$||\widetilde{f}(t,\omega)|| + ||\widetilde{g}(t,x,\omega)|| \leq K\left(1 + \max\{||f(t,\omega)||, ||Y_t(\omega)||, ||Z_t(\omega)||\}\right) =: \widetilde{K}_t(\omega), \tag{2.9}$$

where the $\widetilde{K}_t$ is nonanticipative with $\int_0^T \mathbb{E}\widetilde{K}_s^2 ds < \infty$. Thus, from [6, Chapter 5, Theorem 2.9] the SDE (2.5) has a unique strong solution X_t, which is pathwise continuous. (The solution is also mean-square continuous, which is shown within the proof and is what we need in the sequel.)

We will now show that $Y_t \le X_t \le Z_t$ for $t \in [0,T]$, w.p.1. Suppose that there exists an interval $(t_1,t_2) \subset [0,T]$ such that $X_{t_1} = Y_{t_1}$ and $X_t \le Y_t$ for $t \in (t_1,t_2)$. Then

$$X_t - Y_t \ge \int_{t_1}^{t} r(s,X_s,\omega)\,ds + \int_{t_1}^{t} \big(g(p(s,X_s,\omega)) - g(Y_s)\big)\,dW_s = \int_{t_1}^{t} r(s,X_s,\omega)\,ds \ge 0 \tag{2.10}$$

for all $t \in (t_1,t_2)$, since $p(t,X_t,\omega) \equiv Y_t$ in (t_1,t_2), so $r(t,X_t,\omega) = (Y_t - X_t)/(1+\|X_t\|^2) \ge 0$, which gives a contradiction. Thus, $X_t \ge Y_t$ for all $t \in [0,T]$. Using the same argument we can also prove that $X_t \le Z_t$. That is, X_t is in fact a strong solution of the original SDE (2.5). □

3. SDEs with discontinuous drift coefficient

We now restrict attention to the autonomous SDE

$$dX_t = f(X_t)\,dt + g(X_t)\,dW_t, \quad t \in [0,T], \tag{3.1}$$

and assume that the drift coefficient is increasing, that is, $f(x) \le f(y)$ whenever $x \le y$ (where the inequalities are interpreted componentwise), but need not be continuous. In addition, we assume that the diffusion coefficient is Lipschitz continuous. This applies in particular to the scalar SDE (1.2) with the Heaviside drift coefficient. We show that the SDE (3.1) has a strong solution whenever it has an upper and a lower solution.

Theorem 3.1. *Suppose that $f, g : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ both satisfy the linear growth bound (2.4) and, in addition, that f is increasing and g is Lipschitz continuous. Moreover, suppose that the SDE (3.1) has mean-square continuous upper and lower solutions Z_t and Y_t on $[0,T]$ with $\int_0^T \mathbb{E}\|f(Y_t)\|^2 dt < \infty$, $\int_0^T \mathbb{E}\|f(Z_t)\|^2 dt < \infty$, and $Y_t \le Z_t$ for $t \in [0,T]$, w.p.1.*

Then the SDE (3.1) has at least one mean-square continuous strong solution X_t. Moreover, $Y_t \le X_t \le Z_t$ for $t \in [0,T]$, w.p.1.

Proof. We define by $\mathcal{X}$ the space of all d-dimensional nonanticipative mean-square continuous stochastic process $X = \{X_t, t \in [0,T]\}$ satisfying $\sup_{0\le s\le T} \mathbb{E}\|X_s\|^2 < \infty$ with the norm $\|X\| := (\sup_{0\le s\le T} \mathbb{E}\|X_s\|^2)^{1/2}$, which is a Banach space.

We denote by $\mathcal{K}$ the order interval $[Y,Z]$ in $\mathcal{X}$, that is, consisting of all X in $\mathcal{X}$ with $Y_t \le X_t \le Z_t$ for $t \in [0,T]$, w.p.1, which is closed and bounded in the above norm. Using the Lebesgue dominated convergence theorem, one can prove that a monotone sequence that belongs to $\mathcal{K}$ converges in $\mathcal{X}$. Thus, $\mathcal{K}$, with the above norm, is a regularly ordered metric space (for the definition, see [4, page 117]).

For any process $U \in \mathcal{K}$, it is clear that Y and Z are also mean-square continuous lower and upper solutions for the SDE

$$dX_t = f(U_t(\omega))\,dt + g(X_t)\,dW_t, \quad t \in [0,T]. \tag{3.2}$$

Thus, by Theorem 2.2, for any $\mathcal{F}_0$-measurable X_0 with $\mathbb{E}\|X_0\|^2 < \infty$ and $Y_0 \le X_0 \le Z_0$, the SDE (3.2) has a mean-square continuous (in fact, pathwise continuous) unique strong solution X_t, which satisfies $Y_t \le X_t \le Z_t$ for all $t \in [0,T]$, w.p.1.

We define an operator $S : \mathcal{H} \to \mathcal{H}$ where $X = S(U)$ is the unique mean-square continuous strong solution of the SDE (3.2) corresponding to the stochastic process $U \in \mathcal{H}$. We will apply [4, Corollary 3.2] to show that S has a fixed point, which is then the desired solution. For this we only have to prove that S is an increasing map.

We will prove that if $U^{(1)}$ and $U^{(2)}$ are stochastic processes in $\mathcal{H}$ with $U_t^{(1)} \le U_t^{(2)}$ for all $t \in [0,T]$ and if $X^{(1)} = S(U^{(1)})$, $X^{(2)} = S(U^{(2)})$, then $X_t^{(1)} \le X_t^{(2)}$ for all $t \in [0,T]$.

Let us choose stochastic processes $U^{(1)}$, $U^{(2)}$ in $\mathcal{H}$ with $U_t^{(1)} \le U_t^{(2)}$ for all $t \in [0,T]$ and define $X^{(1)} = S(U^{(1)})$. Since the drift coefficient f is an increasing function, $X_t^{(1)}$ is a lower solution of the SDE

$$X_t = X_0 + \int_0^t f(U_s^{(2)})\,ds + \int_0^t g(X_s)\,dW_s. \tag{3.3}$$

But this problem has an upper solution, namely, the stochastic process Z_t. Thus, by Theorem 2.2, the SDE (3.3) has a mean-square continuous strong solution $X_t^{(2)}$, which satisfies $X_t^{(1)} \le X_t^{(2)} \le Z_t$. Now $X^{(2)} = S(U^{(2)})$, so S is an increasing map as required and thus, by [4, Corollary 3.2], has a fixed point $X^* = S(X^*) \in \mathcal{H}$, that is,

$$X_t^* = X_0 + \int_0^t f(X_s^*)\,ds + \int_0^t g(X_s^*)\,dW_s \tag{3.4}$$

with $Y_t \le X_t^* \le Z_t$ for all $t \in [0,T]$, w.p.1. In particular, X_t^* is nonanticipative and mean-square continuous. □

We can apply Theorem 3.1 to the scalar SDE (1.2) with the Heaviside drift coefficient $f(x) = H(x)$ and diffusion coefficient $g(x) \equiv 1$. First we note that $H(x)$ is an increasing function and then that

$$X_0 + \int_0^t dW_s \le X_0 + \int_0^t H(X_s)\,ds + \int_0^t dW_s \le X_0 + \int_0^t 1\,ds + \int_0^t dW_s \tag{3.5}$$

for any sample path continuous, nonanticipative stochastic process X_t. Hence $Y_t := X_0 + W_t$ and $Z_t := X_0 + W_t + t$ are lower and upper solutions for the Heaviside SDE (1.2). Thus the Heaviside SDE (1.2) has at least one mean-square continuous strong solution X_t^* taking values between those of these lower and upper solutions, specifically with

$$X_0 + W_t \le X_t^* \le X_0 + 1 + W_t, \quad t \in [0,T], \text{ w.p.1.} \tag{3.6}$$

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