SOME LIMIT THEOREMS CONNECTED WITH BROWNIAN LOCAL TIME

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Received 26 October 2004; Revised 11 April 2005; Accepted 12 April 2005

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion and let $(L_t^x;\ t \geq 0,\ x \in \mathbb{R})$ be a continuous version of its local time process. We show that the following limit $\lim_{\varepsilon\downarrow 0}(1/2\varepsilon)\int_0^t \{F(s,B_s - \varepsilon) - F(s,B_s+\varepsilon)\}\,ds$ is well defined for a large class of functions $F(t,x)$, and moreover we connect it with the integration with respect to local time L_t^x. We give an illustrative example of the nonlinearity of the integration with respect to local time in the random case.

1. Introduction

1.1. The local time of the Brownian motion B at the point a is defined as follows:

$$L_t^a = \mathbb{P}\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t 1_{(|B_s-a|\leq\varepsilon)}\,ds, \tag{1.1}$$

which equivalently could be written as follows:

$$L_t^a = \mathbb{P}\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \big(1_{(B_s-\varepsilon\leq a)} - 1_{(B_s+\varepsilon\leq a)}\big)\,ds. \tag{1.2}$$

Here we are, more generally, interested in the limit in L^1:

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \{F(s,B_s-\varepsilon) - F(s,B_s+\varepsilon)\}\,ds \tag{1.3}$$

for some function F.

Our motivation comes from the desire to connect Chitashvili and Mania results [1] with those of Eisenbaum [2].

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 26961, Pages 1–5
DOI 10.1155/JAMSA/2006/26961

1.2. We give an example which illustrates that the integration with respect to $(L_t^x;\ 0 \le t \le 1,\ x \in \mathbb{R})$ does not admit a linear extension in the random case (see Section 3.2 for details) and in particular local time is not a 1-integrator, which is also proved by Eisenbaum [2].

2. Notation and preliminaries

Let $B = (B_t)_{t\ge 0}$ be a standard Brownian motion and let $(L_t^x;\ t \ge 0,\ x \in \mathbb{R})$ be a continuous version of its local time process. Let $(\mathcal{F}_t)_{t\ge 0}$ denote the natural filtration generated by B. Without loss of generality, we restrict our attention to functions defined on $[0,1]\times\mathbb{R}$.

For a measurable function f from $[0,1]\times\mathbb{R}$ into $\mathbb{R}$, define the norm $\|\cdot\|$ by

$$\|f\| = 2\left(\int_0^1\int_{\mathbb{R}} f^2(s,x)e^{-x^2/2s}\frac{ds\,dx}{\sqrt{2\pi s}}\right)^{1/2} + \int_0^1\int_{\mathbb{R}} |xf(s,x)|\,e^{-x^2/2s}\frac{ds\,dx}{s\sqrt{2\pi s}}. \tag{2.1}$$

Let $\mathcal{H}$ be the set of functions f such that $\|f\| < \infty$.

In Eisenbaum [2], it is shown that the integration with respect to L is possible in the following sense. Let f_Δ be an elementary function on $[0,1]\times\mathbb{R}$, meaning that

$$f_\Delta(t,x) = \sum_{(s_i,x_j)\in\Delta} f_{i,j}1_{(s_i,s_{i+1}]}(t)1_{(x_j,x_{j+1}]}(x), \tag{2.2}$$

where $\Delta = \{(s_i,x_j), 1 \le i \le n, 1 \le j \le m\}$ is an $[0,1]\times\mathbb{R}$ grid, and, for every (i,j), f_{ij} is in $\mathbb{R}$. For such a function, integration with respect to L is defined by

$$\int_0^1\int_{\mathbb{R}} f_\Delta(s,x)dL_s^x = \sum_{(s_i,x_j)\in\Delta} f_{i,j}\left(L_{s_{i+1}}^{x_{j+1}} - L_{s_i}^{x_{j+1}} - L_{s_{i+1}}^{x_j} + L_{s_i}^{x_j}\right). \tag{2.3}$$

Let f be an element of $\mathcal{H}$. For any sequence of elementary functions $(f_{\Delta_k})_{k\in\mathbb{N}}$ converging to f in $\mathcal{H}$, the sequence $(\int_0^1\int_{\mathbb{R}} f_{\Delta_k}(s,x)dL_s^x)_{k\in\mathbb{N}}$ converges in L^1. The limit obtained does not depend on the choice of the sequence (f_{Δ_k}) and represents the integral $\int_0^1\int_{\mathbb{R}} f(s,x)dL_s^x$.

Theorem 2.1 (see [2]). *Let $(A(x,t);\ x \in \mathbb{R},\ 0 \le t \le 1)$ be a continuous random process taking values in $\mathbb{R}$, such that for any t in $[0,1]$ and any ω, $A(\cdot,t)$ is absolutely continuous with respect to dx. Note $\partial A/\partial x$ its derivative and ask $\partial A/\partial x$ to be continuous. Then $\int_0^1\int_{\mathbb{R}} A(x,s)dL_s^x$ exists and the following hold:*

(i) *for any couple (a,b) in $\mathbb{R}^2$ with $a < b$*

$$\int_0^t\int_b^a A(x,s)dL_s^x = -\int_0^t \frac{\partial A}{\partial x}(B_s,s)\,ds + \int_0^t A(b,s)d_sL_s^b - \int_0^t A(a,s)d_sL_s^a; \tag{2.4}$$

(ii)

$$\int_0^1\int_{\mathbb{R}} A(x,s)dL_s^x = -\int_0^1 \frac{\partial A}{\partial x}(B_s,s)\,ds; \tag{2.5}$$

(iii)

$$\left(\int_0^t\int_b^a A(x,s)dL_s^x\right)(\omega) = \int_0^t\int_b^a A(x,s)(\omega)dL_s^x(\omega). \tag{2.6}$$

3. Main results

3.1. Deterministic case

Theorem 3.1. *Let F be a bounded element of $\mathscr{H}$. The following equalities hold in L^1:*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\{F(s,B_s)-F(s,B_s-\varepsilon)\}ds=-\int_0^t\int_{\mathbb{R}}F(s,x)dL_s^x; \tag{3.1}$$

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\{F(s,B_s+\varepsilon)-F(s,B_s)\}ds=-\int_0^t\int_{\mathbb{R}}F(s,x)dL_s^x; \tag{3.2}$$

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t\{F(s,B_s-\varepsilon)-F(s,B_s+\varepsilon)\}ds=\int_0^t\int_{\mathbb{R}}F(s,x)dL_s^x. \tag{3.3}$$

Remark 3.2. (1) If we take $F(t,x)=1_{(x\le a)}$ in (3.1), we have the very definition of L_t^a.

(2) Eisenbaum [2] has shown that for any Borelian function $b(t)$,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t 1_{(|B_s-b(s)|<\varepsilon)}ds=\int_0^t\int_{\mathbb{R}}1_{(-\infty,b(s))}(x)dL_s^x\quad\text{in } L^1, \tag{3.4}$$

which corresponds to (3.3) with $F(t,x)=1_{(x\le b(t))}$.

Proof. Define $H_\varepsilon(t,x)=(1/\varepsilon)\int_{x-\varepsilon}^x F(t,y)dy$. Then $H_\varepsilon\to F$ in $\mathscr{H}$ as $\varepsilon\downarrow 0$. On the one hand, $(\partial/\partial x)H_\varepsilon(t,x)=(1/\varepsilon)\{F(t,x)-F(t,x-\varepsilon)\}$. It follows that (see Eisenbaum [2, Theorem 5.1(ii)]) $\int_0^t\int_{\mathbb{R}}H_\varepsilon(s,x)dL_s^x=-(1/\varepsilon)\int_0^t\{F(s,B_s)-F(s,B_s-\varepsilon)\}ds$. On the other hand, $\int_0^t\int_{\mathbb{R}}H_\varepsilon(s,x)dL_s^x\to\int_0^t\int_{\mathbb{R}}F(s,x)dL_s^x$ in L^1. □

Corollary 3.3 (see [3]). *The following relation holds in L^1:*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t g(s)I(b(s)-\varepsilon<B_s<b(s)+\varepsilon)ds=\int_0^t g(s)dL_s^b \tag{3.5}$$

for a continuous function $g:[0,t]\to\mathbb{R}$ and a continuous curve $b(\cdot)$ with bounded variation on $[0,t]$.

Proof. We apply Theorem 3.1 to the function $F(t,x)=g(t)I(x<b(t))$. It follows that $(1/2\varepsilon)\int_0^t g(s)I(b(s)-\varepsilon<B_s<b(s)+\varepsilon)ds\to\int_0^t\int_{\mathbb{R}}g(s)I(x<b(s))dL_s^x$ in L^1 as $\varepsilon\downarrow 0$. We conclude using (see [4, Corollary 2.9]) that for the *continuous* function g, we have $\int_0^t g(s)\partial_s L_s^{b(s)}=\int_0^t g(s)dL_s^b$. □

3.2. Random function case. Let a,b be in $\mathbb{R}$ with $a<b$. Let $\mathcal{M}$ be the set of elementary processes A such that

$$A(s,x)=\sum_{(s_i,x_j)\in\Delta}A_{ij}1_{s_i,s_{i+1}]}(s)1_{(x_j,x_{j+1}]}(x), \tag{3.6}$$

where $(s_i)_{1\le i\le n}$ is a subdivision of $(0,1]$, $(x_j)_{1\le j\le m}$ is a finite sequence of real numbers in $(a,b]$, $\Delta=\{(s_i,x_j),1\le i\le n,1\le j\le m\}$, and, is A_{ij} an $\mathscr{F}_{s_j}$-measurable random variable such that $|A_{ij}|\le 1$ for every (i,j).

Eisenbaum [2] asked the following question: does integration with respect to $(L_t^x;\ 0 \le t \le 1,\ x \in \mathbb{R})$ admit a *linear* extension to $\mathcal{P}$ the field generated by $\mathcal{M}$, verifying the following property?

If $(A_n)_{n\ge 0}$ converges a.e. to $A(t,x)$, then $(\int_0^1 \int_a^b A_n(s,x)dL_s^x)_{n\ge 0}$ converges in L^1 to $\int_0^1 \int_a^b A(s,x)dL_s^x$.

She only obtained a negative answer to the following *weaker* question:

$$\text{Is the set } \left\{ \int_0^1 \int_a^b A(s,x)dL_s^x,\ A \in \mathcal{M} \right\} \text{ bounded in } L^1? \tag{3.7}$$

Consequently, integration with respect to $(L_t^x;\ 0 \le t \le 1,\ x \in \mathbb{R})$ does not admit a *continuous* extension in L^1.

Here we give an *illustrative example*, thanks to a result obtained by Walsh, which shows the lack of a *linear* extension.

Let us define $A_\varepsilon(t,x) = (1/\varepsilon)\int_{x-\varepsilon}^x L_t^y\,dy$ and $\widetilde{A}_\varepsilon(t,x) = (1/\varepsilon)\int_x^{x+\varepsilon} L_t^y\,dy$. We see easily that $A_\varepsilon(t,x)$ (resp., $\widetilde{A}_\varepsilon(t,x)$) converges a.e. to L_t^x, nevertheless we have

$$\lim_{\varepsilon\downarrow 0} \int_0^t \int_{\mathbb{R}} A_\varepsilon(s,x)dL_s^x \neq \lim_{\varepsilon\downarrow 0} \int_0^t \int_{\mathbb{R}} \widetilde{A}_\varepsilon(s,x)dL_s^x. \tag{3.8}$$

Remark 3.4. The integrals $\int_0^t \int_{\mathbb{R}} A_\varepsilon(s,x)dL_s^x$ and $\int_0^t \int_{\mathbb{R}} \widetilde{A}_\varepsilon(s,x)dL_s^x$ are well defined thanks to Theorem 2.1, however, one does not know whether $\int_0^t \int_{\mathbb{R}} L_s^x dL_s^x$ is well defined or not.

Let us recall, for the convenience of the reader, Walsh's theorem about the decomposition of $A(t,B_t) := \int_0^t 1_{\{B_s \le B_t\}}\,ds$.

Theorem 3.5 (see [5]). *$A(t,B_t)$ has the decomposition*

$$A(t,B_t) = \int_0^t L_s^{B_s} dB_s + X_t, \tag{3.9}$$

where

$$X_t = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{L_s^{B_s} - L_s^{B_s-\varepsilon}\}ds = t + \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{L_s^{B_s+\varepsilon} - L_s^{B_s}\}ds. \tag{3.10}$$

The limits exist in probability, uniformly for t in compact sets.

Our example follows by recalling the following property:

$$\int_0^t \int_{\mathbb{R}} A_\varepsilon(s,x)dL_s^x = -\frac{1}{\varepsilon} \int_0^t \{L_s^{B_s} - L_s^{B_s-\varepsilon}\}ds. \tag{3.11}$$

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