ON FUZZY VOLTERRA INTEGRAL EQUATIONS WITH DEVIATING ARGUMENTS

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We investigate the problem of existence of solutions of fuzzy Volterra integral equations with deviating arguments. The results are obtained by using the Darbo fixed point theorem.

1. Introduction

In 1982, Dubois and Prade [4, 5] first introduced the concept of integration of fuzzy functions. Kaleva [7] studied the measurability and integrability for the fuzzy set-valued mappings of a real variable whose values are normal, convex, upper semicontinuous, and compactly supported by fuzzy sets in $\mathbb{R}^n$. Existence of solutions of fuzzy integral equations has been studied by several authors [1, 2, 7, 8]. Subrahmanyam and Sudarsanam [13] proved existence theorems for fuzzy functional equations. They have used the embedding theorem of Kaleva [8], which is a generalization of the classical Rådström embedding theorem [11], and the Darbo fixed point theorem in the convex cone. Recently, Balachandran and Prakash [2, 3] studied the existence of solutions of nonlinear fuzzy Volterra-Fredholm integral equations.

In this paper, we prove the existence of solutions of fuzzy Volterra integral equations with deviating arguments. The results, which generalize the results of [1, 2], are established with the help of the Darbo fixed point theorem. Further, we study the maximal solution of the fuzzy delay Volterra integral equation.

2. Preliminaries

Let $P_K(\mathbb{R}^n)$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}^n$. Addition and scalar multiplication in $P_K(\mathbb{R}^n)$ are defined as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}, \tag{2.1}$$

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where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Then it is clear that $(P_K(\mathbb{R}^n), d)$ becomes a metric space. Let $I = [t_0, t_0 + a] \subset \mathbb{R}$ $(a > 0)$ be a compact interval and let E^n be the set of all $u : \mathbb{R}^n \to [0,1]$ such that u satisfies the following conditions:

(i) u is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy and convex, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = \mathrm{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space. For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq 0\}$. Then from (i)–(iv), it follows that the α-level set $[u]^\alpha \in P_K(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

If $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, then using Zadeh's extension principle, we can extend g to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u,v)(z) = \sup_{z = g(x,y)} \min\{u(x), v(y)\}. \tag{2.2}$$

It is well known that $[\tilde{g}(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^n$, $0 \leq \alpha \leq 1$, and continuous function g. Further, we have $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[ku]^\alpha = k[u]^\alpha$, where $k \in \mathbb{R}$. Define $D : E^n \times E^n \to [0, \infty)$ by the relation $D(u,v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha)$, where d is the Hausdorff metric defined in $P_K(\mathbb{R}^n)$. Then D is a metric in E^n.

Further, we know that [10]

(i) (E^n, D) is a complete metric space,

(ii) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$,

(iii) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u, v \in E^n$ and $\lambda \in \mathbb{R}$.

It can be proved that $D(u+v, w+z) \leq D(u,w) + D(v,z)$ for $u, v, w, z \in E^n$.

Definition 2.1 [7]. A mapping $F : I \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$, the set-valued map $F_\alpha : I \to P_K(\mathbb{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue-measurable when $P_K(\mathbb{R}^n)$ has the topology induced by the Hausdorff metric d. A mapping $F : I \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \leq h(t)$ for every $x \in F_0(t)$.

Definition 2.2 [10]. The integral of a fuzzy mapping $F : I \to E^n$ is defined levelwise by $[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt =$ the set of all $\int_I f(t)dt$ such that $f : I \to \mathbb{R}^n$ is a measurable selection for F_α for all $\alpha \in [0,1]$.

Definition 2.3 [1]. A strongly measurable and integrably bounded mapping $F : I \to E^n$ is said to be integrable over I if $\int_I F(t)dt \in E^n$.

Note that if $F : I \to E^n$ is strongly measurable and integrably bounded, then F is integrable. Further, if $F : I \to E^n$ is continuous, then it is integrable.

Theorem 2.4. *Let $F, G : I \to E^n$ be integrable and $c \in I$, $\lambda \in \mathbb{R}$. Then*

(i) $\int_{t_0}^{t_0+a} F(t)dt = \int_{t_0}^{c} F(t)dt + \int_{c}^{t_0+a} F(t)dt$,

(ii) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,

(iii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,

(iv) $D(F,G)$ *is integrable,*
(v) $D(\int_I F(t)dt, \int_I G(t)dt) \leq \int_I D(F(t),G(t))dt$.

Now, we state the results about the Kuratowski measure of noncompactness [9].

Let $C[I,X]$ denote the Banach space of abstract continuous functions from I to X, and the norm $\|x\| = \max_{t\in I}\|x(t)\|$. For $H \subset C[I,X]$, we denote

$$\begin{aligned} H(t) &= \{x(t) : x \in H\} \subset X, \\ H(I) &= \{x(t) : x \in H,\ t \in I\} = \bigcup_{t\in I} H(t) \subset X. \end{aligned} \tag{2.3}$$

For a bounded subset A of X, the Kuratowski measure of noncompactness is defined as $\alpha(A) = \inf\{\epsilon > 0 : A$ can be covered by a finite number of sets, each with diameter $\leq \epsilon\}$.

The following results are proved in [6].

Proposition 2.5. *Let α be the Kuratowski measure of noncompactness and suppose that A and B are two arbitrary bounded subsets of X; then*

(i) $\alpha(A) = 0$ *if and only if A is relatively compact,*
(ii) $\alpha(A) \leq \alpha(B)$ *if $A \subseteq B$,*
(iii) $\alpha(A) = \alpha(\overline{\mathrm{co}}(A))$, *where* $\mathrm{co}(A)$ *denotes the convex hull of A,*
(iv) $\alpha(A \bigcup B) = \max\{\alpha(A), \alpha(B)\}$,
(v) $\alpha(tA) = |t|\alpha(A)$, *where* $tA = \{tx : x \in A\}$,
(vi) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, *where* $A+B = \{x+y : x \in A$ *and* $y \in B\}$.

Proposition 2.6. *Suppose $H \subset C[I,X]$ is bounded and equicontinuous; then* $\alpha(H) = \alpha(H(I)) = \max_{t\in I}\alpha(H(t))$.

If $A \subset X$ is bounded and the mapping $f : I \times A \to X$ is bounded and uniformly continuous, then

$$\alpha\big(f(I\times B)\big) = \max_{t\in I} \alpha\big(f(t,B)\big) \quad \forall B \subset A. \tag{2.4}$$

Let $x \in C[I,X]$, and $x(t)$ is differentiable (*the Fréchet derivative exists*). Then $x(t_0+a) - x(t_0) \in a\,\overline{\mathrm{co}}\{x'(t) : t \in I\}$.

3. Main results

Let $C[I,E^n]$ denote the space of continuous fuzzy set-valued mappings from I into E^n. Clearly, $C[I,E^n]$ is a convex cone. Consider the following fuzzy Volterra integral equation with deviating arguments:

$$x(t) = x_0(t) + f\big(t, x(\sigma_1(t))\big) + \int_{t_0}^{\sigma_2(t)} k(t,s) g\big(s, x(\sigma_3(s))\big)ds, \tag{3.1}$$

where $x_0 \in C[I,\Omega]$, $f,g \in C[I\times\Omega,\Omega]$, $k \in C[I\times I, I]$, $\sigma_i : I \to I$ and $\sigma_i(t) < t$ for $i = 1,2,3$, and Ω is an open subset of (E^n, D).

By [8, Theorem 2.1], the embedding j from (E^n, D) onto its range $j(E^n) \subset X$ is an isometric isomorphism, and so the embedding $j : C[I,E^n] \hookrightarrow C[I,X]$ is also an isometric isomorphism. Thus, for a subset $E \subset C[I, E^n]$, if $\Phi : E \to E$ is a continuous operator, then $j\Phi j^{-1} : jE \to jE$ is also continuous. We need the following lemma.

LEMMA 3.1 [12]. *Let $E \subset C[I, E^n]$ be a nonempty closed bounded convex subset. If $\Phi : E \to E$ is a continuous operator with $\alpha(j(\Phi B)) \le \rho(\alpha(jB))$ for every $B \subseteq E$, where $\rho : [0,\infty) \to [0,\infty)$ is a right-continuous function with $\rho(r) < r$, for all $r > 0$, then Φ has a fixed point in E.*

LEMMA 3.2 [8]. *Let $F : I \to E^n$ be a fuzzy set-valued mapping provided that $jF : I \to j(E^n) \subset X$ is Bochner-integrable on $t \in I$. Then $\int_I F(t)dt \in E^n$ and $j\int_I F(t)dt = \int_I jF(t)dt$.*

Now, we prove the existence theorem for the fuzzy delay Volterra integral equation (3.1).

THEOREM 3.3. *Assume that*

(i) *there exists a constant $K > 0$ such that $K = \max_{t,s\in I} |k(t,s)|$;*
(ii) *for any bounded subset $B \subseteq \Omega$, f and g are bounded and uniformly continuous on $I \times B$ and there exist constants $N > 0$ and $M > 0$ such that $D(f(t,x(\sigma_1(t))),\hat{0}) \le N$ and $D(g(s,x(\sigma_3(s))),\hat{0}) \le M$ for all $t,s \in I$, $x \in \Omega$, $\sigma \in I$, where $\hat{0}$ denotes the zero fuzzy number;*
(iii) *$\alpha(jf(I \times B)) \le \rho_1(\alpha(jB))$ and $\alpha(jg(I \times B)) \le \rho_2(\alpha(jB))$, where $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, is a right-continuous and monotone nondecreasing function with $\rho_i(r) < r/2$, for all $r > 0$, and $B \subseteq \Omega$ is a bounded subset.*

Then there exists a solution to (3.1) on $[t_0, t_0 + \epsilon]$ for some $\epsilon > 0$.

Proof. Choose $\eta > 0$ such that $B_0 = \{y \in E^n : D(y, x_0(t_0)) \le \eta\} \subset \Omega$. Since $x_0(t)$ is continuous, there is a $\delta > 0$ such that $t_0 \le t \le t_0 + \delta$ implies $D(x_0(t), x_0(t_0)) < \eta/2$.

Let $\epsilon = \min\{a, \delta, (\eta - 2N)/2MK, 1/K\}$, $I_0 = [t_0, t_0 + \epsilon]$, and define $E \subseteq C[I_0, \Omega]$ by

$$E = \left\{x \in C[I_0, \Omega] : \sup_{t\in I_0} D(x(t), x_0(t)) \le \frac{\eta}{2}\right\}. \tag{3.2}$$

Clearly, E is a bounded closed convex subset of $C[I_0, \Omega]$ and $x_0 \in E$. We define the operator $\Phi : C[I_0, \Omega] \to C[I_0, \Omega]$ by

$$(\Phi x)(t) = x_0(t) + f(t, x(\sigma_1(t))) + \int_{t_0}^{\sigma_2(t)} k(t,s) g(s, x(\sigma_3(s))) ds \quad \forall t,s \in I_0. \tag{3.3}$$

Then

$$\begin{aligned} D((\Phi x)(t), x_0(t)) &= D\left(x_0(t) + f(t, x(\sigma_1(t))) + \int_{t_0}^{\sigma_2(t)} k(t,s) g(s, x(\sigma_3(s))) ds, x_0(t)\right) \\ &= D\left(f(t, x(\sigma_1(t))) + \int_{t_0}^{\sigma_2(t)} k(t,s) g(s, x(\sigma_3(s))) ds, \hat{0}\right) \\ &\le D(f(t, x(\sigma_1(t))), \hat{0}) + \int_{t_0}^{\sigma_2(t)} D(k(t,s) g(s, x(\sigma_3(s))), \hat{0}) ds \end{aligned}$$

$$\begin{aligned}
&\leq D\left(f(t,x(\sigma_1(t))),\hat{0}\right)+K\int_{t_0}^{\sigma_2(t)} D\left(g(s,x(\sigma_3(s))),\hat{0}\right)ds\\
&\leq N+K\epsilon M\\
&\leq \frac{\eta}{2} \quad \text{for } x\in E,\ t,s\in I_0.
\end{aligned} \tag{3.4}$$

Thus, we have $\Phi E\subseteq E$ and ΦE is uniformly bounded on I_0.

If $\{x_n\}\subseteq E$ satisfies that $\sup_{t\in I_0} D(x_n(t),x(t))\to 0$ as $n\to\infty$, then $x\in E$ and by the uniform continuity of f and g, we have

$$\begin{aligned}
&\sup_{t\in I_0} D\left(f(t,x_n(\sigma_1(t))),f(t,x(\sigma_1(t)))\right)\longrightarrow 0 \quad \text{as } n\longrightarrow\infty,\\
&\sup_{s\in I_0} D\left(g(s,x_n(\sigma_3(s))),g(s,x(\sigma_3(s)))\right)\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{3.5}$$

Thus, from Theorem 2.4(v), it follows that Φ is a continuous operator. Further, by Proposition 2.6, we have

$$\alpha\left(jf(I_0\times B)\right)=\max_{t\in I_0}\alpha\left(jf(t,B)\right),\qquad \alpha\left(jg(I_0\times B)\right)=\max_{s\in I_0}\alpha\left(jg(s,B)\right), \tag{3.6}$$

for any bounded subset $B\subseteq\Omega$. Now, for $x\in E$, $t,\tau\in I_0$, and $\tau<t$, we have

$$\begin{aligned}
D\left(\Phi x(t),\Phi x(\tau)\right)&=D\bigg(x_0(t)+f(t,x(\sigma_1(t)))\\
&\qquad+\int_{t_0}^{\sigma_2(t)}k(t,s)g(s,x(\sigma_3(s)))ds,x_0(\tau)+f(\tau,x(\sigma_1(\tau)))\\
&\qquad+\int_{t_0}^{\sigma_2(\tau)}k(\tau,s)g(s,x(\sigma_3(s)))ds\bigg)\\
&\leq D\left(x_0(t),x_0(\tau)\right)+D\left(f(t,x(\sigma_1(t))),f(\tau,x(\sigma_1(\tau)))\right)\\
&\quad+D\left(\int_{t_0}^{\sigma_2(t)}k(t,s)g(s,x(\sigma_3(s)))ds,\int_{t_0}^{\sigma_2(\tau)}k(\tau,s)g(s,x(\sigma_3(s)))ds\right)\\
&\leq D\left(x_0(t),x_0(\tau)\right)+D\left(f(t,x(\sigma_1(t))),f(\tau,x(\sigma_1(\tau)))\right)\\
&\quad+\int_{\sigma_2(\tau)}^{\sigma_2(t)}D\left(k(t,s)g(s,x(\sigma_3(s))),\hat{0}\right)ds\\
&\quad+\int_{t_0}^{\sigma_2(\tau)}D\left(k(t,s)g(s,x(\sigma_3(s))),k(\tau,s)g(s,x(\sigma_3(s)))\right)ds\\
&\longrightarrow 0 \quad \text{as } t\longrightarrow\tau.
\end{aligned} \tag{3.7}$$

Hence, $\{\Phi x(t):x\in E\}$ is uniformly bounded and equicontinuous on I_0. Further, condition (ii) implies that $\|jg(s,x(\sigma_3(s)))\|$ is Lebesgue-integrable and $jg(s,x(\sigma_3(s)))$ is strongly measurable for all $s \in I_0$. Therefore, it follows that $jg(s,x(\sigma_3(s)))$ is Bochner-integrable for all $s \in I_0$. Hence, by Lemma 3.2 and Proposition 2.6, we have

$$j(f(t,x(\sigma_1(t)))) = jf(t,x(\sigma_1(t))) \in \overline{\text{co}}\{jf(t,x(\sigma_1(t))) : t \in [t_0,t]\}, \tag{3.8}$$

$$j\left(\int_{t_0}^{\sigma_2(t)} g(s,x(\sigma_3(s)))ds\right) = \int_{t_0}^{\sigma_2(t)} jg(s,x(\sigma_3(s)))ds \in \epsilon\overline{\text{co}}\{jg(s,x(\sigma_3(s))) : s \in [t_0,t]\}. \tag{3.9}$$

From (3.6), (3.7), (3.8), (3.9), and condition (iii), we get

$$\begin{aligned}
\alpha(j\{(\Phi B)(t)\}) &= \alpha\left(jx_0(t) + jf(t,x(\sigma_1(t))) + j\left\{\int_{t_0}^{\sigma_2(t)} k(t,s)g(s,x(\sigma_3(s)))ds\right\} : x \in B\right) \\
&\leq \alpha(j\{f(t,x(\sigma_1(t))) : x \in B\}) \\
&\quad + \alpha\left(j\left\{\int_{t_0}^{\sigma_2(t)} Kg(s,x(\sigma_3(s)))ds : x \in B,\ \sigma_i \in I_0\right\}\right) \\
&= \alpha(jf(t,x(\sigma_1(t))) : x \in B) + K\alpha\left(\int_{t_0}^{\sigma_2(t)} jg(s,x(\sigma_3(s)))ds : x \in B\right) \\
&\leq \alpha(jf(t,x(\sigma_1(t))) : t \in I_0,\ x \in B) \\
&\quad + K\epsilon\alpha(\overline{\text{co}}\{jg(s,x(\sigma_3(s))) : s \in I_0,\ x \in B\}) \\
&= \alpha(jf(I_0 \times B)) + K\epsilon\alpha(\{jg(s,x(\sigma_3(s))) : s \in I_0,\ x \in B\}) \\
&\leq \alpha(jf(I_0 \times B)) + K\epsilon\alpha(jg(I_0 \times B)) \\
&\leq \rho_1(\alpha(jB)) + K\epsilon\rho_2(\alpha(jB)) \\
&\leq \rho_1(\alpha(jB)) + \rho_2(\alpha(jB)) \\
&= \rho(\alpha(jB)),
\end{aligned} \tag{3.10}$$

where $\rho = \rho_1 + \rho_2$. So, by Lemma 3.1, Φ has a fixed point in E and the fixed point of Φ is a solution to (3.1). □

4. Maximal solution

In this section, we prove the maximal solution of the fuzzy delay Volterra integral equation (3.1). E^n constitutes a convex cone for the addition and the nonnegative multiplication in E^n, hence the partial ordering in E^n can be introduced by $x \leq y$ if and only if there exists a $z \in E^n$ such that $y = x + z$ for $x, y \in E^n$.

If $x \leq y$ and $x \neq y$, then we write $x < y$; if $x \leq y$ and $j(y - x) \in \text{Int}(j(E^n))$, then we write $x \ll y$, where $\text{Int}(j(E^n)) \subseteq X$ denotes the set constructed by all the interior points of $j(E^n)$. It is easy to see that $j(E^n)$ is also a closed convex cone in X, and the conjugate cone of $j(E^n)$ is represented by $(j(E^n))^* = \{\varphi \in X^* : \varphi(\omega) \geq 0, \text{ for all } \omega \in j(E^n)\}$, and $\text{Int}(j(E^n))^* = \{\varphi \in X^* : \varphi(\omega) > 0, \text{ for all } \omega \in j(E^n)\}$.

Lemma 4.1 [12]. (i) *$\omega \in j(E^n)$ if and only if for all $\varphi \in (j(E^n))^*$, $\varphi(\omega) \geq 0$.*

(ii) *Let $\omega \in \partial(j(E^n))$; then there exists $\varphi \in \mathrm{Int}(j(E^n))^*$ such that $\varphi(\omega) = 0$, where $\partial(j(E^n)) \subseteq X$ denotes the boundary of $j(E^n)$.*

Definition 4.2 [12]. Let $f : E^n \to E^n$ be a fuzzy set-valued operator if $x \leq y$ implies $f(x) \leq f(y)$ for any $x, y \in E^n$. Then f is said to be fuzzy monotone nondecreasing.

Theorem 4.3. *Assume that*

(i) *for any fixed $t, s \in I$, $f, g \in C[I \times E^n, E^n]$, $\sigma_i : I \to I$ and $x_0, u, v \in C[I, E^n]$;*

(ii) *$f(t, u(\sigma_1(t)))$ and $g(t, u(\sigma_3(t)))$ are fuzzy monotone nondecreasing in $u \in E^n$;*

(iii) *for any fixed $t \in I$, the real functions $h_1(s) = D(g(s, u(\sigma_3(s))), \hat{0})$ and $h_2(s) = D(g(s, v(\sigma_3(s))), \hat{0})$ are Lebesgue-integrable.*

Then

$$\begin{gathered} u(t) \leq x_0(t) + f(t, u(\sigma_1(t))) + \int_{t_0}^{\sigma_2(t)} k(t,s) g(s, u(\sigma_3(s))) ds, \\ v(t) \gg x_0(t) + f(t, v(\sigma_1(t))) + \int_{t_0}^{\sigma_2(t)} k(t,s) g(s, v(\sigma_3(s))) ds, \\ u(t) \ll v(t_0) \end{gathered} \tag{4.1}$$

imply that $u(t) \ll v(t)$, $t \in I$.

Proof. Suppose that the conclusion is not true, then the set

$$\begin{aligned} Z &= \{t \in I : u(t) \ll v(t) \text{ does not hold}\} \\ &= \{t \in I : ju(t) \ll jv(t) \text{ does not hold}\} \neq \varnothing. \end{aligned} \tag{4.2}$$

Let $t_1 = \inf Z$; it is easy to see that $t_1 > t_0$, and for any $t \in [t_0, t_1)$, $jv(t) - ju(t) \in \mathrm{Int}(j(E^n))$ and $jv(t_1) - ju(t_1) \in \partial(j(E^n))$. So, by Lemma 4.1, there exists $F \in \mathrm{Int}(j(E^n))^*$ such that

$$F(jv(t_1) - ju(t_1)) = 0. \tag{4.3}$$

The functions $jf(t, u(\sigma_1(t)))$, $jf(t, v(\sigma_1(t)))$, $k(t,s)$ are continuous for fixed $t \in I$, and hence they are strongly measurable. Further, by (iii), it follows that $jg(s, u(\sigma_3(s)))$ and $jg(s, v(\sigma_3(s)))$ are Bochner-integrable in $s \in I$. From (3.9) and (ii) and by Lemma 3.2, we have

$$\begin{aligned} F(ju(t_1)) &\leq F\left[jx_0(t_1) + jf(t, u(\sigma_1(t_1))) + j\left\{ \int_{t_0}^{\sigma_2(t_1)} k(t_1,s) g(s, u(\sigma_3(s))) ds \right\} \right] \\ &= F\left[jx_0(t_1) + jf(t, u(\sigma_1(t_1))) + \int_{t_0}^{\sigma_2(t_1)} k(t_1,s) jg(s, u(\sigma_3(s))) ds \right] \\ &\leq \left[jx_0(t_1) + jf(t, v(\sigma_1(t_1))) + \int_{t_0}^{\sigma_2(t_1)} k(t_1,s) jg(s, v(\sigma_3(s))) ds \right] \\ &= F\left[jx_0(t_1) + j(f(t, v(\sigma_1(t_1)))) + j\left\{ \int_{t_0}^{\sigma_2(t_1)} k(t_1,s) g(s, v(\sigma_3(s))) ds \right\} \right] \\ &< F(jv(t_1)). \end{aligned} \tag{4.4}$$

This is a contradiction to (4.3), and hence the proof. □

THEOREM 4.4. *Let f, g, and k be as in Theorem 3.3, and for any fixed $t,s \in I$, $f(t,u)$ and $g(t,u)$ are monotone nondecreasing on $u \in \Omega$. Then there exists an $\epsilon > 0$ so that the maximal solution to (3.1) exists on $[t_0, t_0 + \epsilon]$.*

The proof is similar to that of [12, Theorem 4.2] and hence omitted.

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