

WHAT RANDOM VARIABLE GENERATES A BOUNDED POTENTIAL?

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It is known that if a predictable nondecreasing process generates a bounded potential, then its final value satisfies the Garsia inequality. We prove the converse, that is, a random variable satisfying the Garsia inequality generates a bounded potential. We also propose some useful relations between the Garsia inequality and the Cramer conditions, and different ways how to construct a potential.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\{\mathcal{F}_t, t \geq 0\}$ be a filtration satisfying the standard conditions: \mathcal{F}_0 contains \mathbb{P} zero sets of σ -field \mathcal{F} , $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, $0 \leq s \leq t$, $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t$. Also, let $\{X_t, \mathcal{F}_t, t > 0\}$ be the potential with the Doob-Meyer decomposition of the form

$$X_t = M_t - A_t, \quad (1.1)$$

where M_t is a martingale, A_t is nondecreasing integrable right-continuous predictable process, $A_0 = 0$. Let $A_\infty := \lim_{t \rightarrow \infty} A_t$. It follows from (1.1) that X_t also admits the decomposition

$$X_t = E(A_\infty - A_t | \mathcal{F}_t), \quad (1.2)$$

and we say that the process A_t generates the potential X_t . Garsia [2] established that in the case when X_t is bounded, that is, $0 \leq X_t \leq c_0$, the random variable (r.v.) $\xi := A_\infty$ satisfies the Garsia inequality

$$EG(\xi) \leq c_0 Eg(\xi), \quad (1.3)$$

where $g = g(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is any nondecreasing nonnegative function, $G(t) = \int_0^t g(u) du$. In particular, with $g(t) = t^{n-1}$, $n \geq 1$, we obtain from the Garsia inequality that ξ satisfies the Cramer conditions, that is, $E\xi^n \leq c_0^n \cdot n!$, $n \geq 1$.

Our aim is to investigate what sort of potential can be generated by a r.v. that satisfies the Garsia inequality or the Cramer conditions.

The paper is organized as follows. Section 2 contains relations between the Garsia inequality and the Cramer conditions on the one side, and the properties of the distribution function (d.f.) of corresponding r.v. on the other side. Example 2.6 demonstrates that the Garsia inequality and the Cramer conditions are not equivalent. In Section 3 we prove that any r.v. satisfying the Garsia inequality predictably generates a nontrivial bounded potential (corresponding definitions are provided in this section). In Section 4 we transform some r.v. in such a way that the resulting r.v. predictably generates a bounded potential.

2. The asymptotic properties of a distribution function corresponding to a random variable that satisfies the Garsia inequality or the Cramer conditions

Throughout the paper, ξ is a nonnegative unbounded integrable r.v. with the d.f. $F_\xi(x) = \mathbb{P}\{\xi < x\}$. Note that the integral $\int_x^\infty (1 - F_\xi(u))du$ exists and is positive for any $x \geq 0$, $E\xi = \int_0^\infty (1 - F_\xi(u))du$, and $1 - F_\xi(x)$ is a left-continuous function.

Denote

$$\lambda(x) := (1 - F_\xi(x)) \left(\int_x^\infty (1 - F_\xi(u))du \right)^{-1} = (E\xi - x \mid \xi \geq x)^{-1}. \tag{2.1}$$

LEMMA 2.1. (1) *The positive, left-continuous function λ defined by (2.1) also satisfies the relations*

$$1 - F_\xi(x) = E\xi \cdot \lambda(x) \exp \left\{ - \int_0^x \lambda(u)du \right\}, \tag{2.2}$$

$$\int_0^x \lambda(u)du = - \ln \left(\int_x^\infty (1 - F_\xi(u))du \cdot (E\xi)^{-1} \right), \tag{2.3}$$

$$\int_0^\infty \lambda(u)du = \infty. \tag{2.4}$$

(2) *Let the function λ satisfy representation (2.2). Then it is unique, positive, left-continuous, and satisfies (2.1), (2.3), and (2.4).*

Proof. (1) Let λ be defined by (2.1). Denote $Q(x) := \int_x^\infty (1 - F_\xi(u))du$. Then $\int_0^x \lambda(u)du = \int_{Q(x)}^{Q(0)} (du/u) = \ln(Q(0)/Q(x))$, yielding (2.3) and also $E\xi \cdot \lambda(x) \exp \left\{ - \int_0^x \lambda(u)du \right\} = E\xi \cdot \lambda(x) \cdot (Q(x)/Q(0)) = \lambda(x)Q(x) = 1 - F_\xi(x)$, that is (2.2). Evidently, $Q(\infty) = 0$ and therefore $\int_0^\infty \lambda(u)du = \ln(Q(0)/0) = \infty$.

(2) Let the function λ be defined by (2.2). Denote $\Lambda(x) := \int_0^x \lambda(u)du$, and obtain from (2.2) that

$$\begin{aligned} \int_0^x (1 - F_\xi(u))du &= E\xi \cdot \int_0^x \exp \{ - \Lambda(u) \} \lambda(u)du \\ &= E\xi \cdot (1 - \exp \{ - \Lambda(x) \}). \end{aligned} \tag{2.5}$$

Since $E\xi = \int_0^\infty (1 - F_\xi(u))du$, we obtain from (2.5) that $E\xi \cdot \exp\{-\Lambda(x)\} = \int_x^\infty (1 - F_\xi(u))du$. Therefore, from (2.2),

$$1 - F_\xi(x) = \lambda(x) \cdot \int_x^\infty (1 - F_\xi(u))du \tag{2.6}$$

and, also, $E\xi \exp\{-\Lambda(\infty)\} = 0$. Hence $\Lambda(\infty) = \infty$. We establish (2.1) and (2.4). Relation (2.3) follows from (2.1) (see part (1) of the proof). \square

THEOREM 2.2. *Let ξ be a nonnegative unbounded integrable r.v. The following assertions are equivalent:*

- (i) r.v. ξ satisfies Garsia inequality (1.3);
- (ii) for any $x \geq 0$, $\lambda(x) \geq c_0^{-1}$, where λ is defined by (2.1).

Remark 2.3. It follows from (2.1) that (ii) is equivalent to

$$(iii) \sup_{x \geq 0} E(\xi - x \mid \xi \geq x) \leq c_0.$$

Proof. (1) Let r.v. ξ satisfy Garsia inequality (1.3). Then, in particular, $E\xi \leq c_0$. Consider a nonnegative nondecreasing function g such that $Eg(\xi) < \infty$. Then,

$$\begin{aligned} 0 &\leq c_0 Eg(\xi) - EG(\xi) \\ &= c_0 \int_0^\infty \left(g(0) + \int_{0+}^u dg(v) \right) F_\xi(du) \\ &\quad - \int_0^\infty \int_0^u \left(g(0) + \int_{0+}^r dg(v) \right) dr F_\xi(du) \\ &= (c_0 - E\xi)g(0) + \int_{0+}^\infty \left(c_0(1 - F_\xi(u)) - \int_u^\infty (1 - F_\xi(v))dv \right) dg(u). \end{aligned} \tag{2.7}$$

The integrand on the right-hand side of (2.7) is left-continuous. So, if (2.7) holds for any nonnegative nondecreasing g with $Eg(\xi) \leq \infty$, then obviously this integrand is nonnegative, that is,

$$c_0(1 - F_\xi(u)) - \int_u^\infty (1 - F_\xi(v))dv \geq 0 \quad \text{for each } u \geq 0. \tag{2.8}$$

Inequality (2.8), by Lemma 2.1, is equivalent to $\lambda(u) \geq c_0^{-1}$, $u \geq 0$.

(2) Let $\lambda(x) \geq c_0^{-1}$, $x \geq 0$. From $\lambda(0) = (E\xi)^{-1}$, we obtain $E\xi \leq c_0$. From the latter and relation (2.1), we deduce that the right-hand side of (2.7) is nonnegative for any nonnegative nondecreasing g , and for such g with $Eg(\xi) < \infty$, it equals $c_0 Eg(\xi) - EG(\xi)$. Thus we arrive at the Garsia inequality. \square

Remark 2.4. If we consider $F_\xi(x) = \mathbb{P}\{\xi \leq x\}$, that is, right-continuous $F_\xi(x)$, and ξ has no atom at zero, the proof of Theorem 2.2 will be the same. But if ξ has an atom at zero and $F_\xi(x)$ is right-continuous, then we must consider $\lambda(x)$ with some additional term,

namely,

$$\begin{aligned} \lambda(x) &:= (1 - F_\xi(x) + P\{\xi = 0\}I\{x = 0\}) \left(\int_x^\infty (1 - F_\xi(u)) du \right)^{-1} \\ &= (E(\xi - x \mid \xi \geq x))^{-1}. \end{aligned} \tag{2.9}$$

To avoid these technical difficulties, we therefore consider left-continuous functions.

THEOREM 2.5. *Let ξ be the same as in [Theorem 2.2](#). The following assertions are equivalent:*

- (iv) *the r.v. ξ satisfies the Cramer conditions;*
- (v) *$\lim_{x \rightarrow \infty} (1/x) \int_0^x \lambda(u) du > 0$, where λ is defined by [\(2.1\)](#).*

Proof. It is well known (see [\[3\]](#)) that the Cramer conditions hold if and only if there exists an $r > 0$ such that $E \exp\{r\xi\} < \infty$. In turn, the last inequality is equivalent to

$$1 - F_\xi(x) \leq c \exp\{-rx\}, \quad x \geq 0, \tag{2.10}$$

for some $0 < r < \infty$ and $0 < c < \infty$, since $E \exp\{r\xi\} \geq \int_x^\infty \exp\{ru\} dF_\xi(u) \geq \exp\{rx\}(1 - F_\xi(x))$, and $E \exp\{\bar{r}\xi\} = 1 + \bar{r} \int_0^\infty (1 - F_\xi(u)) \exp\{\bar{r}u\} du$, $\bar{r} \in (0, r)$.

By the same arguments, we obtain that [\(2.10\)](#) is equivalent to

$$\int_x^\infty (1 - F_\xi(u)) du \leq c \exp\{-rx\}, \quad x \geq 0, \tag{2.11}$$

for some $0 < r < \infty$ and $0 < c < \infty$. From relation [\(2.3\)](#), we deduce that the Cramer conditions are equivalent to the inequality

$$\int_0^x \lambda(u) du \geq rx + c_1, \quad x \geq 0, \tag{2.12}$$

for some $r > 0$, $c_1 \in \mathbb{R}$, that is, equivalent to (v). □

As mentioned above, the Cramer conditions follow from the Garsia inequality. The next example demonstrates that the converse is not true.

Example 2.6. There exists an integrable nonnegative r.v. ξ satisfying the Cramer conditions but not the Garsia inequality. The idea is the following. If we choose a nonnegative left-continuous function $\lambda(x)$ such that $\ln \lambda(x) - \int_0^x \lambda(u) du$ decreases to $-\infty$, and $\int_0^x \lambda(u) du \geq rx + c_1$, $x \geq 0$, for some $r > 0$, $c_1 \in \mathbb{R}$, but $\lim_{x \rightarrow \infty} \lambda(x) = 0$, then the right-hand side of [\(2.2\)](#) will be a tail of some d.f. $F_\xi(x)$. This d.f. will satisfy the Cramer conditions by [Theorem 2.5](#), but the Garsia inequality will not hold by assumption (ii) of [Theorem 2.2](#).

Three conditions mentioned above will be satisfied if the function λ is a solution of the boundary value problem

$$-\lambda'(x) + \lambda^2(x) = \lambda(x)\varphi(x), \quad x \geq 0, \quad \lambda(0) = \frac{1}{3}, \tag{2.13}$$

where $\varphi(x) = \sum_{n \geq 0} I_{[2n, 2n+1)}(x)$.

The corresponding d.f. $F_\xi(x)$ can be calculated from (2.2) as

$$\begin{aligned}
 1 - F_\xi(x) &= E\xi \cdot \lambda(0) \cdot \exp \left\{ \int_{0+}^x (d \ln \lambda(y) - \lambda(y) dy) \right\} \\
 &= \frac{1}{2} E\xi \exp \left\{ - \int_0^x \varphi(y) dy \right\} \sim c \exp \left\{ - \frac{1}{2} x \right\},
 \end{aligned}
 \tag{2.14}$$

whence the Cramer conditions hold.

The solution of (2.13) on the intervals $[2n, 2n + 1)$ and $[2n + 1, 2n + 2)$ equals

$$\lambda(x) = \begin{cases} \lambda(2n)e^{2n}(\lambda(2n)e^{2n} + (1 - \lambda(2n))e^x)^{-1}, & x \in [2n, 2n + 1), \\ \lambda(2n + 1)((2n + 1) \cdot \lambda(2n + 1) + 1 - \lambda(2n + 1)x)^{-1}, & x \in [2n + 1, 2n + 2), \end{cases}
 \tag{2.15}$$

supposing that $\lambda \in C(\mathbb{R}_+)$. Evidently, $\lambda(x)$ increases on $[2n + 1, 2n + 2)$, and decreases on $[2n, 2n + 1)$, $n \geq 0$. The values of $\lambda(2n + 1)$ at the points $x_n = 2n + 1$ of a local minimum can be estimated from the relations $\lambda(1) = (1 + 2e)^{-1} < 1/6$; $\lambda(2n + 1) = (1 + ((\lambda(2n))^{-1} - 1)e)^{-1}$, $\lambda(2n) = \lambda(2n - 1)(1 - \lambda(2n - 1))^{-1}$, whence $\lambda(2n + 1) = (1 + ((\lambda(2n - 1))^{-1} - 2)e)^{-1}$. Suppose that $\lambda(2n - 1) < 1/(n + 5)$ (it is true for $n = 1$), then $\lambda(2n + 1) < (1 + (n + 3)e)^{-1} < 1/(n + 6)$. It means that $\lambda(2n + 1) \rightarrow 0, n \rightarrow \infty$, and $\underline{\lim}_{x \rightarrow \infty} \lambda(x) \leq \underline{\lim}_{n \rightarrow \infty} \lambda(2n + 1) = 0$, so the Garsia inequality does not hold.

THEOREM 2.7. *Let nonnegative unbounded r.v. ξ satisfy the Cramer conditions. Then $\overline{\lim}_{x \rightarrow \infty} \lambda(x) > 0$.*

Proof. Choose an $r > 0$ such that

$$E \exp \{ r\xi \} < \infty. \tag{2.16}$$

Then from (2.2) and Raikov theorem (see [3]), for any $0 < \bar{r} < r$, $(1 - F_\xi(x)) \exp \{ \bar{r}x \} \rightarrow 0, x \rightarrow \infty$, and

$$\begin{aligned}
 E \exp \{ \bar{r}\xi \} &= \bar{r} \int_0^\infty \exp \{ \bar{r}x \} (1 - F_\xi(x)) dx \\
 &= \bar{r} \int_0^\infty \exp \left\{ \bar{r}x - \int_0^x \lambda(u) du \right\} \lambda(x) dx \cdot E\xi.
 \end{aligned}
 \tag{2.17}$$

Suppose that $\lambda(x) \rightarrow 0, x \rightarrow \infty$. Choose any $\varepsilon \in (0, \bar{r})$. Then there exists $x_0 > 0$ such that $\lambda(x) < \varepsilon$, for $x > x_0$, whence

$$\begin{aligned}
 &\int_0^\infty \exp \left\{ \bar{r}x - \int_0^x \lambda(u) du \right\} \lambda(x) dx \\
 &\geq \exp \left\{ \int_0^{x_0} (r - \lambda(u)) du \right\} \cdot \int_0^\infty \exp \{ (r - \varepsilon)(x - x_0) \} \lambda(x) dx \\
 &> \exp \left\{ \int_0^{x_0} (r - \lambda(u)) du \right\} \cdot \int_0^\infty \lambda(x) dx = \infty.
 \end{aligned}
 \tag{2.18}$$

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Relations (2.17) and (2.18) contradict (2.16). So our assumption is false, whence the proof follows. \square

The next example demonstrates that the inequality $\overline{\lim}_{x \rightarrow \infty} \lambda(x) > 0$ is not equivalent to the Cramer conditions.

Example 2.8. Denote

$$\begin{aligned} x_n &:= \frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2}, \\ y_n &:= \frac{(n+1)n(2n+1)}{6} + \frac{n(n-1)}{2}, \quad n \geq 1. \end{aligned} \quad (2.19)$$

Let $\lambda(0) = 1$,

$$\lambda(x) = \begin{cases} \frac{1}{\sqrt{n}}, & x_n < x \leq y_n \\ \frac{1}{\sqrt{n}} + \frac{1}{n}(x - y_n), & y_n \leq x \leq x_{n+1}, \quad n \geq 1. \end{cases} \quad (2.20)$$

Then $\overline{\lim}_{x \rightarrow \infty} \lambda(x) \geq \lim_{n \rightarrow \infty} \lambda(x_{n+1}) = \lim_{n \rightarrow \infty} (1 + 1/\sqrt{n}) = 1$.

Denote $\psi(x) := \lambda'(x) - \lambda^2(x)$. Then $\psi(x) \leq 1/n - (1/\sqrt{n})^2 = 0$ and λ has only negative jumps. Therefore, $\lambda(x) \exp\{-\int_0^x \lambda(u) du\}$ is nonincreasing. It means that $1 - F_\xi(x)$ is considered as the left-hand side of (2.1), with $E\xi = 1$ being nonincreasing. Moreover, for $x_n < x < x_{n+1}$, $n \geq 2$,

$$\begin{aligned} & \frac{1}{\sqrt{n}} \exp \left\{ - \sum_{k=1}^n \left(\frac{1}{\sqrt{k}} k^2 + \frac{k}{2} + \sqrt{k} \right) \right\} \\ & \leq 1 - F_\xi(x) \leq \left(\frac{1}{\sqrt{n}} + 1 \right) \exp \left\{ - \sum_{k=1}^{n-1} \left(\frac{1}{\sqrt{k}} k^2 + \frac{k}{2} + \sqrt{k} \right) \right\}. \end{aligned} \quad (2.21)$$

From the right-hand side of (2.21), $1 - F_\xi(x) \leq \exp\{-(2/5)(n-1)^{5/2}\}$, and from the left-hand side of (2.21),

$$1 - F_\xi(x) > \frac{1}{\sqrt{n}} \exp \left\{ - \frac{6}{5}(n+1)^{5/2} \right\}, \quad x_n < x \leq x_{n+1}. \quad (2.22)$$

Evidently, $x_n > (n-1)^3/3$, $x_{n+1} < (5/6)(n+1)^3$. So, for sufficiently large x ,

$$1 - F_\xi(x) \leq \exp \left\{ - \frac{2}{5} \left(\left(\frac{6}{5}x \right)^{1/3} - 2 \right)^{5/2} \right\} \sim \exp \{ - \alpha x^{5/6} \} \quad (2.23)$$

for some constant $\alpha > 0$. Therefore ξ is integrable r.v. On the other hand,

$$1 - F_\xi(x) > ((3x)^{1/3} + 1)^{-1/2} \exp \left\{ -\frac{6}{5}((3x)^{1/3} + 2)^{5/2} \right\} \sim \beta x^{-1/6} \exp \{ -\gamma x^{5/6} \} \quad (2.24)$$

for some constants $\beta, \gamma > 0$. It means that $E \exp\{r\xi\} = \infty$ for any $r > 0$; so ξ does not satisfy the Cramer conditions.

3. What random variable generates a bounded potential?

Definition 3.1. A r.v. ξ is said to generate a bounded potential if there exists a filtration $\{G_t, t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies the standard conditions, and a bounded right-continuous G_t -adapted potential X_t that admits the expansion (compare with (1.2))

$$X_t = E(A_\infty - A_t | G_t), \quad (3.1)$$

with a G_t -adapted nondecreasing integrable process A_t such that $A_\infty = \xi$.

[Definition 3.1](#) means, in particular, that ξ is G_∞ -adapted.

Definition 3.2. A r.v. ξ is said to generate a nontrivial bounded potential if the potential X_t in expansion (3.1) can be chosen in such a way that $\mathbb{P}\{X_t > 0\} > 0$ for any $t > 0$.

Definition 3.3. A r.v. ξ is said to predictably generate a bounded potential if the process A_t in expansion (3.1) can be chosen to be G_t -predictable.

Note that for a given X_t , the predictable process in (3.1) is unique.

Denote G_0 the family of all \mathbb{P} zero sets of \mathcal{F} .

LEMMA 3.4. *Any integrable r.v. ξ generates a bounded potential.*

Proof. Choose any $t_0 > 0$ and set $A_t = \xi I\{t \geq t_0\}$, $G_t = \overline{G_0} \vee \Omega$ for $t < t_0$, $G_t = \mathcal{F}$ for $t \geq t_0$. Then $A_\infty = \xi$, $E(A_\infty - A_t | G_t) = 0$ for $t \geq t_0$ and $E(A_\infty - A_t | G_t) = E\xi$ for $t < t_0$. So, $X_t = E\xi I\{t < t_0\}$ is a bounded potential. (Note that X_t is trivial in terms of [Definition 3.2.](#)) □

THEOREM 3.5. *Let a r.v. ξ satisfy Cramer conditions. Then ξ generates a nontrivial bounded potential.*

Proof. According to [Theorem 2.7](#) we can choose some $c > 0$ and a sequence $\pi = \{t_n, n \geq 0\}$, $t_0 = 0$, $t_n \uparrow \infty$ such that $\int_{t_n}^\infty (1 - F_\xi(u)) du (1 - F_\xi(t_n))^{-1} \leq c$.

Define

$$\begin{aligned} G_t &:= \begin{cases} \overline{G_0}, & 0 \leq t < t_1, \\ \sigma\{\xi \wedge t_k, 1 \leq k \leq n\}, & t_n \leq t < t_{n+1}, \end{cases} \\ A_t &:= \begin{cases} 0, & 0 \leq t < t_1, \\ \xi \wedge t_n, & t_n \leq t < t_{n+1}, \end{cases} \end{aligned} \quad (3.2)$$

where $n \geq 1$. Then G_t satisfies the standard conditions, A_t is a G_t -adapted nonnegative nondecreasing right-continuous process, and $A_\infty = \xi$. Now, let $X_t := E(A_\infty - A_t | G_t)$.

Since for any $k < m$ and $\alpha < t_k$, the event $\{\xi \wedge t_k < \alpha\} = \{\xi < \alpha \wedge t_k\} = \{\xi \wedge t_m < \alpha \wedge t_k\}$, we have that $G_t = \sigma\{\xi \wedge t_n\}$ if $t_n \leq t < t_{n+1}$. Therefore, for $t_n \leq t < t_{n+1}$,

$$\begin{aligned} X_t &= E(\xi - \xi \wedge t_n \mid \xi \wedge t_n) \\ &= \int_{t_n}^{\infty} (u - t_n) dF_{\xi}(u) (1 - F_{\xi}(u))^{-1} I\{\xi > t_n\} \\ &= \int_{t_n}^{\infty} (1 - F_{\xi}(u)) du (1 - F_{\xi}(u))^{-1} I\{\xi > t_n\} \leq c I\{\xi > t_n\} \leq c. \end{aligned} \tag{3.3}$$

We conclude that the potential X_t is bounded and $\mathbb{P}\{X_t > 0\} > \mathbb{P}\{\xi > t_n\} > 0$, $t_n \leq t < t_{n+1}$; so X_t is nontrivial. \square

THEOREM 3.6. *Let a r.v. ξ satisfy the Garsia inequality. Then ξ predictably generates a non-trivial potential.*

Proof. Let $G_t = \sigma\{\xi \wedge s, s \leq t\}$. Analogously to the proof of [Theorem 3.5](#), we can establish that $G_t = \sigma\{\xi \wedge t\}$. Also, let $A_t := \xi \wedge t$, $A_{\infty} = \xi$. According to [Theorem 2.2](#) and representation [\(2.1\)](#),

$$X_t := E(A_{\infty} - A_t \mid G_t) = E(\xi - \xi \wedge t \mid \xi \wedge t) I\{\xi > t\} = (\lambda(t))^{-1} I\{\xi > t\} \leq c_0. \tag{3.4}$$

Evidently, $\mathbb{P}\{X_t > 0\} > 0$, $t > 0$, and A_t is continuous. Therefore, it is a predictable process and the proof follows. \square

Remark 3.7. The σ -fields G_t participating in the proof of [Theorem 3.6](#) were considered by Dellacherie [[1](#)], where the dual predictable projection of the process $A = I_{[\xi, \infty[}$ with respect to $\{G_t, t \geq 0\}$ was constructed.

The next example demonstrates that the same r.v. ξ can generate rather different potentials with respect to the same filtration $\{G_t, t \geq 0\}$.

Example 3.8. Let $\{P_t, t \geq 0\}$ be a homogeneous Poisson process with the intensity λ_0 , $\theta(x) = \exp\{-\lambda_1 x\}$, $\lambda_0, \lambda_1 > 0$. Define $\xi := \int_0^{\infty} \theta(u) P_u du = \lambda_1^{-1} \int_0^{\infty} \theta(u) dP_u$. Then $E\xi = \lambda_0 \int_0^{\infty} u \theta(u) du = \lambda_0 \lambda_1^{-2}$.

Also, let $G_t = \sigma\{P_s, 0 \leq s \leq t\}$. If we put $A_t^0 := \int_0^t \theta(u) P_u du$, $A_{\infty}^0 := \xi$, then the process A_t is predictable, and the potential $X_t^0 := E(A_{\infty}^0 - A_t^0 \mid G_t) = P_t \int_t^{\infty} \theta(u) du = \lambda_1^{-1} \theta(t) P_t$ is nontrivial and unbounded for any $t > 0$. With $A_t^1 := \lambda_1^{-1} \int_0^t \theta(u) dP_u$, $A_{\infty}^1 = \xi$, A_t^1 is not predictable, and the potential $X_t^1 := E(A_{\infty}^1 - A_t^1 \mid G_t) = \lambda_0 \lambda_1^{-1} \int_t^{\infty} \theta(u) du = \lambda_0 \lambda_1^{-1} \theta(t)$ is non-trivial and bounded.

4. Some additional results

The next theorem states that in the case when $\overline{\lim}_{x \rightarrow \infty} \lambda(x) > 0$, an integrable r.v. ξ can be transformed into a r.v. $\psi(x)$ that predictably generates a bounded potential. Note that in this case we can choose the sequence $\pi = \{t_n, n \geq 0\}$, $t_0 = 0$, $t_n \uparrow \infty$ such that

$$\lambda(t_n) \geq c > 0, \quad \delta_{\pi} := \inf_{n \geq 1} (t_n - t_{n-1}) > 0. \tag{4.1}$$

THEOREM 4.1. *Let $\overline{\lim}_{x \rightarrow \infty} \lambda(x) > 0$, let $\pi\{t_n, n \geq 0\}$ be chosen as in (4.1). Denote by $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an increasing piece-wise linear function such that $\psi(t_n) = n$.*

Then $\psi(\xi)$ predictably generates a nontrivial bounded potential.

Proof. The d.f. of the r.v. $\psi(\xi)$ equals $F_\psi(x) = \mathbb{P}\{\psi(\xi) \leq x\} = \mathbb{P}\{\xi \leq \psi^{-1}(x)\}$. Therefore,

$$\begin{aligned} & \int_n^\infty (1 - F_\psi(u)) du \cdot (1 - F_\psi(n))^{-1} \\ &= \int_n^\infty (1 - F_\xi(\psi^{-1}(u))) du \cdot (1 - F_\xi(t_n))^{-1} \\ &= \int_{t_n}^\infty (1 - F_\xi(u)) \psi'(u) du \cdot (1 - F_\xi(t_n))^{-1}. \end{aligned} \tag{4.2}$$

Furthermore, for $t \in (t_{n-1}, t_n)$,

$$\psi'(t) = (\psi(t_n) - \psi(t_{n-1})) (t_n - t_{n-1})^{-1} = (t_n - t_{n-1})^{-1} \leq \delta_\pi^{-1}. \tag{4.3}$$

We obtain from (4.2) and (4.3) that there exists $c_1 > 0$ such that $(\lambda_\psi(n))^{-1} := \int_n^\infty (1 - F_\psi(u)) du (1 - F_\psi(n))^{-1} \leq c_1$. Note that $x(1 - F_\psi(x)) = x(1 - F_\xi(\psi^{-1}(x))) = \psi^{-1}(x)(1 - F_\xi(\psi^{-1}(x))) \cdot (x(\psi^{-1}(x))^{-1})$, and $\psi^{-1}(x)(1 - F_\xi(\psi^{-1}(x))) \rightarrow 0, x \rightarrow \infty, x(\psi^{-1}(x))^{-1} \leq (n+1)/\psi^{-1}(n) = (n+1)/t_n < (n+1)/\delta_\pi n \leq c_2, n < t \leq n+1$.

It means that $\int_0^\infty u dF_\psi(u) = \int_0^\infty (1 - F_\psi(u)) du < \infty$, that is, ψ is integrable.

Now, let $G_t = \sigma\{\psi(\xi) \wedge n\}, n \leq t < n+1$, and

$$A_t := \begin{cases} 0, & 0 \leq t < 2, \\ \psi(\xi) \wedge (n-1), & n \leq t < n+1. \end{cases} \tag{4.4}$$

Then A_t is G_t -predictable, $A_\infty = \psi(\xi)$, and for any $t > 0$,

$$\begin{aligned} X_t &:= E(A_\infty - A_t | G_t) \\ &= E(\psi(\xi) - \psi(\xi) \wedge n | \psi(\xi) \wedge n) + (\psi(\xi) \wedge n - \psi(\xi) \wedge (n-1)) \leq c_1 + 1. \end{aligned} \tag{4.5}$$

□

The next result demonstrates that an integrable r.v. ξ with $\overline{\lim}_{x \rightarrow \infty} \lambda(x) > 0$ generates an “almost bounded” potential.

THEOREM 4.2. *Let $\overline{\lim}_{x \rightarrow \infty} \lambda(x) > 0$. Then for any $\varepsilon > 0$, the r.v. ξ predictably generates a nontrivial potential that is bounded on the set $\mathbb{R}_+ \setminus B$ where $m(B) < \varepsilon$ ($m(\cdot)$ is a Lebesgue measure on \mathbb{R}).*

Proof. Consider $G_t := \sigma\{\xi \wedge t_n\}, t_n - \varepsilon/2^n \leq t < t_{n+1} - \varepsilon/2^{n+1}$, and A_t being the same as in the proof of Theorem 3.5. Then A_t is G_t -predictable. Furthermore, $X_t := E(A_\infty - A_t | G_t) = E(\xi - \xi \wedge t_n | \xi \wedge t_n) \leq c$ for $t_n \leq t < t_{n+1} - \varepsilon/2^{n+1}$. □

Remark 4.3. Let the r.v. ξ be such that $0 = \underline{\lim}_{x \rightarrow \infty} \lambda(x) < \overline{\lim}_{x \rightarrow \infty} \lambda(x)$. Consider G_t and A_t from the proof of [Theorem 3.5](#) and construct the compensator (dual predictable projection) A_t^π of A_t . Evidently,

$$\begin{aligned} A_t^\pi &= \sum_{k=1}^n E(\xi \wedge t_k - \xi \wedge t_{k-1} | \xi \wedge t_{k-1}) \\ &= \sum_{k=1}^n \int_{t_{n-1}}^{t_n} (x - t_{n-1}) dF_\xi(x) (1 - F_\xi(t_{n-1}))^{-1} I\{\xi > t_{n-1}\}, \quad t_n \leq t < t_{n+1}, \quad (4.6) \\ A_\infty^\pi &= \sum_{k=1}^n \int_{t_{n-1}}^{t_n} (x - t_{n-1}) dF_\xi(x) (1 - F_\xi(t_{n-1}))^{-1} I\{\xi > t_{n-1}\}. \end{aligned}$$

Since $A_t - A_t^\pi$ is a G_t -martingale, we obtain that

$$X_t := E(A_\infty - A_t | G_t) = E(A_\infty^\pi - A_t^\pi | G_t) \leq c. \quad (4.7)$$

The last relation means that A_∞^π satisfies the Garsia inequality.

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References

- [1] C. Dellacherie, *Capacités et Processus Stochastiques*, Springer-Verlag, Berlin, 1972.
- [2] A. M. Garsia, *Martingale Inequalities: Seminar Notes on Recent Progress*, W. A. Benjamin, Reading, Massachusetts, 1973.
- [3] B. Ramachandran, *Advanced Theory of Characteristic Functions*, Statistical Publishing Society, Calcutta, 1967.

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