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A NON-MARKOVIAN QUEUEING SYSTEM WITH A VARIABLE NUMBER OF CHANNELS

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In this paper we study a queueing model of type GI/M/$\tilde{m}_a/\infty$ with m parallel channels, some of which may suspend their service at specified random moments of time. Whether or not this phenomenon occurs depends on the queue length. The queueing process, which we target, turns out to be semi-regenerative, and we fully explore this utilizing semi-regenerative techniques. This is contrary to the more traditional supplementary variable approach and the less popular approach of combination semi-regenerative and supplementary variable technique. We pass to the limiting distribution of the continuous time parameter process through the embedded Markov chain for which we find the invariant probability measure. All formulas are analytically tractable.

Keywords: Multi-Channel Queue, Variable Number of Channels, Limiting Distribution, Semi-Regenerative Process, Queueing Process.

AMS (MOS) subject classification: 60K25, 60K10, 65K10, 90B25.

1 Introduction

This paper analyzes a multi-channel queueing system with a random number of channels, infinite capacity waiting room, general input, and exponentially distributed service times. The total number of channels does not exceed m, but at any given time not all of them are "active." The latter implies that even if a particular channel is busy servicing a customer, the service at some later time can become suspended. We assume that there is a certain sequence $\{T_n\}$ of stopping times relative to the queueing process at which a decision is being made for every busy channel to continue, suspend, or activate service. This policy literally makes the total quantity of servers random and it affects both the servicing and queueing processes.

More formally, if Z_t denotes the number of customers in the system at any given time t, the number of active channels at T_n (where $\{T_n\}$ is the arrival process) is a binomial random variable with parameters $(Z_{T_n},\ a)$ where $0 \leq a \leq 1$, provided that $Z_t \leq m$. Unless service is interrupted, each of the customers is being processed at an exponentially distributed time and service durations on each of the active channels are independent. The input is a regular renewal process. For this system we use the symbolic notation GI/M/$\tilde{m}_a/\infty$.

The system, as it is, generalizes the classical GI/M/m/∞ queue by making it, in some real-world applications, more versatile. We can easily associate it with any mail order servicing system where once the order has been taken, it can be suspended at any moment of time for an unaware customer (who believes he is being processed) for various reasons, most commonly due to unavailable items. An item can also be back-ordered. The company is trying to shop around and find the item, and this takes time and effort. A similar situation occurs in Internet service where jobs are being routinely suspended for numerous reasons by Internet providers.

In this problem setting we make the consequences of this interruption policy milder, in which suspensions take place only if the buffer (or waiting room) is empty, but all or a few channels are occupied. It makes perfect sense to reduce waiting time for many customers from the buffer.

The queueing systems with variable number of channels have been investigated in the past literature. Saati [4] describes such one as a fully exponential system. Most commonly, the queues with unreliable servers (which can break down at any time) and priorities can also fall into this category. Our system is different firstly because it is non-Markovian and secondly because service interruptions do not take place fully arbitrarily, but with some probabilities upon certain random times. The closest problem to ours is the system in Rosson/Dshalalow [3] with no buffer.

In the present paper, we focus our attention on the queueing process, which turns out to be semi-regenerative relative to the sequence $\{T_n\}$ of arrivals. We start with the embedded process over $\{T_n\}$ and turn to the analysis of the queue as a semi-regenerative process, which to the best of our knowledge has not been analyzed this way even in the case of the basic GI/M/m/∞ system, and thus this method is by itself novel for the class of multi-channel queues. We have used this approach in Rosson/Dshalalow [3] for the case of a more rudimentary GI/M/m/0 system with a random number of channels. The paper is organized as follows. Section 2 formalizes the model more rigourously. Section 3 deals with an embedded Markov chain and the invariant probability measure under a given ergodicity condition. Section 4 analyzes the continuous time parameter queueing process followed by an example presented in Section 5. All formulas are obtained in analytically tractable forms.

2 Description of the System and Notation

2.1 Description of the system

Let $(\Omega, \mathcal{F}, (P^x)_{x=0,1,\ldots}, Z_t;\ t \geq 0) \rightarrow \mathcal{E} = \{0, 1, \ldots\}$ be the stochastic process which describes the evolution of the queueing process in the GI/M/$\tilde{m}_a/\infty$ system introduced in the previous section. In other words, at any time t, Z_t gives the total number of units (or customers) in the system including those being in service. The servicing facility has m permanent channels, of which not all are necessarily active. The buffer (or waiting room) is of an infinite capacity, but the system needs to be "watched" for preserving the equilibrium condition. Customers arrive singly in the system in accordance with a standard renewal process $\{T_n;\, n = 0, 1, ..., T_0 = 0\}$. Inter-renewal times have a common PDF (probability distribution function) $A(x)$, with finite mean $\bar{a}$ and the LST (Laplace-Stieltjes transform) $\alpha(\theta)$. The formation of active or inactive channels is being rendered upon $\{T_n\}$ as follows. If at time T_n- (i.e. immediately preceding the nth arrival) the total number of customers is less than m, each of the channels, including the one that is aimed to accommodate the nth customer (just arrived), can become temporarily inactive with probability b. In such a case, the service of a customer by any such channel becomes suspended until the next stopping time T_{n+1} (of the process Z_t). Every busy channel is active with probability a $(= 1 - b)$ and he is processing a customer a period of time exponentially distributed with parameter μ until T_{n+1} or the end of service, whichever comes first. Thereby, service times at each of the parallel channels within the random interval $[T_n, T_{n+1})$ are conditionally independent, given $X_n := Z_{T_n-}$ and the number of active channels is governed by a binomial random variable ξ_n with parameters $(X_n + 1, a)$ provided that $X_n < m$. If the number of customers in the system upon time T_n- is m or more, then all channels are almost surely active until T_{n+1}.

2.2 Notation

We will be using the following notation throughout the remainder of this paper:

$a =$ probability of each busy channel at T_{n+} to be active, $n = 0, 1, \ldots$.

$b = 1 - a =$ probability of a channel at T_{n+} to be inactive.

$\widehat{\beta}_0 = \gamma_m = \int_0^\infty [p(x)]^m dA(x)$

$\beta_s = \int_0^\infty e^{-m\mu x} \frac{(m\mu x)^s}{s!} dA(x),\ s = 0,\ 1,\ \ldots,$

$p(x) = b + a\ e^{-\mu x}$

$q(x) = (1 - e^{-\mu x})\ a.$

$\alpha_r = \alpha(r\mu),$

$\alpha\ (\theta) = \int_0^\infty e^{-\theta u} A(du).$

$\gamma_r = \int_0^\infty [p(x)]^r dA(x), r\ =\ 0,\ 1,\ \ldots.$

$\widehat{\alpha}_r = \int_0^\infty [p(x)]^r [1 - A(x)]dx.$

$$a_r = \begin{cases} 1, & r = 0, \\ \prod_{j=1}^{r} \frac{\gamma_j}{1-\gamma_j}, & r > 0. \end{cases}$$

3 Embedded Process

As a precursor to the key semi-regenerative approach for treating the one-dimensional distribution of the queueing process

$$(\Omega, \mathcal{F}, (P^x)_{x=0,1,...}, Z_t; \quad t \geq 0) \to \mathcal{E} = \{0, 1, ...\},$$

we begin with the process

$$(\Omega, \mathcal{F}, (P^x)_{x=0,1,\ldots}, X_n = Z_{T_n-}; n = 0, 1, ...) \to \mathcal{E} = \{0, 1, ...\}$$

embedded in (Z_t) over the almost surely strictly monotone increasing sequence $\{T_n\}$ of stopping times relative to (Z_t). As in the case of the usual GI/M/m/∞ queue, (Z_t) is semi-regenerative with respect to $\{T_n\}$, and as a consequence, $\{X_n\}$ is a Markov chain. It is obviously time-homogeneous.We now turn to the TPM (transition probability matrix) of $\{X_n\}$.

3.1 Transition Probabilities

Let

$$q_s^r = \binom{r}{s} a^{r-s} b^s, \; s = 0, ..., r, \quad r = 0, ..., m-1$$

(binomial distribution; commonly denoted by $b(r, 1-a; s)$ in the literature) and let

$$B_{jk} = \int_0^\infty \binom{j}{k} (1 - e^{-\mu x})^{j-k} e^{-\mu k x}\, dA(x), k = 0, 1, \ldots, j+1; \tag{3.1}$$

$$j = 0, 1, ..., m-1.$$

Then,

$$p_{jk} = \sum_{s=0}^{k} q_s^{j+1} B_{j+1-s,k-s} \tag{3.2}$$

$$= \sum_{s=0}^{k} \binom{j+1}{s} a^{j+1-s} b^s \int_0^\infty \binom{j+1-s}{k-s} \left(1 - e^{-\mu x}\right)^{j+1-k} e^{-\mu (k-s) x}\, dA(x),$$

$$k = 0, \ldots, j+1;\; j = 0, \ldots, m-1,$$

where q_s^{j+1} is the probability that s out of $j+1$ busy channels are inactive at T_n, $(1-e^{-\mu x})^{j+1-k}$ is the probability that in $[0,x]$, $j+1-s-(k-s)$ customers are processed, and $e^{-\mu(k-s)x}$ is the probability that in $[0,x]$, $k-s$ customers are not finished while being treated. Clearly, $p_{jk} = 0$ for

$$k > 0,\; j = 0, \ldots, m-1. \tag{3.3}$$

Notice that

$$p_{m-1,m} = \int_0^\infty [p(x)]^m dA(x) = \widehat{\beta}_0 = \gamma_m, \tag{3.4}$$

where

$$\widehat{\beta}_0 = p_{m-1,m} = \sum_{s=0}^{m} B_{m-s,m-s} q_s^m$$

$$= \sum_{s=0}^{m} \int_0^\infty \binom{m-s}{m-s} e^{-\mu(m-s)x} dA(x) \binom{m}{s} a^{m-s} b^s$$

$$= \int_0^\infty (b + a e^{-\mu x})^m dA(x) = \widehat{\beta}_0 = \gamma_m,$$

and that $\widehat{\beta}_0 = \beta_0 = \alpha_m = \int_0^\infty e^{-\mu m x} dA(x)$ if $a = 1$.

The above probabilities (3.1)-(3.4) form the upper block

$$L_1 = \begin{bmatrix} p_{00} & p_{01} & 0 & \cdots & 0 & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots & 0 & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots & 0 & \cdots \\ \vdots & \vdots & \cdots & & & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m-1,0} & p_{m-1,1} & p_{m-1,2} & \cdots & p_{m-1,m-1}\, p_{m-1,m} = \widehat{\beta}_0 & 0 \end{bmatrix} \tag{3.5}$$

of the TPM (transition probability matrix) $L = (p_{ij})_{i,\, j \in \mathcal{E}}$.

The lower block of L with the rows from m and all the way down are identical to that for the system GI/M/m/∞:

$$L_2 = \begin{bmatrix} p_{m,0} & p_{m,1} & \cdots & p_{m,m} & \beta_1 & \beta_0 & 0 & 0 & \cdots \\ p_{m+1,0} & p_{m+1,1} & \cdots & p_{m+1,m} & \beta_2 & \beta_1 & \beta_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{3.6}$$

with

$$\beta_j = \int_{x=0}^{\infty} e^{-\mu m x} \frac{(m\mu x)^j}{j!}\, dA(x)\, j = 0, 1, \ldots, \tag{3.7}$$

and

$$p_{ij} = \int_{x=0}^{\infty} \begin{pmatrix} m \\ j \end{pmatrix} e^{-\mu j x} \int_{y=0}^{x} (e^{-yx} - e^{-\mu x})^{m-j} m\mu \frac{(m\mu y)^{i-m}}{(i-m)!} dy dA(x), \tag{3.8}$$

$$i = m, \ldots;\ j = 0, 1, \ldots, m.$$

The PGF (probability generation function) of the i^{th} row of L_1 is

$$p_i(z) = \int_0^\infty [q\,(x) + p\,(x) z]^{i+1} dA(x),\ i = 0, \ldots, m-1 \tag{3.9}$$

where

$$q(x) = (1 - e^{-\mu x})a \tag{3.10}$$

and

$$p(x) = b + ae^{-\mu x}. \tag{3.11}$$

We can easily deduce that L represents an irreducible and aperiodic MC (Markov chain).

3.2 The Invariant Probability Measure

According to Abolnikov and Dukhovny [1], for $m\mu\alpha > 1$, there exists an invariant probability measure $P := (P_0, P_1, \ldots)$ of $\{X_n\}$ as a unique positive solution of the matrix equation

$$\begin{cases} P = PL \\ (P, 1) = 1, \quad P \in R^{\mathcal{E}}. \end{cases} \tag{3.12}$$

(3.12) also reads

$$P_k = \sum_{i=k-1}^{\infty} p_{ik} P_i, \quad k = 0, \quad 1, \ldots, \tag{3.13}$$

with

$$p_{-10} = P_{-1} = 0, \tag{3.14}$$

and

$$\sum_{k=0}^{\infty} P_i = 1. \tag{3.15}$$

For $k \geq m$, (3.13), in light of (3.6-3.8), leads to

$$P_k = \begin{cases} \widehat{\beta}_0\, P_{m-1} + \sum_{i \geq m} \beta_{i+1-k} P_i, & k = m, \\ \sum_{i=k-1} \beta_{i+1-k} P_i, & k > m. \end{cases} \tag{3.16}$$

Let us now consider equations (3.16) for $k \geq m$. We will seek the solution of (3.16) in the form

$$P_k = A\,\delta^{k-m},\ k \geq m \tag{3.17}$$

where A will be evaluated from $(P, 1) = 1$. Inserting (3.17) into (3.16) gives,

$$A\delta^{k-m} = \sum_{s=0}^{\infty} \beta_s A \delta^{s+k-1-m}, \quad k > m, \tag{3.18}$$

which yields

$$\delta = \alpha(m\mu - m\mu\delta) \tag{3.19}$$

where δ, according to Takács [6], is a unique, real positive root of (3.19), strictly less than 1 when meeting the ergodicity condition $m\mu\alpha > 1$. For $k = m$, (3.16) and (3.17) yield

$$A = \widehat{\beta}_0 P_{m-1} + \sum_{i \geq m}^{\infty} \beta_{i+1-m} A \delta^{i-m} \tag{3.20}$$

$$= \widehat{\beta}_0 P_{m-1} - A\beta_0 \delta^{-1} + \frac{1}{\delta} \sum_{i \geq m-1}^{\infty} \beta_{i+1-m} A \delta^{i+1-m},$$

which, after some algebra and due to (3.19), reduces to

$$P_{m-1} = A\delta^{-1} \frac{\beta_0}{\widehat{\beta}_0}. \tag{3.21}$$

(If $\widehat{\beta}_0 = \beta_0$ and $P'_{m-1} = 1$, then $P_{m-1} = A\,\delta^{-1}$ as in GI/M/m/∞.)

To determine the unknowns $P_0, P_1, \ldots, P_{m-1}$ let us define the PGF

$$U(z) = \sum_{k=0}^{m-1} z^k P_k. \tag{3.22}$$

The following proposition gives an equation in $U(z)$ and P_i, $i = 0, \ldots, m-1$, which lead to the derivation of the invariant probabilities.

Proposition 3.1: *It holds true that*

$$U(z) = \sum_{i=0}^{m-1} P_i \int_0^\infty [p(x)z + q(x)]^{i+1} dA(x) + AW(z), \tag{3.23}$$

where

$$W(z) = m\mu \int_{x=0}^\infty \int_{y=0}^x e^{m\mu y\delta}(e^{-\mu y} - e^{-\mu x} + ze^{-\mu x}) dy dA(x) - z^m. \tag{3.24}$$

Proof: From (3.13) ,

$$U(z) = \sum_{k=0}^{m-1} z^k P_k = \sum_{k=0}^{m-1} z^k \sum_{i=k-1}^\infty p_{ik}\, P_i = C(z) + D(z), \tag{3.25}$$

where

$$C(z) = \sum_{i=0}^{m-1} P_i \sum_{k=0}^{i+1} p_{ik}\, z^k - p_{m-1,m}\, z^m P_{m-1} \tag{3.26}$$

and

$$D(z) = \sum_{k=0}^{m-1} z^k \sum_{i\geq m}^\infty p_{ik} P_i. \tag{3.27}$$

Because of

$$p_i(z) = \sum_{k=0}^{i+1} p_{ik} z^k = \int_0^\infty [p(x)z + q(x)]^{i+1} dA(x), \;\; i = 0, \ldots, m-1, \tag{3.28}$$

along with (3.21) and (3.28), we have

$$C(z) = \sum_{i=0}^{m-1} P_i \int_0^\infty [p\,(x)z + q(x)]^{i+1} dA\,(x) - \frac{A}{\delta} z^m \alpha_m. \tag{3.29}$$

Now we turn to $D(z)$. Substitution of (3.8) into (3.27) and Fubini's Theorem give

$$D(z) = \int_{x=0}^\infty \int_{y=0}^x \left(\sum_{k=0}^{m-1} \} \sum_{i\geq m}^\infty \binom{m}{j} e^{-\mu jx} (e^{-yx} - e^{-\mu x})^{m-j} \right. \tag{3.30}$$

$$\left. \times\, m\mu \frac{(m\mu y)^{i-m}}{(i-m)!} z^k P_i \right) dy dA(x).$$

Furthermore, with $P_k = A\,\delta^{k-m}$, we have

$$D(z) = A \int_{x=0}^\infty \int_{y=0}^x \sum_{i\geq m}^\infty \frac{(m\mu y\delta)^{i-m}}{(i-m)!} e^{-\mu my}$$

$$\left(\sum_{k=0}^m \binom{m}{j} \left[e^{-\mu(x-y)}\right]^j \left[1 - e^{-\mu(\alpha-y)}\right]^{m-j} z^k - e^{m\mu(x-y)} z^m \right) m\mu dy dA(x)$$

$$= A\int_{x=0}^{\infty}\int_{y=0}^{x} e^{m\mu y\delta}e^{-m\mu y}(1-e^{-\mu(x-y)}+ze^{-\mu(x-y)})^m m\mu\,dydA(x)$$

$$-Az^m\int_{x=0}^{\infty}e^{-m\mu x}\int_{y=0}^{x}m\mu\,e^{m\mu y\delta}\,dydA(x)$$

$$= A\int_{x=0}^{\infty}\int_{y=0}^{x} e^{m\mu y\delta}\,(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})^m m\mu\,dydA(x)$$

$$-Az^m\int_{x=0}^{\infty}e^{-m\mu x}\delta^{-1}(e^{m\mu x\,\delta}-1)\,dA(x),$$

so that

$$D(z) = Am\mu\int_{x=0}^{\infty}\int_{y=0}^{x} e^{m\mu y}(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})\,dydA(x) \tag{3.31}$$

$$+\,Az^m\,(\,\frac{\alpha_m}{\delta}-1).$$

By (3.26) and (3.31), we get

$$U(z) = \sum_{i=0}^{m-1}P_i\int_0^{\infty}[p\,(x)z+q(x)]^{i+1}dA(x)+AW(z),$$

where

$$W(z) = m\mu\int_{x=0}^{\infty}\int_{y=0}^{x} e^{m\mu y\delta}(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})\,dydA(x)\;-z^m.$$

□

To obtain P_k's we will use a method similar to that of Takács [6]. Given a polynomial function $f(z)$, define the sequence of linear functionals:

$$\forall\; r=0,1,\ldots,R_rf = \lim_{z\to 1}\frac{1}{r!}f^{(r)}(z). \tag{3.32}$$

Since $U(z-1)$ is identical to the Taylor series of $U(z)$ expanded at 1, applying R_r to the polynomial $U(z)$, $r=0,1,\ldots,m-1$, and then multiplying it scalarly by $(1,z-1,\ldots,(z-1)^{m-1})$ we will thereby reexpand $U(z)$ in a Taylor series at 1 and arrive at its binomial moments:

$$R_rU(z) = \sum_{k=r}^{m-1}P_k\begin{pmatrix}k\\r\end{pmatrix} =: U_r, r=0,1,\ldots,m-1. \tag{3.33}$$

On the other hand, given the binomial moments, we have

$$P_k = \sum_{r=k}^{m-1}\begin{pmatrix}r\\k\end{pmatrix}(-1)^{r-k}U_r, k=0,\ldots,m-1. \tag{3.34}$$

The following proposition is an analog to the result known for the GI/M/m/∞ queue, except for a_r in (3.36) being different.

Proposition 3.2: *The n^{th} binomial moment U_n of $U(z)$ is given by*

$$U_n = \frac{a_n\left(\frac{A\delta^{-1}\frac{\beta_0}{\hat{\beta}_0}}{a_{m-1}} + \sum_{r=n+1}^{m-1}\frac{\binom{m}{r}^{[m(1-\gamma_r)-r]}}{a_r(1-\gamma_r)[m(1-\delta)-r]}\right)}{\frac{\frac{\beta_0}{\hat{\beta}_0}}{a_{m-1}\delta} + \frac{1}{1-\delta} + \sum_{r=1}^{m-1}\frac{\binom{m}{r}^{[m(1-\gamma_r)-r]}}{a_r(1-\gamma_r)[m(1-\delta)-r]}},\,, n = 0,\ldots,m-1 \tag{3.35}$$

with $\sum_{r=m}^{m-1} = 0$, *where*

$$a_r = \begin{cases} 1 & r = 0, \\ \prod_{j=1}^{r}\frac{\gamma_j}{1-\gamma_j} & r > 0, \end{cases} \tag{3.36}$$

and

$$\gamma_r = \int_0^\infty [p(x)]^r dA(x) \tag{3.37}$$

Proof: Applying R_r to (3.25) we have

$$U_r = R_r U(z) = \lim_{z\to 1}\frac{1}{r\,!}\frac{d^r}{dz^r}U(z) \tag{3.38}$$

From (3.26),

$$R_r C(z) = \lim_{z\to 1}\frac{1}{r\,!}\frac{d^r}{dz^r}\,C(z) \tag{3.39}$$

$$= \lim_{z\to 1}\frac{1}{r\,!}\frac{d^r}{dz^r}\sum_{i=0}^{m-1}P_i\int_0^\infty\,[p\,(x)z + q(x)]^{i+1}$$

and due to

$$R_r[p(x)z + q(x)]^{i+1} = \begin{cases} \binom{i+1}{r}p(x)^r & r\,\le i+1 \\ 0 & r > i+1, \end{cases} \tag{3.40}$$

we have

$$U_r = \sum_{j=0}^{m-1}P_i\int_0^\infty R_r[p(x)z + q(x)]^{i+1}dA(x) + AR_rW(z)$$

$$= \sum_{i=r-1}^{m-1}P_i\binom{i+1}{r}\gamma_r - AW_r,$$

where

$$\gamma_r = \int_0^\infty [p(x)]^r dA(x)$$

and

$$W_r := R_rW(z).$$

With the usual combinatorics,

$$\sum_{i=r-1}^{m-1}P_i\binom{i+1}{r}\gamma_r = \gamma_r\,(U_r + U_{r-1}). \tag{3.41}$$

From (3.41) we have

$$U_r = \gamma_r(U_r + U_{r-1}) - AW_r, \tag{3.42}$$

or in the form

$$U_r = \frac{\gamma_r}{1-\gamma_r}\,U_{r-1} - \frac{A}{1-\gamma_r}W_r\,,\ r = 1, \ldots,\ m-1. \tag{3.43}$$

From (3.24)

$$R_r W(z) = W_r \tag{3.44}$$

$$= \lim_{z\to 1} \frac{1}{r\,!}\frac{d^r}{dz^r} m\mu \int_{x=0}^{\infty}\int_{y=0}^{x} e^{m\mu y\,\delta}(e^{-\mu y} - e^{-\mu x} + ze^{-\mu x})dy\,dA(x)\ - z^m$$

$$= \binom{m}{r} m\mu \int_{x=0}^{\infty} e^{-r\mu x}\int_{y=0}^{x} e^{(m\mu\delta-(m-r)\mu)y}\,dy\,dA\,(\,x\,)\ - \binom{m}{r},$$

$$r = 1, 2, \ldots, m-1.$$

Simplifying, we obtain

$$W_r = \binom{m}{r}\frac{m}{m\,(1-\delta)-r}\left(\int_0^{\infty} e^{-(1-\delta)m\mu x}dA(x) - \int_0^{\infty} e^{-r\mu x}\,dA\,(x)\right) - \binom{m}{r} \tag{3.45}$$

$$= \binom{m}{r}\frac{m(1-\gamma_r)-r}{m\,(1-\delta)-r}\,, \quad r = 1, 2, \ldots, m-1.$$

Denoting

$$a_r = \begin{cases} 1, & r = 0, \\ \prod_{j\,=\,1}^{r} \frac{\gamma_j}{1-\gamma_j}, & r > 0, \end{cases}$$

we have

$$\frac{\gamma_j}{1-\gamma_j} = \frac{a_r}{a_{r-1}}, \tag{3.46}$$

and dividing by a_r both sides of (3.43), we have

$$\frac{U_r}{a_r} = \frac{1}{a_{r-1}}U_{r-1} - \frac{A}{a_r(1-\gamma_r)}W_r,\ r = 1, \ldots,\ m-1. \tag{3.47}$$

Assuming $0 \le n \le m-2$, we add up equation (3.47) for $r = n+1, \ldots, m-1$, arriving at

$$\sum_{r=n+1}^{m-1}\frac{U_r}{a_r} = \sum_{r=n+1}^{m-1}\frac{1}{a_{r-1}}\,U_{r-1} - \sum_{r=n+1}^{m-1}\frac{A}{a_r(1-\ \gamma_r)}W_r\,,\ r = 1, \ldots,\ m-1, \sum_{r=m}^{m-1} = 0$$

and thus

$$U_n = \frac{a_n U_{m-1}}{a_{m-1}} + a_n\sum_{r=n+1}^{m-1}\frac{A}{a_r(1-\ \gamma_r)}W_r\,,\ n = 0, 1, \ldots, m-2. \tag{3.48}$$

By (3.22), (3.34) and (3.38), $U_{m-1} = P_{m-1}$, and, by (3.21), $P_{m-1} = A\ \delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}$. Therefore,

$$U_n = \frac{a_n A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}} + a_n \sum_{r=n+1}^{m-1} \frac{A}{a_r(1-\gamma_r)} W_r\ ,\ n = 0, 1, \ldots, m-2. \tag{3.49}$$

$$U_{m-1} = P_{m-1}. \tag{3.50}$$

In (3.49), for $n = 0$ and $a_0 = 1$

$$U_0 = \frac{A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}} + \sum_{r=1}^{m-1} \frac{A}{a_r(1-\gamma_r)} W_r. \tag{3.51}$$

On the other hand, by (3.22) and (3.17),

$$U(1) = \sum_{k=0}^{m-1} P_k = 1 - \sum_{k=m}^{\infty} P_k = 1 - \sum_{k=m}^{\infty} A\,\delta^{k-m}.$$

Hence,

$$U(1) = 1 - \frac{A}{1-\delta}. \tag{3.52}$$

Futhermore,

$$U_k = \frac{U^{(k)}(1)}{k!} = \frac{1}{k!}\left(\frac{d^k U(z)}{dz^k}\right)_{z=1},\ k = 0, 1, \ldots, m-1.$$

In particular, $U_0 = U(1)$. By (3.52), therefore,

$$U_0 = 1 - \frac{A}{1-\delta}.$$

Substitution of $U_0 = 1 - \frac{A}{1-\delta}$ into (3.51) gives

$$A = \left(\frac{\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}\delta} + \frac{1}{1-\delta} + \sum_{r=1}^{m-1} \frac{1}{a_r(1-\gamma_r)} W_r\right)^{-1}. \tag{3.53}$$

Explicitly,

$$A = \left(\frac{\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}\delta} + \frac{1}{1-\delta} + \sum_{r=1}^{m-1} \frac{\binom{m}{r}[m(1-\gamma_r)-r]}{a_r(1-\gamma_r)[m\,(1-\delta)-r]}\right)^{-1}. \tag{3.54}$$

Furthermore, equation (3.49) also holds for $n = m-1$ with $\sum_{r=m}^{m-1} := 0$. We conclude that

$$U_n = Aa_n\left(\frac{A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}} + \sum_{r=n+1}^{m-1} \frac{W_r}{a_r(1-\gamma_r)}\right), \tag{3.55}$$

$$n = 0, 1, \ldots, m-1$$

$$= Aa_n \sum_{r=n+1}^{m} \frac{1}{a_r(1-\gamma_r)} W_r,$$

with

$$W_m := \delta^{-1} \frac{\beta_0}{\widehat{\beta}_0} \gamma_m. \tag{3.56}$$

(Indeed, (3.56) agrees with (3.55) as the following calculations show:

$$\frac{W_m}{a_m(1-\gamma_m)} = \frac{\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}}$$

$$W_m = \frac{\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0} a_m(1-\ \gamma_m)}{a_{m-1}} = \frac{\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0} \gamma_m(1-\gamma_m)}{1-\ \gamma_m} = \delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}\gamma_m.)$$

Substituting equation (3.53) into (3.55) yields

$$U_n = \frac{a_n\left(\frac{A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}} + \sum_{r=n+1}^{m-1} \frac{W_r}{a_r(1-\gamma_r)}\right)}{\frac{\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}\,\delta} + \frac{1}{1-\delta} + \sum_{r=1}^{m-1} \frac{1}{a_r(1-\gamma_r)} W_r}, \tag{3.57}$$

and after substituting equation (3.45) into (3.57) we finally have

$$U_n = \frac{a_n\left(\frac{A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}} + \sum_{r=n+1}^{m-1} \frac{\binom{m}{r}^{[m(1-\gamma_r)-r]}}{a_r(1-\gamma_r)[m\,(1-\delta)-r]}\right)}{\frac{\frac{\beta_0}{\widehat{\beta}_0}}{a_{m-1}\,\delta} + \frac{1}{1-\delta} + \sum_{r=1}^{m-1} \frac{\binom{m}{r}^{[m(1-\gamma_r)-r]}}{a_r(1-\gamma_r)[m\,(1-\delta)-r]}}$$

$$\sum_{r=0}^{m-1} = 0.$$

We are done with the proposition. □

4 Continuous Time Parameter Process

The continuous time parameter queueing process is our main objective, which we target in the present section. The tools we are exploring are quite different from that used in our past experience (cf. Dshalalow [2]). Back then, we extended the process $(\Omega, \mathcal{F}, (P^x)_{x=0,1,\ldots}, Z_t;\, t \geq 0)$ from just being semi-regenerative to a two- (and more) variate process by using typical supplementary variable techniques, forming Kolmogorov's partial differential equations and Laplace transforms, then combining all of these with some semi-regenerative tools to yield a compact result. In this particular case, the past approach does not work since the additional information about arrival processes still does not make the two-variate process Markov, and any further efforts in this direction are cumbersome and counterproductive. So, we instead try to use a similar idea as that utilized in our recent paper [3] on semi-regenerative analysis directly applied to $(\Omega, \mathcal{F}, (P^x)_{x=0,1,\ldots}, Z_t;\, t \geq 0)$. Knowledge of invariant probability measures for embedded processes is crucial information needed for the upcoming analysis. We present some formal concepts of semi-regenerative processes pertinent to the models studied here.

According to Section 3,

$$(\Omega, \mathcal{F}, (P^x)_{x=0,1,\ldots}, Z_t;\, t \geq 0) \to \mathcal{E} = \{0, 1, \ldots\}$$

is a semi-regenerative process relative to the sequence $\{T_n\}$ of stopping times. By the key convergence theorem for semi-regenerative processes, for each initial state $x = 0, 1, \ldots,$ the limiting probability

$$\lim_{t\to\infty} P^x\{Z_t = j\} = \pi_j \tag{4.1}$$

exists if $m\mu\alpha > 1$ and is given by the expression

$$\pi_j = \frac{1}{\overline{a}} \sum_{i\in\mathcal{E}} P_i \int_0^\infty K_{ij}(t)dt, j \in \mathcal{E} \tag{4.2}$$

where $\{K_{ij}(t)\}$ is the semi-regenerative kernel of the process (Z_t) (cf. [2]) whose entries are defined as

$$K_{ij}(t) = P\{Z_t = j,\, T_1 > t \,\big|\, Z_0 = i\}, t \geq 0; i, j \in \mathcal{E}. \tag{4.3}$$

Observe that $K_{ij}(t) = P^i\{Z_t = j \,\big|\, T_1 > t\}\,[1 - A(x)]$, and thus $h_{ij} = \int_0^\infty K_{ij}(t)dt$ looks like the transition probability p_{ij} except that $dA(x)$ is replaced with $[1 - A(x)]dx$.

Notice that

$$h_{m-1,m} = \int_0^\infty [p(x)]^m [1 - A(x)]\, dx = \widehat{\alpha}_m, \tag{4.4}$$

and

$$h_{ij} = \int_{x=0}^\infty \left(\begin{array}{c} m \\ j \end{array} \right) e^{-\mu j x} \int_{y=0}^x (e^{-yx} - e^{-\mu x})^{m-j} \tag{4.5}$$

$$m\mu \frac{(m\mu y)^{i-m}}{(i-m)\,!} dy\, [1 - A(x)]\, dx, i = m, \ldots; j = 0, 1, \ldots, m.$$

Let us now consider equations (4.2) for the index values $j \geq m$. The following theorem states that, except for π_m, the result for π_j, $j \geq m$, is similar to that of [6].

Theorem 4.1: *The part of the limiting distribution $\pi_m, \pi_{m+1}, \ldots$ of the queueing process Z_t exists if $m\mu\alpha > 1$, it is independent of any initial state, and is given by*

$$\pi_j = \frac{1}{\overline{a}m\mu} P_{j-1}, j = m+1, m+2, \ldots,$$

and

$$\pi_m = \frac{1}{\overline{a}m\mu} P_{m-1}[m\mu\widehat{\alpha}_m + \widehat{\beta}_0], j = m.$$

Proof:

Case I: Let $j > m$. Then from (4.2),

$$\overline{a}\pi_j = \sum_{i=j-1}^{\infty} P_i \triangle_{i+1-j}, \tag{4.6}$$

where

$$\triangle_s = \int_0^{\infty} \frac{(m\mu x)^s}{s!} e^{-m\mu x}[1 - A(x)]\,dx, s \geq 0, \tag{4.7}$$

and

$$P_i = A\delta^{i-m}, i = j-1. \tag{4.8}$$

Inserting (4.8) into (4.6) we get

$$\overline{a}\pi_j = \sum_{s=0}^{\infty} A\delta^{s+j-1-m} \triangle_s. \tag{4.9}$$

Then,

$$\overline{a}\pi_j = A\delta^{j-1-m} \int_0^{\infty} e^{-m\mu\, x(1-\delta)}[1 - A(x)]dx \tag{4.10}$$

$$= A\delta^{j-1-m} \frac{1 - \alpha(m\mu - m\mu\delta)}{m\mu(1-\delta)}.$$

Since

$$\delta = \alpha\,(m\mu - m\mu\delta),$$

we can simplify $\overline{a}\pi_j$ to

$$\overline{a}\pi_j = A\delta^{j-1-m} \frac{1}{m\mu} = \frac{1}{m\mu} P_{j-1}. \tag{4.11}$$

Therefore,

$$\pi_j = \frac{1}{\overline{a}m\mu} P_{j-1},\; j = m+1, m+2, \ldots. \tag{4.12}$$

Case II: Let $j = m$. Then

$$\overline{a}\pi_m = \sum_{i=m-1}^{\infty} P_i h_{im} \tag{4.13}$$

$$= P_{m-1} h_{m-1,m} + \sum_{i=m}^{\infty} P_i \triangle_{i+1-m}. \tag{4.14}$$

Now, since $P_{m-1} = A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}, h_{m-1,m} = \widehat{\alpha}_m = \int_0^{\infty}[p\,(x)\,]^m[1 - A\,(x)]dx$ and $\sum_{i=m}^{\infty} P_i \triangle_{i+1-m} = A\delta^{-1}\frac{1}{m\mu}$, we have

$$\overline{a}\pi_m = A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}\widehat{\alpha}_m - A\delta^{-1}\triangle_0 + A\delta^{-1}\frac{1}{m\mu}, \tag{4.15}$$

where

$$\triangle_0 = \int_0^{\infty} e^{-m\mu x}[1 - A(x)]\,dx = \frac{1 - \alpha_m}{m\mu} \tag{4.16}$$

(using integration by parts for Stieltjes integrals as in Appendix, Theorem A.1) and

$$A\delta^{-1}\frac{1}{m\mu} = A\sum_{i=m-1}^{\infty}\delta^{-i-m}\,\triangle_{i+1-m}. \tag{4.17}$$

Therefore,

$$\overline{a}\pi_m = A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}\widehat{\alpha}_m - \frac{1-\alpha_m}{m\mu}A\,\delta^{-1} + A\delta^{-1}\frac{1}{m\mu}$$

and the latter simplifies to

$$\overline{a}\pi_m = A\,\delta^{-1}[\frac{\beta_0}{\widehat{\beta}_0}\widehat{\alpha}_m + \frac{\alpha_m}{m\mu}],$$

or also in the form: (since $\beta_0 = \alpha_m$)

$$\overline{a}\pi_m = A\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0}[\widehat{\alpha}_m + \frac{\widehat{\beta}_0}{m\mu}].$$

Since $A\,\delta^{-1}\frac{\beta_0}{\widehat{\beta}_0} = P_{m-1}$, we have

$$\pi_m = \frac{1}{\overline{a}m\mu}P_{m-1}[m\mu\widehat{\alpha}_m + \widehat{\beta}_0]. \tag{4.18}$$

[In the special case, $\widehat{\alpha}_m = \frac{1-\alpha_m}{m\mu}$ and $\widehat{\beta}_0 = \beta_0 = \alpha_m$, we have $m\mu\widehat{\alpha}_m + \widehat{\beta}_0 = 1$ and thus $\pi_m = \frac{1}{\overline{a}m\mu}P_{m-1}$.] □

To determine the unknown probabilities $\pi_0, \pi_1, \ldots \pi_{m-1}$ let us define the PGF

$$\widehat{U}(z) = \sum_{k=0}^{m-1}\pi_j\,z^j, \tag{4.19}$$

where π_j satisfies (4.2).

Proposition 4.2: *It holds true that*

$$\overline{a}\widehat{U}(z) = \sum_{i=0}^{m-1}P_i\int_0^{\infty}[q(x)+p(x)z]^{i+1}[1-A(x)]\,dx \,+ A\Phi(z), \tag{4.20}$$

where

$$\Phi(z) = m\mu\int_{x=0}^{\infty}\int_{y=0}^{x}(ze^{-\mu x}+e^{-\mu y}-e^{-\mu x})^m e^{m\mu y\delta}dy\,[1-A(x)]\,dx \tag{4.21}$$

$$-z^m\frac{\alpha_m}{m\mu\delta}.$$

Proof: Substituting equation (4.2) into (4.19) yields

$$\overline{a}\widehat{U}(z) = \sum_{j=0}^{m-1}z^j\sum_{i=j-1}^{m-1}h_{ij}P_i = F(z)+\varphi(z), \tag{4.22}$$

where

$$F(z) = \sum_{i=0}^{m-1} P_i \sum_{j=0}^{i+1} h_{ij} z^j - h_{m-1,\,m}\, z^m \; P_{m-1} \tag{4.23}$$

and

$$\varphi(z) = \sum_{j=0}^{m-1} z^j \sum_{i=m}^{\infty} h_{ij} P_i. \tag{4.24}$$

Because

$$h_i(z) = \sum_{j=0}^{i+1} h_{ij}\, z^j = \int_0^\infty [p(x)z + q(x)]^{i+1} [1 - A(x)]\, dx, \tag{4.25}$$

$$i = 0, \ldots, m-1,$$

with (3.21), (4.4), and (4.25), we have

$$F(z) = \sum_{i=0}^{m-1} P_i \int_0^\infty [q(x) + p(x)z]^{i+1} [1 - A(x)] dx - A\delta^{-1} \frac{\alpha_m}{\gamma_m}\, \widehat{\alpha}_m z^m. \tag{4.26}$$

Now we turn to $\varphi(z)$. Substitution of (4.5) into (4.24) and Fubini's Theorem give

$$\varphi(z) = \sum_{j=0}^{m-1} z^j A m\mu \sum_{i \geq m}^{\infty} \int_{x=0}^{\infty} \binom{m}{j} e^{-\mu j x} \int_{y=0}^{x} (\, e^{-\mu y} - e^{-\mu x}\,)^{m-j} \tag{4.27}$$

$$\frac{(m\mu y \delta)^{i-m}}{(i-m)!}\, dy\, [1 - A(x)] dx$$

$$= \sum_{j=0}^{m-1} A m\mu \int_{x=0}^{\infty} \int_{y=0}^{x} (z e^{-\mu x} + e^{-\mu y} - e^{-\mu x})^m\, e^{m\mu y\, \delta} dy\, [1 - \; A(x)] dx$$

$$- A m\mu\, z^m \int_{x=0}^{\infty} e^{-m\mu x} \frac{e^{m\mu\delta x} - 1}{m\mu\delta} [1 - A(x)]\, dx,$$

which can be simplified to

$$\varphi(z) = \sum_{j=0}^{m-1} A m\mu \int_{x=0}^{\infty} \int_{y=0}^{x} (z e^{-\mu x} + e^{-\mu y} - e^{-\mu x})^m e^{m\mu y \delta} dy [1 - A(x)] dx - \Gamma, \tag{4.28}$$

where

$$\Gamma = A m\mu\, z^m \frac{1}{m\mu\delta} \int_0^\infty \{ e^{-m\mu(1-\delta)x} - e^{-m\mu x} \} [1 - A(x)]\, dx. \tag{4.29}$$

The latter can be simplified to

$$\Gamma = A m\mu\, z^m \frac{1}{m\mu\delta} \{ \frac{1 - \alpha(m\mu - m\mu\delta)}{m\mu(1-\delta)} - \frac{1 - \alpha(m\mu)}{m\mu} \} = A z^m \frac{\alpha_m}{m\mu\delta}. \tag{4.30}$$

Therefore,

$$\varphi(z) = \sum_{j=0}^{m-1} A m\mu \int_{x=0}^{\infty} \int_{y=0}^{x} (z e^{-\mu x} + e^{-\mu y} - e^{-\mu x})^m\, e^{m\mu y\, \delta} dy\, [1 - \; A(x)] dx \tag{4.31}$$

$$-Az^m \frac{\alpha_m}{m\mu\delta}.$$

By (4.26) and (4.31), we finally have (4.20) and (4.21) valid. $\square$

Lemma 4.3: *The r^{th} binomial moment $\widehat{u}_r$ of $\widehat{U}(z)$ satisfies the following formula:*

$$\overline{a}\widehat{u}_r = \frac{\widehat{\alpha}_r}{\gamma_r} U_r + A\left(\frac{\widehat{\alpha}_r}{\gamma_r} + \frac{1}{r\mu}\right) W_r - \rho, \tag{4.32}$$

where

$$\rho = A\delta^{-1}\binom{m}{r}\alpha_m\left(\frac{\widehat{\alpha}_m}{\gamma_m} + \frac{1}{m\mu}\right), \tag{4.33}$$

and

$$W_r = \binom{m}{r}\frac{m(1-\gamma_r)-r}{m(1-\delta)-r}, \quad r = 1,\ldots,m-1, \tag{4.34}$$

and U_r is defined as in (3.43).

Proof: We use a method similar to that in Section 3.

If $\widehat{u}_r \left(= \sum_{k=r}^{m-1} \pi_k \binom{k}{r}, r = 1, ..., m-1\right)$ denotes the r^{th} binomial moment of $\widehat{U}(z)$, then

$$\overline{a}R_r\widehat{U}(z) = \overline{a}\widehat{u}_r = \overline{a}\lim_{z\to 1}\frac{1}{r\,!}\frac{d^r}{dz^r}\widehat{U}(z) \tag{4.35}$$

$$= R_rF(z) - A\delta^{-1}z^m\frac{\alpha_m\,\widehat{\alpha}_m}{\gamma_m} + R_r\Phi(z),$$

where

$$R_rF(z) = \lim_{z\to 1}\frac{1}{r\,!}\frac{d^r}{dz^r}F(z)$$

$$= \lim_{z\to 1}\frac{1}{r!}\frac{d^r}{dz^r}\sum_{i=0}^{m-1}P_i\int_0^\infty [q(x)\,+\,p\,(x)z]^{i+1}[1-A(x)]\,dx$$

$$-A\,\delta^{-1}\frac{\alpha_m}{\gamma_m}\,\widehat{\alpha}_m\,z^m. \tag{4.36}$$

$$R_r[p(x)z+q(x)]^{i+1} = \begin{cases}\binom{i+1}{r}p(x)^r, & r \le i+1,\\ 0, & r > i+1.\end{cases}$$

With the usual combinatorics we arrive at

$$\sum_{i=r-1}^{m-1}P_i\binom{i+1}{r}\widehat{\alpha}_r = \widehat{\alpha}_r(U_r + U_{r-1}), \tag{4.37}$$

where

$$\widehat{\alpha}_r = \int_0^\infty [p(x)]^r][1-A(x)]dx. \tag{4.38}$$

(In the appendix, we provide an alternative expression for $\widehat{\alpha}_r$.)

Let

$$\Phi_r = R_r\Phi(z). \tag{4.39}$$

Then,

$$\Phi_r = \lim_{z\to 1}\frac{1}{r!}\frac{d^r}{dz^r}Am\mu\int_{x=0}^{\infty}\int_{y=0}^{x}(z\,e^{-\mu x}+e^{-\mu y}-e^{-\mu x})^m$$

$$e^{m\mu y\delta}dy[1-A(x)]\,dx - z^m\frac{\alpha_m}{m\mu\delta}$$

$$= A\begin{pmatrix} m \\ r \end{pmatrix}\left(\frac{m}{m(\delta-1)+r}\frac{1-\alpha(m\mu(1-\delta))}{m\mu(1-\delta)}-\frac{1-\alpha_r}{r\mu}\right)-Az^m\frac{\alpha_m}{m\mu\delta}$$

$$= A\begin{pmatrix} m \\ r \end{pmatrix}\frac{1}{r\mu}\frac{(1-\alpha_r)m-r}{m(1-\ \delta)-r} = A\frac{1}{r\mu}W_r - Az^m\frac{\alpha_m}{m\mu\delta}. \tag{4.40}$$

From (4.24), (4.36), (4.37) and (4.40) we obtain

$$\overline{a}\widehat{u}_r = \widehat{\alpha}_r(U_r+U_{r-1}) + A\frac{1}{r\mu}W_r - \rho. \tag{4.41}$$

Substituting U_r from (3.43) into (4.41), we have

$$\overline{a}\widehat{u}_r = \widehat{\alpha}_r\left(U_r+\frac{1-\gamma_r}{\gamma_r}U_r+\frac{AW_r}{\gamma_r}\right)+A\frac{1}{r\mu}W_r-\rho$$

$$= \frac{\widehat{\alpha}_r}{\gamma_r}U_r + \widehat{\alpha}_r\frac{AW_r}{\gamma_r} + A\frac{1}{r\mu}W_r - \rho.$$

Finally,

$$\overline{a}\widehat{u}_r = \frac{\widehat{\alpha}_r}{\gamma_r}U_r + A\left(\frac{\widehat{\alpha}_r}{\gamma_r}+\frac{1}{r\mu}\right)W_r - \rho,$$

where

$$\rho = A\delta^{-1}\begin{pmatrix} m \\ r \end{pmatrix}\alpha_m\left(\frac{\widehat{\alpha}_m}{\gamma_m}+\frac{1}{m\mu}\right).$$

The proof of Lemma 4.3 is therefore completed. □

The below corollary finalizes our efforts to get $\pi_0, \pi_1, \ldots, \pi_{m-1}$.

Corollary 4.4: *The limiting probabilities $\pi_0, \pi_1, \ldots, \pi_{m-1}$ satisfy the following formulas:*

$$\pi_k = \sum_{r=k}^{m-1}\begin{pmatrix} r \\ k \end{pmatrix}(-1)^{r-k}\frac{1}{\overline{a}}\left(\frac{\widehat{\alpha}_r}{\gamma_r}\,U_r\,+A\left(\frac{\widehat{\alpha}_r}{\gamma_r}+\frac{1}{r\mu}\right)W_r-\rho\right), \tag{4.42}$$

$$k\ = 0, 1, \ldots, m, -1,$$

where U_r, and $\widehat{\alpha}_r$ are defined by (3.43) *and* (4.38), *respectively.* □

In the following sections we will be concerned with a special case.

5 Special Case

Consider the special case of GI/M/$\tilde{m}_a$/∞, with $a = 1$ and $p(x) = e^{-\mu x}$, which corresponds to the system GI/M/m/∞. Now, we have:

$$\gamma_m = \alpha_m, \widehat{\alpha}_m = \frac{1-\alpha_m}{m\mu}. \tag{5.1}$$

Thus

$$h_{m-1,m}P_{m-1} = A\delta^{-1}\frac{1-\alpha_m}{m\mu}. \tag{5.2}$$

Equation (5.1) immediately leads to

$$\Theta = \frac{1-\alpha_m}{m\mu} + \frac{\alpha_m}{m\mu} = \frac{1}{m\mu}. \tag{5.3}$$

Substituting (5.3) into the equation $\rho = A\,\delta^{-1}\begin{pmatrix} m \\ r \end{pmatrix}\Theta$ we have

$$\rho = A\delta^{-1}\begin{pmatrix} m \\ r \end{pmatrix}\frac{1}{m\mu}, \tag{5.4}$$

$$\widehat{\alpha}_r = \frac{1-\alpha_r}{r\mu},\ \gamma_r = \alpha_{r.} \tag{5.5}$$

Then, substituting equations (5.4) and (5.5) into (4.32) we have

$$\overline{a}\widehat{u}_r = \frac{1-\alpha_r}{r\mu\alpha_r}U_r + A\left(\frac{1-\alpha_r}{r\mu\alpha_r} + \frac{1}{r\mu}\right)W_r - A\,\delta^{-1}\begin{pmatrix} m \\ r \end{pmatrix}\frac{1}{m\mu}, \tag{5.6}$$

where

$$U_r = \frac{\alpha_r}{1-\alpha_r}U_{r-1} - AW_r\frac{1}{1-\alpha_r}. \tag{5.7}$$

So,

$$\overline{a}\widehat{u}_r = \frac{1}{r\mu}U_{r-1} - \begin{pmatrix} m \\ r \end{pmatrix}\frac{1}{m\mu}P_{m-1}, r = 1,\ldots,m-1, \tag{5.8}$$

and

$$\overline{a}\widehat{u}_r = \frac{1}{r\mu}U_{r-1} - \frac{(m-1)!}{(m-1-(r-1))r(r-1)!\mu}P_{m-1} \tag{5.9}$$

$$= \frac{1}{r\mu}\left(U_{r-1} - \begin{pmatrix} m-1 \\ r-1 \end{pmatrix}P_{m-1}\right)$$

$$= \frac{1}{r\mu}\left(\sum_{k=r-1}^{m-2}P_k\begin{pmatrix} k \\ r-1 \end{pmatrix} - \begin{pmatrix} m-1 \\ r-1 \end{pmatrix}P_{m-1}\right) \tag{5.10}$$

Therefore,

$$\overline{a}r\mu = \sum_{k=r}^{m-1}\begin{pmatrix} k \\ r \end{pmatrix}\pi_k = \sum_{k=r-1}^{m-2}\begin{pmatrix} k \\ r-1 \end{pmatrix}P_k \tag{5.11}$$

and

$$\pi_{m-1} = \frac{1}{\overline{a}\,(m-1)\mu}P_{m-2}. \tag{5.12}$$

Suppose

$$\pi_k = \frac{1}{\overline{a}\,k\mu}P_{k-1},\ k = m-1,\ldots,r+1. \tag{5.13}$$

Then

$$\overline{a}\,r\mu = \sum_{k=r}^{m-1}\begin{pmatrix} k \\ r \end{pmatrix}\pi_k = \pi_r\,\overline{a}\,r\mu + \overline{a}\,r\mu,\ \sum_{k=r+1}^{m-1}\begin{pmatrix} k \\ r \end{pmatrix}\pi_k \tag{5.14}$$

$$= \pi_r \overline{a}\, r\mu + \sum_{k=r+1}^{m-1} \binom{k-1}{r-1} P_{k-1}. \tag{5.15}$$

After cancellation,

$$\pi_r \overline{a} r\mu = P_{r-1} \tag{5.16}$$

which proves by induction for any $r = 1, \ldots, m-1$ that if $a = 1$, then (4.32) will be reduced to GI/M/m/∞.

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Appendix

The following theorem gives an alternative expression for $\widehat{\alpha}_r$ of (4.38) under some relatively weak, sufficient conditions. This expression can on occasions be more beneficial.

Theorem A.1: *Given* $\overline{a_2} = E\,[\,(\,T_{n+1} - T_n\,)^2] < \infty$, *it holds true that*

$$\widehat{\alpha}_r = \int_{x=0}^{\infty} \int_{u=0}^{\infty} [p(u)]^r du\, dA(x), r = 1, \ldots, m-1. \tag{A.1}$$

Proof: Recalling (4.38), where $\widehat{\alpha}_r = \int_0^\infty [\,p(u)\,]^r [1 - A(x)]\,dx$, and letting

$$F(t) = \int_{u=0}^{t} [\,p\,(u)]^r du$$

and

$$G(t) = A(x) - 1,$$

we can then use the integration by parts formula for Stieltjes integrals,

$$F(\infty)\,G(\infty) - F(\,0-)\,G(0-) = \int_{0-}^{\infty} FdG + \int_{0-}^{\infty} GdF,$$

to show that

$$\int_{0-}^{\infty} G\, dF = -\widehat{\alpha}_r.$$

Observe that

$$\int_{0}^{\infty} F dG = \int_{0}^{\infty} \int_{0}^{t} [p\,(u)]^{\,r} du\,)\, dA(x) \;\leq \int_{0}^{\infty} x\, dA(x) = \overline{a}.$$

Since $F(0-) = 0$, we need to show that under the condition that $\overline{a_2} = E[(T_{n+1} - T_n\,)^2\,] < \infty$ imposed on the PDF $A(x)$,

$$-F(\infty)\, G(\infty) = \lim_{t\to\infty}[1 - A\,(x)] \int_{u=0}^{t} [\,p(u)\,]^r\, du = 0.$$

Since $[p\,(u)]^r \leq 1$, $F(t) \;\leq\; t$, and thus,

$$\lim_{t\to\infty}[1 - A(x)]\, F(t) \leq \lim_{t\to\infty} x[1 - A(x)].$$

Now,

$$\lim_{t\to\infty} x[1 - A(x)] = 0$$

if

$$\Lambda = \int_{0}^{\infty} x[1 - A(x)]\, dx <\, \infty.$$

The latter holds if and only if $E\,[(T_{n+1} - T_n)^2] < \infty$. Indeed,

$$\Lambda = \int_{t=0}^{\infty} x \int_{u=t}^{\infty} dA(u)\, dx \;= \int_{u=0-}^{\infty} dA\,(u) \int_{t=0}^{u} x\; dx = \frac{1}{2}\; \overline{a_2}\,.$$

Remark A.2: Evidently,

$$\lim_{t\to\infty} x[1 - A(x)] = 0 \tag{A.2}$$

can hold while $\overline{a_2} = \infty$, and so condition (A.2) is exactly what we need and it is weaker. However, $\overline{a_2} <\, \infty$ is more tame. In most special cases, however, (A.2) holds true (i.e. if $1 - A(x) \to 0$ and $x\, \to \infty$, the latter should go faster to zero than $\frac{1}{x}$ does). For instance, if $A(x)$ is k - Erlang, then

$$x[1 - A(x)] = x\, \sum_{j=0}^{k-1} e^{-\lambda x} \frac{(\,\lambda x\,)^j}{j!}$$

obviously vanishes when $x\, \to \infty$, for any $k = 1, 2, \ldots$, and so will do any convex linear combination of Erlang distributions.