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FILTERING WITH A LIMITER (IMPROVED PERFORMANCE)

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We consider a filtering problem for a Gaussian diffusion process observed via discrete-time samples corrupted by a non-Gaussian white noise. Combining the Goggin's result [2] on weak convergence for conditional expectation with diffusion approximation when a sampling step goes to zero we construct an asymptotic optimal filter. Our filter uses centered observations passed through a limiter. Being asymptotically equivalent to a similar filter without centering, it yields a better filtering accuracy in a prelimit case.

Key words: Kalman Filter, Limiter, Asymptotic Optimality.
AMS subject classifications: 60G35, 93E11.

1. Introduction

Any computer implementation of filtering leads to a so-called "continuous-discrete time model" when a filtered continuous-time signal has to be estimated from discrete-time noisy samples. In this paper, we analyze such a filtering problem with a small sampling step, for which the use of a limit model, corresponding to a sampling step going to zero, is natural. It is clear that asymptotic optimality for the "filtering estimate" obtained via limit model plays a crucial role. To get an accomplished result and compare it to other well-known results, we restrict ourselves to a consideration of a simple model with a fixed sample step Δ so that a grid of times is: $t_0 = 0$, $t_k = k\Delta$, $k \geq 1$. An observation signal at these points of time is defined as:

$$Y_{t_0} = 0$$

$$Y_{t_k} - Y_{t_{k-1}} = AX_{t_{k-1}}\Delta + \xi_k\sqrt{\Delta}, \quad k \geq 1,$$

where X_t is an unobservable signal and $\xi_k\sqrt{\Delta}$ is a noise (A is a known constant). In this setting, we assume that $\{\xi_k, k \geq 1\}$ forms an i.i.d. sequence of random variables, independent of the process X_t, with $E\xi_1 = 0$ and $E\xi_1^2 = B^2$. For further convenience, introduce right continuous, having left-sided limits, random process

$$Y_t^{\Delta} = \sum_k I(t_{k-1} \le t < t_k) Y_{t_{k-1}}. \tag{1.1}$$

If Δ is sufficiently small, it makes sense to find a "limit" for Y_t^{Δ} as $\Delta \to 0$. Applying Donsker's theorem [1] one can show (see e.g., [6]) that, independently of the distribution for ξ_1, a diffusion type "limit" exists along with independent of (X_t) Wiener process (W_t):

$$Y_t = \int_0^t AX_s ds + BW_t. \tag{1.2}$$

For the sake of simplicity, assume that the signal X_t is generated by a linear Itô equation (with known parameters a and b) with respect to a Wiener process V_t, independent of ξ_k, $k \ge 1$:

$$X_t = X_0 + \int_0^t aX_s ds + bV_t, \tag{1.3}$$

where X_0 is a Gaussian random variable. For a pair (X_t, Y_t), the optimal filtering estimate in the mean square sense is defined by the Kalman filter ($\widehat{X}_0 = EX_0$, $P_0 = E(X_0 - EX_0)^2$) with

$$\begin{aligned} d\widehat{X}_t &= a\widehat{X}_t dt + \frac{P_t A}{B^2}(dY_t - A\widehat{X}_t dt) \\ \dot{P}_t &= 2aP_t + b^2 - \frac{P_t^2 A^2}{B^2}. \end{aligned} \tag{1.4}$$

Although $\widehat{X}_t$ is defined via Itô's integration "$P_t A/B^2 dY_t$", the first equation in (1.4) can be used for finding a continuous functional $\pi_t(y)$, $t \ge 0$, $y = (y_t)_{t \ge 0} \in \boldsymbol{C}_{[0,\infty)}$[1] such that $\widehat{X}_t = \pi_t(Y)$. It is clear that such functional is defined by the integral equation

$$\pi_t(y) = \widehat{X}_0 + \int_0^t \left(a - \frac{P_s A}{B^2}\right)\pi_s(y) ds + \frac{AP_t}{B^2} y_t - \int_0^t \frac{A\dot{P}_s}{B^2} y_s ds \tag{1.5}$$

and, in addition, it is well defined not only for $\boldsymbol{C}_{[0,\infty)}$ but for $\boldsymbol{D}_{[0,\infty)}$ as well. Moreover, for functions from $\boldsymbol{D}_{[0,\infty)}$ of locally bounded variations (namely, as line of Y_t^{Δ}), $\pi_t(y)$ is defined by the first equation in (1.4) with replacing Y_t by y_t. Therefore, following Kushner [3], one can take $\widehat{X}_t^{\Delta} := \pi_t(Y^{\Delta})$ as a filtering estimate for prelimit observation. A weak convergence $(Y_t^{\Delta}) \overset{law}{\to} (Y_t)$ for a fixed t, the continuity of $\pi_t(y)$ in the local supremum topology, and the uniform integrability for $(\pi_t(Y^{\Delta}))^2$ allows us to conclude that

$$\lim_{\Delta \to 0} E(X_t - \pi_t(Y^{\Delta}))^2 = E(X_t - \widehat{X}_t)^2 (= P_t).$$

Let us compare now $\pi_t(Y^{\Delta})$ with the optimal filtering estimate $\pi_t^{\Delta} = E(X_t \mid Y_{[0,t]}^{\Delta})$. Assume that a probability density function of the random variable ξ_1 has a finite Fisher information, say, $\mathfrak{I}_p$. Under some technical conditions, it is shown in [2] that $\pi_t^{\Delta} \overset{law}{\to} E(X_t \mid Y_{[0,t]}, Y_{[0,t]}^p)$, where a pair (Y_t, Y_t^p) is defined as: $Y_0 = Y_0^p = 0$ and

[1] $\boldsymbol{C}_{[0,\infty)}$ and $\boldsymbol{D}_{[0,\infty)}$ are the spaces of continuous and right continuous functions with left-sided limits, respectively.

$$dY_t = AX_t dt + \sqrt{B^2 - (1/\mathfrak{I}_p)}dW'_t + \frac{1}{\sqrt{\mathfrak{I}_p}} dW_t,$$
$$dY^p_t = AX_t dt + \frac{1}{\sqrt{\mathfrak{I}_p}} dW_t,$$

with independent Wiener processes W'_t, W_t independent of the X_t process. In turn, in [4] (see also [9]), it is shown that

$$E(X_t \mid Y_{[0,t]}, Y^p_{[0,t]}) = E(X_t \mid Y^p_{[0,t]}), \quad P\text{-a.s.}$$

All these facts enable us to express the optimal asymptotic accuracy

$$\lim_{\Delta \to 0} E(X_t - \pi^\Delta_t)^2 = E\Big(X_t - E(X_t \mid Y^p_{[0,t]})\Big)^2 (: = P^0_t) \tag{1.6}$$

as the filtering mean square error for the pair (X_t, Y^p_t) or, in other words, to define P^0_t as a solution of the Ricatti equation (compare with (1.4))

$$\dot{P}^0_t = 2aP^0_t + b^2 - \mathfrak{I}_p (P^0_t A)^2$$

subject to $P^0_0 = P_0$. We compare now P_t and P^0_t. Due to the Cramer-Rao inequality, we have $\mathfrak{I}_p > \frac{1}{B^2}$ (unless ξ_1 is Gaussian with $\mathfrak{I}_p = \frac{1}{B^2}$). Therefore, by the comparison theorem for ordinary differential equations we obtain (unless of Gaussian ξ_1)

$$P_t > P^0_t, \quad t > 0. \tag{1.7}$$

This fact shows that in the non-Gaussian case, the lower bound P^0_t is unattainable for any linear filter with prelimit observations.

In the case of a finite Fisher information, the authors of [4] proposed a nonlinear filter, for which P^0_t is asymptotically attainable. To describe the structure of this filter, let us denote by $p(x)$ the probability density function of ξ_1. Since $\mathfrak{I}_p < \infty$, it is assumed that $p(x)$ is a smooth function such that the function $G(x) = -\frac{p'}{p}(x)$ is well defined. With that $G(x)$, let us define the new observation (compare (1.1))

$$Y^{p,\Delta}_t = \sum_k I(t_{k-1} \le t < t_k) Y^{p,\Delta}_{t_{k-1}}, \tag{1.8}$$

where $Y^{p,\Delta}_0 = 0$ and $Y^{p,\Delta}_{t_k} - Y^{p,\Delta}_{t_{k-1}} = \frac{\sqrt{\Delta}}{\mathfrak{I}_p} G\left(\frac{Y_{t_k} - Y_{t_{k-1}}}{\sqrt{\Delta}}\right)$ and the filtering estimate $\pi^p_t(Y^{p,\Delta})$ with $\pi^p_t(y)$, $y \in \boldsymbol{D}_{[0,\infty)}$ defined by the linear integral equation (compare (1.5))

$$\pi^p_t(y) = \widehat{X}_0 + \int_0^t \Big(a - \mathfrak{I}_p P^0_s A^2\Big) \pi^p_s(y) ds + \mathfrak{I}_p A P^0_t y_t - \int_0^t \mathfrak{I}_p A \dot{P}^0_s y_s ds. \tag{1.9}$$

The analysis of the prelimit mean square error $\varepsilon_t := E(X_t - \pi^p_t(Y^{p,\Delta}))^2$ leads to the following structure: $\varepsilon_t = P^0_t + k_t \Delta^2 + o(\Delta^2)$.

In this paper, we show the existence of an asymptotically optimal filter with the mean square error $\varepsilon'_t = P^0_t + k'_t \Delta^2 + o(\Delta^2)$ for $k'_t < k_t$.

The paper is organized as follows. In Section 2, we describe the proposed filter and give both the diffusion approximation and a proof of asymptotic optimality. In Section 3, we analyze prelimit quality of filtering and demonstrate the results obtained via simulation.

2. The Filter

The filter, given in (1.9), is inspired by the Kalman filter corresponding to the pair (X_t, Y_t^p). We introduce now another pair $(X_t, X_t^{p,c})$ with the centered observation process

$$Y_t^{p,c} = Y_t^p - \int_0^t A\widehat{X}_s^{p,c}ds = \int_0^t A(X_t - \widehat{X}_s^{p,c})ds + \frac{1}{\sqrt{\mathfrak{I}_p}}W_t, \tag{2.1}$$

where $\widehat{X}_t^{p,c} = E(X_t \mid Y_{[0,t]}^{p,c})$. For this pair, a generalized Kalman filter is well defined (see e.g., Ch. 12 in [5]): $X_0^{p,c} = EX_0, P_0^{p,c} = E(X_0 - EX_0)^2$ and

$$d\widehat{X}_t^{p,c} = a\widehat{X}_t^{p,c}dt + \mathfrak{I}_p P_t^{p,c} A dY_t^{p,c}$$

$$\dot{P}_t^{p,c} = 2aP_t^{p,c} + b^2 - \mathfrak{I}_p(P_t^{p,c}A)^2. \tag{2.2}$$

Since P_t^0 and $P_t^{p,c}$ are defined by the same Ricatti equation, we have $P_t^{p,c} \equiv P_t^0$. It should be noted also that $\frac{Y_t^{p,c}}{\mathfrak{I}_p}$ is so-called, *innovation Wiener process* and thus the new observation $Y_t^{p,c}$ is a zero mean random process.

We use (2.1) and (2.2) to construct a new nonlinear filter for prelimit observation. (2.2) implies (compare (1.9))

$$\pi_t^{p,c}(y) = \widehat{X}_0 + \int_0^t a\pi_s^{p,c}(y)ds + \mathfrak{I}_p AP_t^0 y_t - \int_0^t \mathfrak{I}_p A\dot{P}_s^0 y_s ds \tag{2.3}$$

and, consequently, we take $\pi_t^{p,c}(Y^{p,c,\Delta})$ as a new filtering estimate, where

$$Y_t^{p,c,\Delta} = \sum_k I(t_{k-1} \le t < t_k) Y_{t_{k-1}}^{p,c,\Delta}, \tag{2.4}$$

and

$$Y_0^{p,c,\Delta} = 0$$

$$Y_{t_k}^{p,c,\Delta} - Y_{t_{k-1}}^{p,c,\Delta} = \frac{\sqrt{\Delta}}{\mathfrak{I}_p} G\left(\frac{Y_{t_k} - Y_{t_{k1}}}{\sqrt{\Delta}} - A\pi_{t_{k-1}}^{p,c}(Y^{p,c,\Delta})\sqrt{\Delta}\right). \tag{2.5}$$

2.1 Diffusion approximation for $(X_t, Y_t^{p,c,\Delta}, \pi_t^{pc}(Y^{p,c,\Delta}))$

For brevity, we will write $\mathcal{W} - \lim_{n\to\infty}$ to denote weak convergence in the Skorokhod-Lindvall and the local supremum topologies (see e.g., Ch. 6 in [6]). Recall that $G(x) = -\frac{p'}{p}(x)$.

Theorem 2.1: *Assume that $EG^2(\xi_1) < \infty$ and the density $p(x)$ is twice continuously differentiable such that $G'(x)$ is well defined and satisfies the linear growth condition: $|G'(x)| \le c(1 + |x|)$. Then,*

$$\mathcal{W} - \lim_{n\to\infty}(X_t, Y_t^{p,c,\Delta}, \pi_t^{p,c}(Y^{p,c,\Delta}))_{t\ge 0} = (X_t, Y_t^{p,c}, \widehat{X}_t^{p,c})_{t \ge 0}.$$

We start with an auxiliary result. Define an non-decreasing right continuous function $L_t^\Delta = \Delta[t/\Delta]$, where $[t]$ is the integer part of t, and right continuous with left-sided limits random process

$$M_t^\Delta = \frac{\sqrt{\Delta}}{\mathfrak{I}_p} \sum_{k=1}^{L_t^\Delta/\Delta} G(\xi_k). \tag{2.6}$$

Introduce the process $\widetilde{Y}_t^{p,c,\Delta}$ (for brevity write $s-\Delta$ instead of $(s-\Delta)\vee 0$):

$$\widetilde{Y}_t^{p,c,\Delta} = \int_0^t A[X_{s-\Delta} - \pi_{s-\Delta}^{p,c}(\widetilde{Y}^{p,c,\Delta})]dL_s^\Delta + M_t^\Delta. \tag{2.7}$$

Lemma 2.1:

$$\mathscr{W}-\lim_{n\to\infty}(X_t, \widetilde{Y}_t^{p,c,\Delta}, \pi_t^{p,c}(\widetilde{Y}^{p,c,\Delta}))_{t\geq 0} = (X_t, Y_t^{p,c}, \widehat{X}_t^{p,c})_{t\geq 0}.$$

Proof: Since $\pi_t^{p,c}(y)$ is a continuous funcitonal in the local supremum topology and $\pi_t^{p,c}(Y^{p,c}) = \widehat{X}_t^{p,c}$, it is clear that the statement of the lemma follows from

$$\mathscr{W}-\lim_{n\to\infty}(X_t, \widetilde{Y}_t^{p,c,\Delta}))_{t\geq 0} = (X_t, Y_t^{p,c})_{t\geq 0}. \tag{2.8}$$

Therefore, we will only verify (2.8) by applying Theorem 8.3.3 from [6]. Note that $EG(\xi_1) = 0$ and $EG^2(\xi_1) < \infty$. Then M_t^Δ forms a square integrable martingale (with respect to an appropriate filtration). Recall also that

$$dX_t = aX_t dt + bdV_t$$

and

$$dY_t^{p,c} = A[X_t - \pi_t^{p,c}(Y^{p,c})]dt + \frac{1}{\sqrt{\mathfrak{I}_p}}dW_t.$$

Hence, only the following two conditions from the above mentioned theorem have to be shown. For every $T > 0$,

$$P\text{-}\lim_{\Delta\to 0} \sup_{t\leq T}\left| \int_0^t A[X_{s-\Delta} - \pi_{s-\Delta}^{p,c}(\widetilde{Y}^{p,c,\Delta})]dL_s^\Delta - \int_0^t A[X_s - \pi_s^{p,c}(\widetilde{Y}^{p,c,\Delta})]ds\right| = 0$$

$$\lim_{\Delta\to 0} \sup_{t\leq T}\left|\frac{\Delta}{\mathfrak{I}_p^2}\sum_{k=1}^{L_t^\Delta/\Delta} EG^2(\xi_k) - \frac{t}{\mathfrak{I}_p}\right| = 0. \tag{2.9}$$

Since $EG^2(\xi_k) = \mathfrak{I}_p$, the second condition in (2.9) follows from

$$\lim_{\Delta\to 0} \sup_{t\leq T} | L_t^\Delta - t | = 0.$$

To verify the first condition, let us use the estimate

$$\sup_{t<T}\left| \int_0^t A[X_{s-\Delta} - \pi_{s-\Delta}^{p,c}(\widetilde{Y}^{p,c,\Delta})]dL_s^\Delta - \int_0^t A[X_s - \pi_s^{p,c}(\widetilde{Y}^{p,c,\Delta})]ds\right|$$

$$\leq |A| \left\{ \int_0^T | X_{s-\Delta} - X_s | ds + \int_0^T | \pi_{s-\Delta}^{p,c}(\widetilde{Y}^{p,c,\delta}) - \pi_s^{p,c}(\widetilde{Y}^{p,c,\Delta}) | ds \right.$$

$$\left. + \sup_{t<T}\left| \int_0^t X_{s-\Delta} d[L_s^\Delta - s]\right| + \sup_{t\leq T}\left| \int_0^t \pi_{s-\Delta}^{p,c}(\widetilde{Y}^{p,c,\Delta}) d[L_s^\Delta - s]\right| \right\}$$

$$= |A| \{i_1^\Delta + i_2^\Delta + i_3^\Delta + i_4^\Delta\}.$$

Since X_t is a continuous process, $i_1^\Delta \to 0$, P-a.s. as $\Delta \to 0$. Introduce a sequence of piecewise constant processes $X_t^m = X_{[tm]/m}$, $m \geq 1$ ($[x]$ is the integer part of x). Then

$$i_3^\Delta = \sup_{t \leq T} \left| \int_0^t X_{s-\Delta} d[L_s^\Delta - s] \right| \leq \sup_{t \leq T} \left| \int_0^t X_{s-\Delta}^m d[L_s^\Delta - s] \right|$$

$$+ \sup_{t \leq T} \left| \int_0^t [X_{s-\Delta} - X_{s-\Delta}^m] d[L_s^\Delta - s] \right|$$

$$\leq 2 \sup_{t \leq T} |X_t| \sup_{t \leq T} |L_t^\Delta - t| + |\sup_{t < T} |X_t - X_t^m| [L_T^\Delta + T]$$

$\to 0$, P-a.s., if for the limit $\lim\limits_{m \to \infty} \limsup\limits_{\Delta \to 0}$ is taken.

Thus, $i_1^\Delta, i_3^\Delta \to 0$ for $\Delta \to 0$. To verify the same property for i_2^Δ and i_4^Δ, one has to show first that

$$\lim_{C \to \infty} \limsup_{\Delta \to 0} P\left(\sup_{t < T} |\pi_t^{p,c}(\widetilde{Y}^{p,c,\delta})| > C \right) = 0. \tag{2.10}$$

From (2.3), by using Theorem 2.5.3 [6], we obtain that for a fixed $T > 0$, there exists a positive constant ℓ, dependent on T, such that for any $t \leq T$,

$$\sup_{s < t} |\pi_s^{p,c}(y)| \leq \ell \sup_{s \leq t} |y_s|.$$

Denote by $X_t^* = \sup_{s \leq t} |X_t|$, $\widetilde{Y}_t^* = \sup_{s \leq t} |\widetilde{Y}_s^{p,c,\Delta}|$, and $M_t^{\Delta,*} = \sup_{s \leq t} |M_s^\Delta|$. Then by (2.7), for any $t \leq T$, we arrive at the inequality

$$\widetilde{Y}_t^* \leq \Big[|A| X_t^* + M_T^{\Delta,*} \Big] + \ell \int_0^t \widetilde{Y}_{s-}^* dL_s^\Delta,$$

which, by Theorem 2.5.3 in [6], implies that

$$\widetilde{Y}_t^* \leq \left[|A| \sup_{t \leq T} |X_t| + M_T^{\Delta,*} \right] (1 + \ell\Delta)^{t/\Delta}.$$

Therefore, (2.10) holds true provided that

$$\lim_{C \to \infty} \limsup_{\Delta \to 0} P\left(\sup_{t < T} M_T^{\Delta,*} > C \right) = 0,$$

whose validity is due to Doob's inequality (see e.g., Tehorem 1.9.1 in [6]):

$$P\left(\sup_{t \leq T} M_T^{\Delta,*} > C \right) \leq \frac{1}{C^2} E\Big(M_T^\Delta\Big)^2 \left(= \frac{L_T^\Delta}{C^2 \mathfrak{I}_p} \right).$$

Thus, (2.10) and

$$\lim_{C \to \infty} \limsup_{\Delta \to 0} P\left(\sup_{t \leq T} |\widetilde{Y}_t^{p,c,\Delta}| > C \right) = 0 \tag{2.11}$$

are established.

We are now in the position to show that $i_2^\Delta \xrightarrow{P} 0$ as $\Delta \to 0$. Due to (2.3) and (2.7) for every $C > 0$, there exists a positive constant γ, dependent on C, such that on the set $\widetilde{\mathcal{A}}_C = \{\sup_{t \leq T} |\widetilde{Y}_t^{p,c,\Delta}| \leq C\}$,

$$|\pi_t^{p,c}(\widetilde{Y}^{p,c,\Delta}) - \pi_{t-\Delta}^{p,c}(\widetilde{Y}^{p,c,\Delta})| \leq \gamma\Big(\Delta + |M_t^\Delta - M_{t-\Delta}^\Delta|\Big).$$

Consequently, due to (2.10), the required conclusion, on the set $\widetilde{\mathcal{A}}_C$, holds when $\int_0^T |M_s^\Delta - M_{s-\Delta}^\Delta| ds \xrightarrow{P} 0$, as $\Delta \to 0$ and the latter is due to the Cauchy-Schwartz inequality: $E|M_s^\Delta - M_{s-\Delta}^\Delta| \le \sqrt{E|M_s^\Delta - M_{s-\Delta}^\Delta|^2} \le \sqrt{\frac{\Delta}{\mathfrak{I}_p}}$. Therefore, by virtue of (2.11), we obtain the desired property.

The proof for $i_4^\Delta \xrightarrow{P} 0$ is similar to that for i_3^Δ. Moreover, it suffices to check its validity only on the set $\widetilde{\mathcal{A}}_C$ with an arbitrary C. In fact, letting $\pi_t^{p,c,m}(y) = \pi_{[tm]/m}^{p,c}(y)$, for every fixed m we have on the above-mentioned set

$$\sup_{t \le T} \left| \int_0^t \pi_{s-\Delta}^{p,c,m}(\widetilde{Y}^{p,c,\Delta}) d[L_s^\Delta - s] \right| \xrightarrow{P} 0$$

and

$$\int_0^T |\pi_{s-\Delta}^{p,c}(\widetilde{Y}^{p,c,\Delta}) - \pi_{s-\Delta}^{p,c,m}(\widetilde{Y}^{p,c,\Delta})| d[L_s^\Delta + s] \xrightarrow{P} 0,$$

where as the limit $\lim_{m\to\infty}\limsup_{\Delta\to 0}$ is understood. □

Proof of Theorem 2.1: Due to Lemma 2.1 and Theorem 4.1 in [1] (Ch. 1, § 4), the property

$$P - \lim_{\Delta \to 0} \sup_{t \le T} \left| Y_t^{p,c,\Delta} - \widetilde{Y}_t^{p,c,\Delta} \right| = 0, \text{ for all } T > 0 \tag{2.12}$$

yields the statement of the theorem. Below we verify (2.12). We show first that for every $T > 0$,

$$\lim_{C \to \infty} \limsup_{\Delta \to 0} P\left(\sup_{t \le T} |Y_t^{p,c,\Delta}| > C \right) = 0. \tag{2.13}$$

Taking into account the linear growth assumption for $G'(x)$ and using (2.5) and the obtained above estimate $\sup_{s \le t} |\pi_s^{p,c}(y)| \le \ell \sup_{s \le t} |y_s|$, for every $t \le T$, we get

$$\begin{aligned} \sup_{s \le t} |Y_t^{p,c,\Delta}| &\le \left[\sup_{s \le T} |M_s^\Delta| + cL_T^\Delta(1 + |A| \sup_{s \le T} |X_s|) \right] \\ &+ c\ell \int_0^t \sup_{s' \le s} |Y_{s'-}^{p,c,\Delta}| dL_s^\Delta. \end{aligned} \tag{2.14}$$

Hence, due to Theorem 2.5.3 in [6], (2.13) holds true, provided that

$$\lim_{C \to \infty} \limsup_{\Delta \to 0} P\left(\sup_{t \le T} |M_t^\Delta| > C \right) = 0. \tag{2.15}$$

(2.15) is due to the weak convergence $\mathcal{W} - \lim_{\Delta \to 0}(M_t^\Delta)_{t \ge 0} = (\frac{1}{\mathfrak{I}_p} W_t)_{t \ge 0}$, which, in turn, is due to the Donsker theorem (see e.g., [6]).

For further convenience, denote

$$\delta_t = Y_t^{p,c,\Delta} - \widetilde{Y}_t^{p,c,\Delta} \quad \text{and} \quad z_t = \pi_t^{p,c}(Y^{p,c,\Delta}) - \pi_t^{p,c}(\widetilde{Y}^{p,c,\delta})$$

and set

$$u^\Delta(t) = \frac{1}{\mathfrak{I}_p} G'(\xi_k), \quad t_{k-1}^n < t \le t_k^n$$

$$U_t^\Delta = \frac{\Delta}{\mathfrak{I}_p} \sum_{k=1}^{L_t^\Delta/\Delta} A(X_{t_{k-1}} - \pi_{t_{k-1}}^{p,c}(Y^{p,c,\Delta}))$$

$$\times\Big[G'\Big(\theta_k A[X_{t_{k-1}} - \pi^{p,c}_{t_{k-1}}(Y^{p,c,\Delta})]\sqrt{\Delta} + \xi_k\Big) - G'(\xi_k)\Big]. \tag{2.16}$$

By the mean value theorem,

$$Y^{p,c,\Delta}_t = \int_0^t u^{\Delta}(s)A[X_{s-\Delta} - \pi^{p,c}_{s-\Delta}(Y^{p,c,\Delta})]dL^{\Delta}_s + M^{\Delta}_t + U^{\Delta}_t.$$

This presentation and (2.7) imply that

$$\delta_t = U^{\Delta}_t + \int_0^t (u^{\Delta}(s)-1)A[X_{s-\Delta} - \pi^{p,c}_{s-\Delta}(Y^{p,c,\Delta})]dL^{\Delta}_s + \int_0^t Az_{s-\Delta}dL^{\Delta}_s.$$

Denote by $\delta^*_t = \sup_{s\le t}|\delta|$, $z^*_t = \sup_{s\le t}|Z_s|$, and $U^{\Delta,*}_t = \sup_{s\le t}|U^{\Delta}_s|$. Noticing also that $z^*_t \le \ell\delta^*_t$, one can show

$$\delta^*_t = U^{\Delta,*}_t + \sup_{t'\le t}\left|\int_0^t (u^{\Delta}(s)-1)A[X_{s-\Delta} - \pi^{p,c}_{s-\Delta}(Y^{p,c,\Delta})]dL^{\Delta}_s\right| + \int_0^t |A|\,\delta^*_{s-}dL^{\Delta}_s.$$

Therefore, Theorem 2.5.3 of [6] makes us conclude that the statement of the theorem holds true provided that

$$P-\lim_{\Delta\to 0}|U^{\Delta,*}_T| = 0$$
$$P=\lim_{\Delta\to 0}\sup_{t'\le T}\left|\int_0^{t'} (u^{\Delta}(s)-1)A[X_{s-\Delta} - \pi^{p,c}_{s-\Delta}(Y^{p,c,\Delta})]dL^{n}_s\right| = 0. \tag{2.17}$$

For $C>0$, introduce the set $\mathfrak{B}_C = \{|A|[\sup_{t\le T}|X_t| + \sup_{t\le T}|\pi^{p,c}_t(Y^{p,c,\Delta})|] \le C\}$. A method of the proof for Theorem 1 in [4] is also applicable here for checking that, for every $C>0$ ($I_{\mathfrak{B}_C}$ is the indicator function of the set $\mathfrak{B}_C$),

$$P-\lim_{\Delta\to 0}\Big(|U^{\Delta,*}_T|\,I_{\mathfrak{B}_C}\Big) = 0$$

and

$$P-\lim_{\Delta\to 0}\left(\sup_{t'\le T}\left|\int_0^{t'} (u^{\Delta}(s)-1)A[X_{s-\Delta} - \pi^{p,c}_{s-\Delta}(Y^{p,c,\Delta})]dL^{n}_s\right| I_{\mathfrak{B}_C}\right) = 0.$$

Thus, $\lim_{C\to\infty}\limsup_{\Delta\to 0}P(\Omega\backslash\mathfrak{B}_C) = 0$ yields (2.17). □

2.2 Asymptotic optimality

As mentioned, $P^0_t = E\Big(X_t - \widehat{X}^{p,c}_t\Big)^2$ is a lower bound for asymptotic filtering error. So, $\pi^{p,c}_t(Y^{p,c,\Delta})$ is an asymptotic optimal filtering estimate, if

$$\lim_{\Delta\to 0}\Big(X_t - \pi^{p,c}_t(Y^{p,c,\Delta})\Big)^2 = P^0_t. \tag{2.18}$$

Theorem 2.2: *Let the assumptions of Theorem 2.1 be met and* $EG^4(\xi_1) < \infty$.

Then (2.18) *holds.*

Proof: For any fixed $t > 0$, by Theorem 2.1 we have, as $\Delta \to 0$,

$$\left(X_t - \pi_t^{p,c}(Y^{p,c,\Delta})\right)^2 \overset{law}{\to} \left(X_t - \widehat{X}_t^{p,c}\right)^2,$$

and, therefore, only the uniform integrability for $\left(X_t - \pi_t^{p,c}(Y^{p,c,\Delta})\right)^2$ should be verified. We use the sufficient condition $\sup_{\Delta \le 1} E\left(X_t - \pi_t^{p,c}(Y^{p,c,\Delta})\right)^4 < \infty$. Since X_t is a Gaussian variable, only $\sup_{\Delta \le 1} E\left(\pi_t^{p,c}(Y^{p,c,\Delta})\right)^4 < \infty$ has to be verified. The use of inequality $|\pi_t^{p,c}(Y^{p,c,\Delta})| \le \ell \sup_{s \le t} |Y_s^{p,c,\Delta}|$ reduces our verification to

$$\sup_{\Delta \le 1} E \sup_{s \le t} |Y_s^{p,c,\Delta}|^4 < \infty$$

only. Moreover, by virtue of (2.14), for some fixed $T > t$, only the validity of

$$\sup_{\Delta \le 1} E \sup_{s \le t} |M_s^{\Delta}|^4 < \infty, \tag{2.19}$$

needs to be proved. One can use now the fact that the random process M_s^{Δ} is a martingale and apply the Burkholder-Gundy inequality (see e.g., Ch. 1, §9 of [6]): $E\sup_{s \le T} |M_s^{\Delta}|^4 \le C_4 E[M^{\Delta}, M^{\Delta}]_T^2$, where C_4 is a constant, independent of Δ, and in our case, $[M^{\Delta}, M^{\Delta}]_T = \frac{\Delta}{\mathfrak{g}_p^2} \sum_{k=1}^{L_T^{\Delta}/\Delta} G^2(\xi_k)$. Hence,

$$\begin{aligned} E[M^{\Delta}, M^{\Delta}]_T^2 &= \frac{\Delta^2}{\mathfrak{g}_p^4}\left(\sum_{k=1}^{L_T^{\Delta}/\Delta} EG^4(\xi_1) + 2\sum_{k=1}^{L_T^{\Delta}/\Delta} \sum_{i=1}^{k-1} (EG^2(\xi_1))\right) \\ &\le \frac{\Delta L_T^{\Delta}}{\mathfrak{g}_p^4} EG^4(\xi_1) + \frac{(L_T^{\Delta})^2}{\mathfrak{g}_p^2} \le \text{const.} \end{aligned}$$

□

3. Prelimit Analysis

Hereafter, we study prelimit properties of the filter proposed in the preceding sections and compare it to the one obtained in [2] and [4]. We show that centering of the observations by the filtering estimate is advantageous in a pre-asymptotic situation.

Centered limiter: For the sake of simplicity, we analyze the centered filter with the Kalman gain in which $P_t^{p,c}$ is replaced by its limit $P^{p,c} = \lim_{t \to \infty} P_t^{p,c}$, that is the following filter will be investigated (recall that the process $Y_t^{p,c,\Delta}$ is defined in (2.5)):

$$d\widehat{X}_t^{p,c,\Delta} = a\widehat{X}_t^{p,c,\Delta} dt + \mathfrak{g}_p A P^{p,c} dY_t^{p,c,\Delta}. \tag{3.1}$$

Denote by $U_{t_k}^{p,c,\Delta} = E\left(X_{t_k} - \widehat{X}_{t_k}^{p,c,\Delta}\right)^2$ and $B_{t_k}^{p,c,\Delta} = E\left(X_{t_k} - \widehat{X}_{t_k}^{p,c,\Delta}\right)$. Assuming that $G(\cdot)$ is three times continuously differentiable and all its derivatives are bounded and the fourth moment of filtering estimate is finite, for small Δ it can be shown that

$$\begin{aligned} U_{t_{k+1}}^{p,c,\Delta} = {} & (1 + a\Delta - \mathfrak{g}_p A^2 P^{p,c}\Delta)^2 U_{t_k}^{p,c,\Delta} + b_{t_k}^2 \Delta + \left(P^{p,c}A\right)^2 \mathfrak{g}_p \Delta \\ & + \Big\{\left(P^{p,c}A\right)^2 A^2 E[G'(\xi_1) - \mathfrak{g}_p]^2 U_{t_k}^{p,c,\Delta} \end{aligned}$$

$$+2\left(P^{p,c}A\right)^2\frac{1}{2!}A^2EG(\xi_1)G''(\xi_1)U_{t_k}^{p,c,\Delta}$$

$$-2\frac{1}{3!}P^{p,c}A^4EG'''(\xi_1)E(X_{t_k}-\widehat{X}_{t_k}^{p,c,\Delta})^4\Big\}\Delta^2+o(\Delta^2). \tag{3.2}$$

Non-centered limiter: In this case, with $Y_t^{p,\Delta}=\sum_k I(t_{k-1}\le t<t_k)Y_{t_{k-1}}^{p,\Delta}$ and $Y_0^{p,\Delta}=0$,

$$Y_{t_k}^{p,\Delta}-Y_{t_{k-1}}^{p,\Delta}=\frac{\sqrt{\Delta}}{\mathfrak{I}_p}G\left(\frac{Y_{t_k}-Y_{t_{k-1}}}{\sqrt{\Delta}}\right)$$

(compare with (2.5)) we arrive at

$$d\widehat{X}_t^{p,\Delta}=a\widehat{X}_t^{p,\Delta}dt+AP^{p,c}\mathfrak{I}_p\left(dY_t^{p,\Delta}-A\widehat{X}_t^{p,\Delta}dt\right). \tag{3.3}$$

Denote $U_{t_k}^{p,\Delta}=E\left(X_{t_k}-\widehat{X}_{t_k}^{p,\Delta}\right)^2$ and $B_{t_k}^{p,\Delta}=E\left(X_{t_k}-\widehat{X}_{t_k}^{p,\Delta}\right)$. Under the assumptions made for the case of the centered filter we get

$$U_{t_{k+1}}^{p,\Delta}=(1+a\Delta-\mathfrak{I}_pA^2P^{p,c}\Delta)^2U_{t_k}^{p,\Delta}+b_{t_k}^2\Delta+\left(P^{p,c}A\right)^2\mathfrak{I}_p\Delta$$

$$+\Big\{\left(P^{p,c}A\right)^2A^2E[G'(\xi_1)-\mathfrak{I}_p]^2EX_{t_k}^2$$

$$+2\left(P^{p,c}A\right)^2\frac{1}{2!}A^2EG(\xi_1)G''(\xi_1)EX_{t_k}^2$$

$$-2\frac{1}{3!}P^{p,c}A^4EG'''(\xi_1)ED_{t_k}^{p,\Delta}X_{t_k}^3\Big\}\Delta^2+o(\Delta^2). \tag{3.4}$$

Comparison of the limiters: Put $\delta_{t_k}^{\Delta}=U_{t_k}^{p,\delta}-U_{t_k}^{p,c,\Delta}$. (3.2) and (3.4) imply that

$$\delta_{t_{k+1}}^{\Delta}=(1+a\Delta-\mathfrak{I}_pA^2P^{p,c}\Delta)^2\delta_{t_k}^{\Delta}$$

$$+\Big\{\left(P^{p,c}A\right)^2A^2E[G'(\xi_1)-\mathfrak{I}_p]^2\left(EX_{t_k}^2-U_{t_k}^{p,c,\Delta}\right)$$

$$+2\left(P^{p,c}A\right)^2\frac{1}{2!}A^2EG(\xi_1)G''(\xi_1)\left(EX_{t_k}^2-U_{t_k}^{p,c,\Delta}\right)$$

$$-2\frac{1}{3!}P^{p,c}A^4EG'''(\xi_1)\left(ED_{t_k}^{p}X_{t_k}^3-E\left[D_{t_k}^{p,c,\Delta}\right]^4\right)\Big\}\Delta^2+o(\Delta^2)$$

$$=(1+a\Delta-\mathfrak{I}_pA^2P^{p,c}\Delta)^2\delta_{t_k}^{\Delta}+\rho^{\Delta}\Delta^2+o(\Delta^2). \tag{3.5}$$

It can be shown $\delta_{t_k}^{\Delta}$ is asymptotically positive in the sense of $\lim_{\Delta\to 0}\rho^{\Delta}>0$. We have, with D_t^p and $D_t^{p,c}$ being limits for $D_t^{p,\Delta}=X_t-\widehat{X}_t^{p,\Delta}$ and $D_t^{p,c,\Delta}-X_t-\widehat{X}_t^{p,c,\Delta}$ respectively, that

$$\lim_{\Delta\to 0}\rho^{\Delta}=(P^{p,c})^2A^4E[G'(\xi_1)-\mathfrak{I}_p]^2\left(EX_t^2-P_t^{p,c}\right)$$

$$+(P^{p,c})^2A^4EG(\xi_1)G''(\xi_1)\left(EX_t^2-P_t^0\right)$$

$$-\frac{1}{3}P^{p,c}A^4EG'''(\xi_1)\left(ED_t^pX_t^3-E[D_t^{p,c}]^4\right)$$

$$=K_1+K_2-K_3. \tag{3.6}$$

The limit processes (X_t, D_t^p) and $D_t^{p,c}$ are Gaussian and therefore,

$$E[D_t^{p,c}]^4 = 3E[D_t^{p,c}]^2 = 3P_t^{p,c}$$
$$ED_t^p X_{t_k}^3 = 3ED_t^p X_t EX_t^2 = 3P_t^{p,c} EX_t^2.$$

Moreover, since $EG(\xi_1)G''(\xi_1) = \int \frac{p'''p'}{p} dx$ and $EG'''(\xi_1) = \int \frac{p'''p'}{p}(x)dx$, we get $K_2 = K_3$. Hence,

$$\lim_{\Delta \to 0} \rho^{\Delta} = (P^{p,c})^2 A^4 E[G'(\xi_1) - \mathfrak{I}_p]^2 \Big(EX_t^2 - P_t^{p,c}\Big) > 0. \tag{3.7}$$

Examples: We compare here three filters: the Kalman filter and nonlinear filters with and without centering. A Gaussian mixture distribution

$$p(x) = \alpha \frac{1}{\sqrt{1\pi\sigma_1}} \exp\left\{ -\frac{x^2}{2\sigma_1^2} \right\} + (1-\alpha) \frac{1}{\sqrt{2\pi\sigma_2}} \exp\left\{ -\frac{x^2}{2\sigma_2^2} \right\}, \quad 0 < \alpha < 1,$$

was chosen for ξ_1. Typical filtering estimates are plotted in Figures 1-2. For relatively small sampling intervals, both nonlinear filters give the same filtering accuracy and it is better, than the one, obtained by the Kalman filter. See Figure 1.

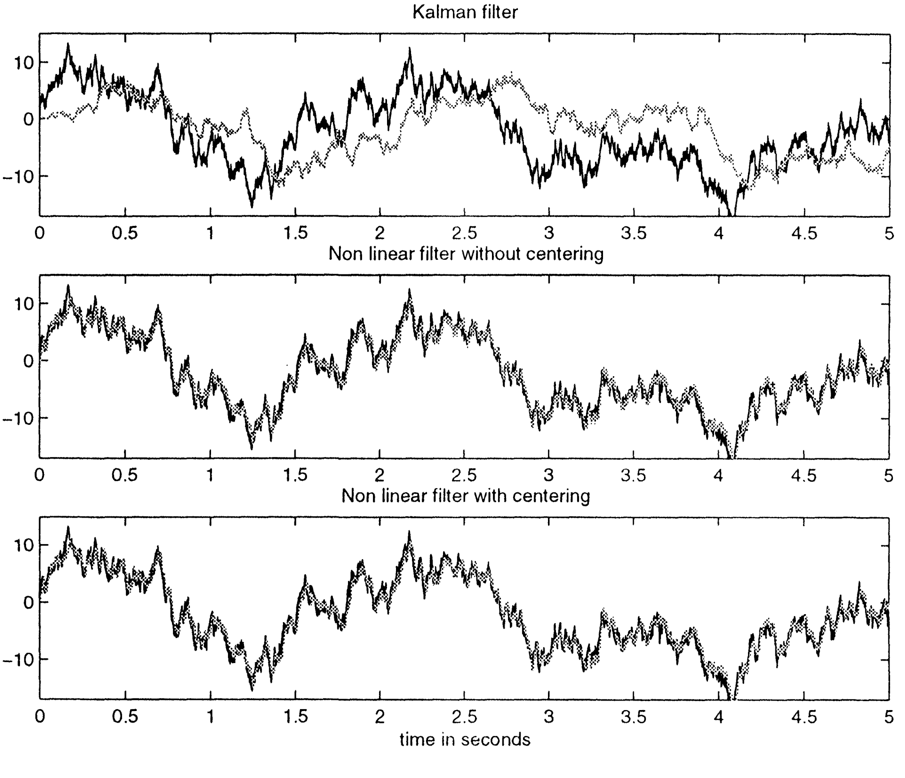

Figure 1: Small sampling interval

An essential increasing of the sampling interval (see Figure 2) causes severe performance degradation of the nonlinear filter without centering, while the filter with centering still gives a satisfactory estimate.

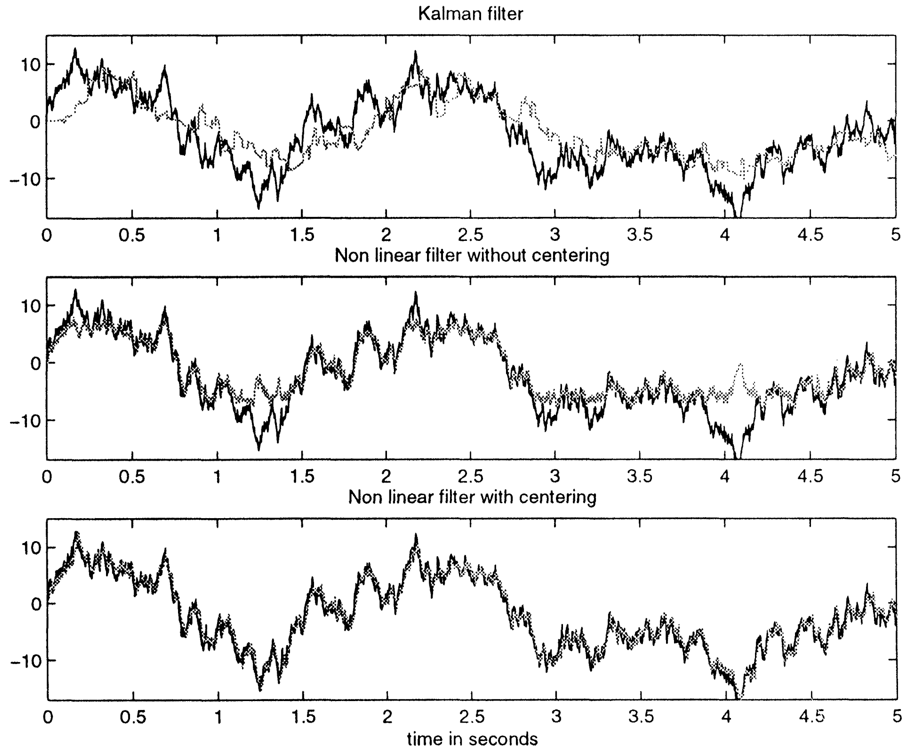

Figure 2: Sampling interval is increased

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