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MONOTONE ITERATIONS FOR DIFFERENTIAL EQUATIONS WITH A PARAMETER

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Consider the problem

$$\begin{cases} y'(t) = f(t, y(t), \lambda), & t \in J = [0, b], \\ y(0) = k_0, \\ G(y, \lambda) = 0. \end{cases}$$

Employing the method of upper and lower solutions and the monotone iterative technique, existence of extremal solutions for the above equation are proved.

Key words: Monotone Iterations, Differential Equations, Monotone Iterative Technique.

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1. Preliminaries

Consider the following differential equation

$$x'(t) = f(t, x(t), \lambda), \quad t \in J = [0, b] \tag{1a}$$

with the boundary conditions

$$x(0) = k_0, \quad x(b) = k_1, \tag{1b}$$

where $f \in C(J \times R \times R, R)$ and $k_0, k_1 \in R$ are given. The corresponding solution of (1) yields a pair of $(x, \lambda) \in C^1(J, R) \times R$ for which problem (1) is satisfied. Problem (1) is called a problem with a parameter.

Conditions on f which guarantee the existence of solutions to (1) are important analysis theorems. Such theorems can be formulated under the assumption that f satisfies the Lipschitz condition with respect to the last two variables with suitable

Lipschitz constants or Lipschitz functions [1-3, 5].

This paper applies the method of lower and upper solutions for proving existence results [4]. Using this technique, we construct monotone sequences, giving sufficient conditions under which they are convergent. Moreover, this method gives a problem solution in a closed set.

Note that $x(b)$ in condition (1b) may appear in a nonlinear way, so it is a reason that we consider the following problem in the place of (1):

$$\begin{cases} y'(t) = f(t, y(t), \lambda), & t \in J = [0, b], \\ y(0) = k_0, \\ G(y, \lambda) = 0. \end{cases} \tag{2}$$

where $f \in C(J \times R \times R, R)$, $G \in C(R \times R, R)$.

2. Main Results

A pair $(v, \alpha) \in C^1(J, R) \times R$ is said to be a lower solution of (2) if:

$$\begin{cases} v'(t) \le f(t, v(t), \alpha), & t \in J, \\ v(0) \le k_0, \\ 0 \le G(v, \alpha), \end{cases}$$

and an upper solution of (2) if the inequalities are reversed.

Theorem 1: *Assume that $f \in C(J \times R \times R, R)$, $G \in C(R \times R, R)$, and:*

1° $y_0, z_0 \in C^1(J, R)$, $\lambda_0, \gamma_0 \in R$, *such that* (y_0, λ_0), (z_0, γ_0) *are lower and upper solutions of problem* (2) *such that* $y_0(t) \le z_0(t), t \in J$ *and,* $\lambda_0 \le \gamma_0$;

2° *f is nondecreasing with respect to the last two variables;*

3° *G is nondecreasing with respect to the first variable;*

4° $G(y, \lambda) - G(y, \beta) \le N(\beta - \lambda)$ *for* $y_0(t) \le y(t) \le z_0(t)$, $t \in J$, $\lambda_0 \le \lambda \le \beta \le \gamma_0$ *with* $N \ge 0$.

Then there exist monotone sequences $\{y_n, \lambda_n\}$, $\{z_n, \gamma_n\}$ such that $y_n(t) \to y(t)$, $z_n(t) \to z(t)$, $t \in J$; $\lambda_n \to \lambda$, $\gamma_n \to \gamma$ as $n \to \infty$; and this convergence is uniformly and monotonically on J. Moreover, (y, λ), (z, γ) are minimal and maximal solutions of problem (2), respectively.

Proof: From the above assumptions, it is known that:

$$\begin{cases} y_0'(t) \le f(t, y_0(t), \lambda_0), & y_0(0) \le k_0, \\ 0 \le G(y_0, \lambda_0), \end{cases} \qquad \begin{cases} z_0'(t) \ge f(t, z_0(t), \gamma_0), & z_0(0) \ge k_0, \\ 0 \ge G(z_0, \gamma_0), \end{cases}$$

and $y_0(t) \le z_0(t)$, $t \in J$, $\lambda_0 \le \gamma_0$. Let (y_1, λ_1), (z_1, γ_1) be the solutions of:

$$\begin{cases} y_1'(t) = f(t, y_0(t), \lambda_0), \quad y_1(0) = k_0, \\ 0 = G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0), \end{cases}$$

and

$$\begin{cases} z_1'(t) = f(t, z_0(t), \gamma_0), \quad z_1(0) = k_0, \\ 0 = G(z_0, \gamma_0) - N(\gamma_1 - \gamma_0), \end{cases}$$

respectively.

Put $p = \lambda_0 - \lambda_1$, so:

$$0 = G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0) \geq -N(\lambda_1 - \lambda_0) = N_p,$$

thus $p \leq 0$ and $\lambda_0 \leq \lambda_1$. Now let $p = \lambda_1 - \gamma_1$. In view of 3° and 4°, we have:

$$\begin{aligned} 0 = G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0) &= G(y_0, \lambda_0) - G(z_0, \gamma_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0) \\ &\leq G(z_0, \lambda_0) - G(z_0, \gamma_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0) \\ &\leq N(\gamma_0 - \lambda_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0) = -Np. \end{aligned}$$

Hence $\lambda_1 \leq \gamma_1$. Set $p = \gamma_1 - \gamma_0$, so that:

$$0 = G(z_0, \gamma_0) - N(\gamma_1 - \gamma_0) \leq -N(\gamma_1 - \gamma_0) = -Np,$$

and thus $\gamma_1 \leq \gamma_0$. As a result, we have:

$$\lambda_0 \leq \lambda_1 \leq \gamma_1 \leq \gamma_0.$$

We shall show that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J. \tag{3}$$

Let $p(t) = y_0(t) - y_1(t)$, $t \in J$, so:

$$p'(t) = y_0'(t) - y_1'(t) \leq f(t, y_0(t), \lambda_0) - f(t, y_0(t), \lambda_0) = 0,$$

and $p(0) = y_0(0) - y_1(0) \leq 0$. This shows that $p(t) \leq 0$, $t \in J$. Therefore $y_0(t) \leq y_1(t)$, $t \in J$. Put $p(t) = y_1(t) - z_1(t)$, $t \in J$. In view of 2°, we have

$$\begin{aligned} p'(t) = y_1'(t) - z_1'(t) &= f(t, y_0(t), \lambda_0) - f(t, z_0(t), \gamma_0) \\ &\leq f(t, z_0(t), \gamma_0) - f(t, z_0(t), \gamma_0) = 0, \end{aligned}$$

and $p(0) = 0$, so $p(t) \leq 0$, $t \in J$, and $y_1(t) \leq z_1(t)$, $t \in J$. Put $p(t) = z_1(t) - z_0(t)$, $t \in J$. We obtain:

$$p'(t) = z_1'(t) - z_0'(t) \leq f(t, z_0(t), \gamma_0) - f(t, z_0(t), \gamma_0) = 0,$$

so $p(t) \leq 0$, $t \in J$, and hence $z_1(t) \leq z_0(t)$, $t \in J$. This shows that (3) is satisfied.

Note that:

$$y_1'(t) - f(t, y_0(t), \lambda_0) \leq f(t, y_1(t), \lambda_1), y_1(0) = k_0,$$

and

$$z_1'(t) - f(t, z_0(t), \gamma_0) \geq f(t, z_1(t), \gamma_1), z_1(0) = k_0.$$

Moreover, in view of 3° and 4°, we have:

$$\begin{aligned} 0 = G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0) &\leq G(y_1, \lambda_0) - N(\lambda_1 - \lambda_0) \\ &= G(y_1, \lambda_0) - G(y_1, \lambda_1) + G(y_1, \lambda_1) - N(\lambda_1 - \lambda_0) \\ &\leq N(\lambda_1 - \lambda_0) + G(y_1, \lambda_1) - N(\lambda_1 - \lambda_0) = G(y_1, \lambda_1), \end{aligned}$$

and

$$\begin{aligned} 0 = G(z_0, \gamma_0) - N(\gamma_1 - \gamma_0) &\geq G(z_1, \gamma_0) - N(\gamma_1 - \gamma_0) \\ &= G(z_1, \gamma_0) - G(z_1, \gamma_1) + G(z_1, \gamma_1) - N(\gamma_1 - \gamma_0) \end{aligned}$$

$$\geq -N(\gamma_1-\gamma_0)+G(z_1,\gamma_1)-N(\gamma_1-\gamma_0)=G(z_1,\gamma_1).$$

Consequently, (y_1,λ_1), (z_1,γ_1) are lower and upper solutions of problem (2).

Let us assume that

$$\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{k-1} \leq \lambda_k \leq \gamma_k \leq \gamma_{k-1} \leq \cdots \leq \gamma_1 \leq \gamma_0,$$
$$y_0(t) \leq y_1(t) \leq \ldots \leq y_{k-1}(t) \leq y_k(t) \leq z_k(t) \leq z_{k-1}(t) \leq \ldots \leq z_1(t) \leq z_0(t),$$
$$t \in J$$

and

$$\begin{cases} y_k'(t) \leq f(t,y_k(t),\lambda_k), \quad y_k(0)=k_0, \\ 0 \leq G(y_k,\lambda_k), \end{cases} \qquad \begin{cases} z_k'(t) \geq f(t,z_k(t),\gamma_k), \quad z_k(0)=k_0, \\ 0 \geq G(z_k,\gamma_k) \end{cases}$$

for some $k>1$. We shall prove that:

$$\begin{cases} \lambda_k \leq \lambda_{k+1} \leq \gamma_{k+1} \leq \gamma_k, \\ y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \quad t \in J, \end{cases} \tag{4}$$

and

$$\begin{cases} y_{k+1}'(t) \leq f(t,y_{k+1}(t),\lambda_{k+1}), \quad y_{k+1}(0)=k_0, \\ 0 \leq G(y_{k+1},\lambda_{k+1}), \end{cases}$$

$$\begin{cases} z_{k+1}'(t) \geq f(t,z_{k+1}(t),\gamma_{k+1}), \quad z_{k+1}(0)=k_0, \\ 0 \geq G(z_{k+1},\gamma_{k+1}), \end{cases}$$

where

$$\begin{cases} y_{k+1}'(t) = f(t,y_k(t),\lambda_k), \quad y_{k+1}(0)=k_0, \\ 0 = G(y_k,\lambda_k)-N(\lambda_{k+1}-\lambda_k), \end{cases}$$

$$\begin{cases} z_{k+1}'(t) = f(t,z_k(t),\gamma_k), \quad z_{k+1}(0)=k_0, \\ 0 = G(z_k,\gamma_k)-N(\gamma_{k+1}-\gamma_k). \end{cases}$$

Put $p=\lambda_k-\lambda_{k+1}$, so:

$$0=G(y_k,\lambda_k)-N(\lambda_{k+1}-\lambda_k) \geq -N(\lambda_{k+1}-\lambda_k)=Np,$$

and hence $\lambda_k \leq \lambda_{k+1}$. Let $p=\lambda_{k+1}-\gamma_{k+1}$. In view of 3° and 4°, we see that:

$$\begin{aligned} 0 &= G(y_k,\lambda_k)-N(\lambda_{k+1}-\lambda_k) \\ &= G(y_k,\lambda_k)-G(z_k,\gamma_k)-N(\lambda_{k+1}-\lambda_k)+N(\gamma_{k+1},\gamma_k) \\ &\leq G(z_k,\lambda_k)-G(z_k,\gamma_k)-N(\lambda_{k+1}-\lambda_k)+N(\gamma_{k+1}-\gamma_k) \\ &\leq N(\gamma_k-\lambda_k)-N(\lambda_{k+1}-\lambda_k)+N(\gamma_{k+1}-\gamma_k) = -Np. \end{aligned}$$

Hence we have $\lambda_{k+1} \leq \gamma_{k+1}$. Now, let $p = \gamma_{k+1} - \gamma_k$. Then:

$$0 = G(z_k, \gamma_k) - N(\gamma_{k+1} - \gamma_k) \leq -Np,$$

so $\gamma_{k+1} \leq \gamma_k$, which shows that the first inequality of (4) is satisfied.

As before, we set $p(t) = y_k(t) - y_{k+1}(t)$, $t \in J$. Then:

$$p'(t) = y_k'(t) - y_{k+1}'(t) \leq f(t, y_k(t), \lambda_k) - f(t, y_k(t), \lambda_k) = 0,$$

and $p(0) = 0$, so $y_k(t) \leq y_{k+1}(t)$, $t \in J$. We observe that for $p(t) = y_{k+1}(t) - z_{k+1}(t)$, $t \in J$, we have

$$p'(t) = y_{k+1}'(t) - z_{k+1}'(t) - f(t, y_k(t), \lambda_k) - f(t, z_k(t), \gamma_k)$$
$$\leq f(t, z_k(t), \gamma_k) - f(t, z_k(t), \gamma_k) = 0$$

which proves that $y_{k+1}(t) \leq z_{k+1}(t)$, $t \in J$. Put $p(t) = z_{k+1}(t) - z_k(t)$, $t \in J$. Then we have:

$$p'(t) = z_{k+1}'(t) - z_k'(t) \leq f(t, z_k(t), \gamma_k) - f(t, z_k(t), \gamma_k) = 0,$$

so $z_{k+1}(t) \leq z_k(t)$, $t \in J$. Therefore:

$$y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \quad t \in J.$$

It is simple to show that (y_{k+1}, λ_{k+1}), (z_{k+1}, γ_{k+1}) are lower and upper solutions of problem (2).

Hence, by induction, we have:

$$\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \gamma_n \leq \ldots \leq \gamma_1 \leq \gamma_0,$$
$$y_0(t) \leq y_1(t) \leq \ldots \leq y_n(t) \leq z_n(t) \leq \ldots \leq z_1(t) \leq z_0(t), \ t \in J$$

for all n. Employing standard techniques [4], it can be shown that the sequences $\{y_n, \lambda_n\}$, $\{z_n, \gamma_n\}$ converge uniformly and monotonically to (y, λ), (z, γ), respectively. Indeed, (y, λ) and (z, γ) are solutions of problem (2) in view of the continuity of f and G, and the definitions of the above sequences.

We have to show that if (u, β) is any solution of problem (2) such that:

$$y_0(t) \leq u(t) \leq z_0(t), \ t \in J, \text{ and } \lambda_0 \leq \beta \leq \gamma_0,$$

then:

$$y_0(t) \leq y(t) \leq u(t) \leq z(t) \leq z_0(t), \ t \in J, \text{ and } \lambda_0 \leq \lambda \leq \beta \leq \gamma \leq \gamma_0.$$

To show this, we suppose that:

$$y_k(t) \leq u(t) \leq z_k(t), \ t \in J, \text{ and } \lambda_k \leq \beta \leq \gamma_k$$

for some k. Put $\beta = \lambda_{k+1} - \beta$. Then, in view of 3° and 4°, we have

$$0 = G(y_k, \lambda_k) - N(\lambda_{k+1} - \lambda_k) \leq G(u, \lambda_k) - N(\lambda_{k+1} - \lambda_k)$$
$$= G(u, \lambda_k) - G(u, \beta) - N(\lambda_{k+1} - \lambda_k)$$
$$\leq N(\beta - \lambda_k) - N(\lambda_{k+1} - \lambda_k) = -Np,$$

so $p \leq 0$, and hence $\lambda_{k+1} \leq \beta$. Let $p = \beta - \gamma_{k+1}$. Then we obtain:

$$0 = G(u, \beta) \leq G(z_k, \beta) = G(z_k, \beta) - G(z_k, \gamma_k) + N(\gamma_{k+1} - \gamma_k)$$

$$\leq N(\gamma_k - \beta) + N(\gamma_{k+1} - \gamma_k) = -Np,$$

and hence $p \leq 0$, so $\beta \leq \gamma_{k+1}$. This shows that:

$$\lambda_{k+1} \leq \beta \leq \gamma_{k+1}.$$

As before, we set $p(t) = y_{k+1}(t) - u(t)$, $t \in J$. In view of 2°, we obtain:

$$p'(t) = y'_{k+1} - u'(t) = f(, y_k(t), \lambda_k) - f(t, u(t), \beta)$$
$$\leq f(t, u(t), \beta) - f(t, u(t), \beta) = 0;$$

hence $p(t) \leq 0$, $t \in J$, and $y_{k+1}(t) \leq u(t)$, $t \in J$. Now let $p(t) = u(t) - z_{k+1}(t)$, $t \in J$. We see that:

$$p'(t) = u'(t) - z'_{k+1}(t) = f(t, u(t), \beta) - f(t, z_k(t), \gamma_k)$$
$$\leq f(t, z_k(t), \gamma_k) - f(t, z_k(t), \gamma_k) = 0,$$

and $p(t) \leq 0$, $t \in J$, so $u(t) \leq z_{k+1}(t)$, $t \in J$. This shows that:

$$y_{k+1}(t) \leq u(t) \leq z_{k+1}(t),\ t \in J.$$

By induction, this proves that the inequalities:

$$y_n(t) \leq u(t) \leq z_n(t),\quad t \in J, \text{ and } \lambda_n \leq \beta \leq \gamma_n$$

are satisfied for all n. Taking the limit as $n \to \infty$, we conclude that:

$$y(t) \leq u(t) \leq z(t),\ t \in J, \text{ and } \lambda \leq \beta \leq \gamma.$$

Therefore, $(y, \lambda), (z, \gamma)$ are minimal and maximal solutions of (2). The proof is complete.

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