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ON THE VARIANCE OF THE NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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The asymptotic estimate of the expected number of real zeros of the polynomial $T(\theta) = g_1 \cos\theta + g_2 \cos 2\theta + \ldots + g_n \cos n\theta$ where g_j $(j = 1, 2, \ldots, n)$ is a sequence of independent normally distributed random variables is known. The present paper provides an upper estimate for the variance of such a number. To achieve this result we first present a general formula for the covariance of the number of real zeros of any normal process, $\xi(t)$, occurring in any two disjoint intervals. A formula for the variance of the number of real zeros of $\xi(t)$ follows from this result.

Key words: Gaussian Random Processes, Number of Real Roots, Number of Crossings, Random Trigonometric Polynomial.

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1. Introduction

Let

$$T(\theta) \equiv T_n(\theta, \omega) = \sum_{j=1}^{n} g_j(\omega) \cos j\theta, \tag{1.1}$$

where $g_1(\omega), g_2(\omega), \ldots, g_n(\omega)$ is a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{A}, \Pr)$, each normally distributed with mean zero and variance one. Much has been written concerning $N_K(0, 2\pi)$, the number of crossings of a fixed level K by $T(\theta)$, in the interval $(0, 2\pi)$. From the work of Dunnage [2] we know that, for all sufficiently large n, the mathematical expectation of $N_0(0, 2\pi) \equiv N(0, 2\pi)$ is asymptotic to $2n/\sqrt{3}$. In [3] and [5] we show that this asymptotic number of crossings remains invariant for any $K \equiv K_n$ such that $K^2/n \to 0$ as $n \to \infty$. However, less information is known about the variance of $N(0, 2\pi)$. The only attempt so far is in [4], where an (fairly large) upper bound is obtained. Indeed this could be justified since the problem with finding the variance consists of different levels of difficulties compared with finding the mean. The degree of difficulty with this challenging problem is reflected in the delicate work of Maslova [8] and Sambandham et al. [7] who have obtained the variance of N for the case of random algebraic polynomial,

$\sum_{j=0}^{n} g_j x^j$; a case involving analysis that is usually easier to handle. Qualls [9] also studied the variance of the number of real roots of a random trigonometric polynomial. However, he studied a different type of polynomial $\sum_{j=0}^{n} a_j \cos j\theta + b_j \sin j\theta$ which has the property of being stationary and for which a special theorem has been developed by Cramér and Leadbetter [1].

Here we look at the random trigonometric polynomial (1.1) as a non-stationary random process. First we are seeking to generalize Cramér and Leadbetter's [1] works concerning factorial moments which are mainly for the stationary case. To evaluate the variance specially, and some other applications generally, it is important to consider the covariance of the number of real zeros of $\xi(t)$ in any two disjoint intervals. To this end, let $\xi(t)$ be a (non-stationary) real-valued separable normal process possessing continuous sample paths, with probability one, such that for any $\theta_1 \neq \theta_2$ the joint normal process $\xi(\theta_1)$, $\xi(\theta_2)$, $\xi'(\theta_1)$ and $\xi'(\theta_2)$ is non-singular. Let (a,b) and (c,d) be any disjoint intervals on which $\xi(t)$ is defined. The following theorem and the formula for the mean number of zero crossings [1, page 85] obtain the covariance of $N(a,b)$ and $N(c,d)$.

Theorem 1: *For any two disjoint intervals, (a,b) and (c,d) on which the process $\xi(\cdot)$ is defined, we have*

$$E\{N(a,b)N(c,d)\} = \int_c^d \int_a^b \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xy| \, p_{\theta_1,\theta_2}(0,0,x,y) dx\, dy\, d\theta_1\, d\theta_2$$

where for $a \le \theta_1 \le b$ and $c \le \theta_2 \le d$, $p_{\theta_1,\theta_2}(z_1,z_2,x,y)$ denotes the four dimensional density function of $\xi(\theta_1)$, $\xi(\theta_2)$, $\xi'(\theta_1)$ and $\xi'(\theta_2)$.

A modification of the proof of Theorem 1 will yield the following theorem which, in reality, is only a corollary of Theorem 1.

Theorem 2: *For $p_{\theta_1,\theta_2}(z_1,z_2,x,y)$ defined as in Theorem 1 we have*

$$EN^2(a,b) = \int_a^b \int_a^b \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xy| \, p_{\theta_1,\theta_2}(0,0,x,y) dx\, dy\, d\theta_1\, d\theta_2.$$

By applying Theorem 2 to the random trigonometric polynomial (1.1) we will be able to find an upper limit for the variance of its number of zeros. This becomes possible by using a surprising and nontrivial result due to Wilkins [12] which reduces the error term involved for $EN(0,2\pi)$ to $O(1)$. We conclude by proving the following.

Theorem 3: *If the coefficients $g_j(\omega)$, $j = 1,2,\ldots,n$ in (1.1) be a sequence of independent random variables defined on probability space $(\Omega, \mathcal{A}, \Pr)$, each normally distributed with mean zero and variance one, then for all sufficiently large n the variance of the number of real zeros of $T(\theta)$ satisfies*

$$var\{N(0,\pi)\} = O(n^{3/2}).$$

2. The Covariance of the Number of Crossings

To obtain the result for the covariance, we shall carry through the analysis for the number of upcrossings, N_u. Indeed, the analysis for the number of downcrossings would be similar and, therefore, the result for the total number of crossings will follow. In order to find $E\{N_u(a,b)N_u(c,d)\}$ we require to refine and extend the proof presented by Cramér and Leadbetter [1, page 205]. However, our proof follows their method and in the following, we highlight the generalization required to obtain our result. Let $a_k = (b-a)k2^{-m} + a$ and similarly $b_l = (d-c)l2^{-m} + c$ for $k, l = 0, 1, 2, \ldots, 2^m - 1$ and we define the random variable $\chi_{k,m}$ and $\chi'_{l,m}$ as

$$\chi_{k,m} = \begin{cases} 1 & \text{if } \xi(a_k) < 0 < \xi(a_{k+1}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi'_{l,m} = \begin{cases} 1 & \text{if } \xi(b_l) < 0 < \xi(b_{l+1}) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

In the following we show that

$$Y_m = \sum_{l=0}^{2^m-1} \sum_{k=0}^{2^m-1} \chi_{k,m}\chi'_{l,m}$$

tends to $N_u(a,b)N_u(c,d)$ as $m \to \infty$ with probability one. See also [1, page 287]. We first note that $E\{N_u(a,b)N_u(c,d)\}$ is finite and therefore $N_u(a,b)N_u(c,d)$ is finite with probability one. Let ν and τ be the number of upcrossings of $\xi(t)$ in (a,b) and (c,d), respectively, and write $t_1, t_2, \ldots, t_\nu$ and $t'_1, t'_2, \ldots, t'_\tau$ for the points of upcrossings of zero by $\xi(t)$ in these two intervals, counted by $N_u(a,b)$ and $N_u(c,d)$. Suppose $I_{s,m}$ and $J_{s',m}$ are the intervals of the form (a_k, a_{k+1}) and (b_k, b_{k+1}) which contains t_s and $t'_{s'}$, $s = 1, 2, \ldots, \nu$ and $s' = 1, 2, \ldots, \tau$, respectively. Then, by continuity of $\xi(t)$, there can be found two sub-intervals for each $I_{s,m}$ and $J_{s',m}$ such that $\xi(t)$ in one is strictly positive and in the other, it is strictly negative. Thus it is apparent that Y_m will count each of $t_s t_{s'}$. That is, $Y_m \geq \nu\tau$, for all sufficiently large m. On the other hand, if $\xi(a_k)\xi(b_{k+1}) < 0$ and $\xi(b_l)\xi(b_{l+1}) < 0$ then $\xi(t)$ must have a zero in (a_k, a_{k+1}) and (b_l, b_{l+1}) and hence $Y_m \leq \nu\tau$ and hence $Y_m \to N_u(a,b)N_u(c,d)$ as $m \to \infty$, with probability one. Now from (2.1) we can see at once that

$$\begin{aligned} E(Y_m) &= \sum_{l=0}^{2^m-1} \sum_{k=0}^{2^m-1} \Pr(\chi_{k,m}\chi'_{l,m} = 1) \\ &= \sum_{l=0}^{2^m-1} \sum_{k=0}^{2^m-1} \Pr(\chi_{k,m} = \chi'_{l,m} = 1). \end{aligned} \tag{2.2}$$

We write η_k for the random variable $2^m[\xi(a_{k+1}) - \xi(a_k)]$ and similarly η'_1 for $2^m[\xi(b_{l+1}) - \xi(b_l)]$, then we have

$$\Pr(\chi_{k,m} = \chi'_{l,m} = 1)$$

$$= \Pr\{0 > \xi(a_k) > 2^{-m}\eta_k, \text{ and } 0 > \xi(b_l) > 2^{-m}\eta'_l\}$$

$$= \int_0^\infty \int_0^\infty \int_0^{2^{-m}x} \int_0^{2^{-m}y} p_{m,k,l}(z_1, z_2, x, y)\, dz_1\, dz_2\, dx\, dy \tag{2.3}$$

where $p_{m,k,l}(z_1, z_2, x, y)$ denotes the four dimensional normal density function for $\xi(a_k)$, $\xi(b_l)$, η_k and η'_k. A simple calculation shows, see [6] or [1, page 207], that if θ_1

and θ_2 are fixed in the interval, (a,b) and (c,d), respectively, and k_m and l_m are such that $a_{k_m} < \theta_1 < a_{k_{m+1}}$ and $b_{l_m} < \theta_2 < b_{l_{m+1}}$ for each m, then all members of the covariance matrix of $p_{m,k,l}(z_1, z_2, x, y)$ will tend to the corresponding members of the covariance matrix of $p_{\theta_1,\theta_2}(z_1, z_2, x, y)$. This co-variance matrix is, indeed, nonsingular. Now let $t = 2^m z_1$ and $r = 2^m z_2$ then from (2.2) and (2.3) we have

$$E(Y_m) = \sum_{l=0}^{2^m-1} \sum_{k=0}^{2^m-1} 2^{-2m} \int_0^\infty \int_0^\infty \int_0^x \int_0^y p_{m,k,l}(2^{-m}t, 2^{-m}r, x, y)\, dt\, dr\, dx\, dy$$

$$= \int_a^b \int_c^d \int_0^\infty \int_0^\infty \int_0^x \int_0^y \Psi_{m,\theta_1,\theta_2}(2^{-m}t, 2^{-m}r, x, y)\, dt\, dr\, dx\, dy\, d\theta_1\, d\theta_2 \tag{2.4}$$

in which $\Psi_{m,\theta_1,\theta_2}(t,r,x,y) = p_{m,k,l}(t,r,x,y)$ for $a_k < \theta_1 < a_{k+1}$ and $b_l < \theta_2 < b_{l+1}$. It follows, similar to [1, page 206], that as $m \to \infty$

$$\Psi_{m,\theta_1,\theta_2}(2^{-m}t, 2^{-m}r, x, y) \to p_{\theta_1,\theta_2}(0,0,x,y)$$

which together with dominated convergence proves Theorem 1.

3. The Variance of the Number of Real Zeros

It will be convenient to evaluate the $EN(N-1)$ rather than the variance itself since $N(N-1)$ can be expressed much more simply. The proof is similar to that established above for covariance, therefore we only point out the generalization required to obtain the result. To avoid degeneration of the joint normal density, $p_{\theta_1,\theta_2}(z_1, z_2, x, y)$, we should omit those zeros in the squares of side 2^{-m} obtained from equal points in the axes (and therefore to evaluate $EN(N-1)$). To this end for any $\mathbf{g} = (g_1, g_2)$ lying in the unit square and $\epsilon > 0$, let $A_{m\epsilon}$ denote the set of all points $\mathbf{g}$ in the unit square such that for all $\mathbf{s}$ belonging to the squares of side 2^{-m} containing $\mathbf{g}$ we have $|s_1 - s_2| > \epsilon$. Let $\lambda_{m\epsilon}$ denote the characteristic function of the set $A_{m\epsilon}$. Finally, similar to the covariance case, let

$$\chi_{k,m} = \begin{cases} 1 & \text{if } \xi(a_k) < 0 < \xi(a_{k+1}) \\ 0 & \text{otherwise} \end{cases}$$

for $k = 0, 1, 2, \ldots, 2^m - 1$, where $a_k = (b-a)k2^{-m} + a$. Now let

$$M_{m\epsilon} = \sum_{k=0}^{2^m-1} \sum_{(l=0,\, l \neq k)}^{2^m-1} \chi_{k,m}\chi_{l,m}\lambda_{m\epsilon}(2^{-m}k, 2^{-m}l). \tag{3.1}$$

Similar to [1, page 205] we show that $M_{m\epsilon}$ is a nondecreasing function of m for any fixed ϵ. It is obvious that $M_{m\epsilon}$ is a nondecreasing function of ϵ for fixed m, and then by two applications of monotone convergence it would be justified to change the order of limits in $\lim_{\epsilon\to 0}\lim_{m\to\infty} M_{m\epsilon}$. To this end, we note that each term of the sums of $M_{m\epsilon}$ corresponds to a square of side 2^{-m}. For fixed $\epsilon > 0$, the typical term is one if both of the following statements are satisfied: (i) every point $\mathbf{s} = (s_1, s_2)$ in the square is such that $|s_1 - s_2| > \epsilon$ and (ii) $\chi_{k,m} = \chi_{l,m} = 1$. When m is increased by one unit, the square is divided into four subsquares, in each of which property (i) still holds. Correspondingly, the typical term of sum is divided into four terms, formed by replacing m by $m+1$ and each k or l by $2k$ and $2l$, for a_{k+1} and a_{l+1}. Since $\chi_{k,m} = \chi_{l,m} = 1$ we must, with probability one, have at least one of these four terms equal one. Hence $M_{m\epsilon}$ is a nondecreasing function of m.

In the following, we show that

$$\lim_{m\to\infty} \lim_{\epsilon\to 0} M_{m\epsilon} = N_u(N_u - 1).$$

We first note that if the typical term in the sum of $M_{m\epsilon}$ is nonzero it follows that $|s_1 - s_2| > 1$, since it is impossible to have $\xi(a_k) < 0 < \xi(a_{k+1})$ and $\xi(a_{k+1}) < 0 < \xi(a_{k+2})$. Therefore, the characteristic function appearing in the formula for $M_{m\epsilon}$ in (3.1) is one and hence

$$\lim_{\epsilon\to 0} M_{m\epsilon} = \sum_{k=0}^{2^m-1} \sum_{(l=0,\, l\neq k)}^{2^m-1} \chi_{k,m}\chi_{l,m}. \tag{3.2}$$

(3.2) is clearly in the form of Y_m defined in Section 2 except that the summations in (3.2) cover all the k and l such that $k \neq l$. Hence from (3.2), we can write

$$\lim_{m\to\infty} \lim_{\epsilon\to 0} M_{m\epsilon} = N_u(N_u - 1).$$

Therefore the same pattern as for the covariance case yields

$$E[N_u(a,b)\{N_u(a,b) - 1\}]$$

$$= \lim_{\epsilon\to 0} \int\int_{D(\epsilon)} \int_0^\infty \int_0^\infty |xy|\, p_{\theta_1,\theta_2}(0,0,x,y)\, dx\, dy\, d\theta_1\, d\theta_2 \tag{3.3}$$

where $D(\epsilon)$ denotes the domain in the two dimensional space with coordinates θ_1, θ_2 such that $a < \theta_1$, $\theta_2 < b$ and $|\theta_1 - \theta_2| > \epsilon$. Now notice that for $\theta_1 = \theta_2 = 0$ the $p_{\theta_1,\theta_2}(0,0,x,y)$ degenerates to just $p_\theta(0,x)$, the two dimensional joint density function of $\xi(\theta)$ and $\xi'(\theta)$. Hence from (3.3) we have

$$E[N_u(a,b)\{N_u(a,b) - 1\}] = \int_a^b \int_a^b \int_0^\infty \int_0^\infty |xy|\, p_{\theta_1,\theta_2}(0,0,x,y)\, dx\, dy\, d\theta_1\, d\theta_2$$

$$- \int_a^b \int_0^\infty |x|\, p_\theta(0,x) d\theta\, dx.$$

Now since $\int_a^b \int_0^\infty |x|\, p_\theta(0,x)\, d\theta$ is $EN_u(a,b)$ the result of Theorem 2 follows.

4. Random Trigonometric Polynomial

To evaluate the variance of the number of real roots of (1.1) in the interval $(0,\pi)$ we use Theorem 2 to consider the interval $(\epsilon', \pi - \epsilon')$. The variances for the intervals $(0,\epsilon')$ and $(\pi - \epsilon', \pi)$ are obtained using an application of Jenson's theorem [10, page

332] or [11, page 125]. We chose $\epsilon' = n^{-1/2}$ which, as we will see later, yields the smallest possible error term. First, for any θ_1 and θ_2 in $(\epsilon', \pi - \epsilon')$, such that $|\theta_1 - \theta_2| > \epsilon$ where $\epsilon = n^{-1/2}$, we evaluate the joint density function of the random variable $T(\theta_1)$, $T(\theta_2)$, $T'(\theta_1)$ and $T'(\theta_2)$. Since for any θ we have

$$\sum_{j=1}^{n} \cos j\theta = [\sin\{(n+1/2)\theta\}/\sin(\theta/2) - 1]/2$$

and also since for the above choice of θ_1 and θ_2, $\theta_1 + \theta_2 < 2(\pi - \epsilon')$ we can show

$$\begin{aligned} A(\theta_1, \theta_2) = \operatorname{cov}\{T(\theta_1), T(\theta_2)\} &= \sum_{j=1}^{n} \cos j\theta_1 \cos j\theta_2 \\ &= [\sin\{(n+1/2)(\theta_1 - \theta_2)\}/\sin\{(\theta_1 - \theta_2)/2\} \\ &\quad + \sin\{(n+1/2)(\theta_1 + \theta_2)\}/\sin\{(\theta_1 + \theta_2)/2\} - 2]/4 \\ &= O(1/\epsilon) + O(1/\epsilon'). \end{aligned} \tag{4.1}$$

Similarly, we can obtain the following two estimates

$$\begin{aligned} C(\theta_1, \theta_2) = \operatorname{cov}\{T'(\theta_1), T(\theta_2)\} &= -\sum_{j=1}^{n} j \sin j\theta_1 \cos j\theta_2 \\ &= (\partial/\partial\theta_1)\{A(\theta_1, \theta_2)\} \\ &= O(n/\epsilon + \epsilon^{-2} + n/\epsilon' + \epsilon'^{-2}) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} B(\theta_1, \theta_2) = \operatorname{cov}\{T'(\theta_1), T'(\theta_2)\} &= \sum_{j=1}^{n} j^2 \sin j\theta_1 \sin j\theta_2 \\ &= (\partial/\partial\theta_2)\{C(\theta_1, \theta_2)\} \\ &= O(n^2/\epsilon + n/\epsilon^2 + \epsilon^{-3} + n^2/\epsilon' + n/\epsilon'^2 + \epsilon'^{-3}). \end{aligned} \tag{4.3}$$

Also in the lemma in [3, page 1405] we obtain

$$\operatorname{var}(T(\theta_1)) = n/2 + O(\epsilon'^{-1}), \quad \operatorname{var}(T'(\theta_1)) = n^3/6 + O(n^2/\epsilon' + n/\epsilon'^2 + \epsilon'^{-3})$$

and

$$\operatorname{cov}\{T(\theta_1), T'(\theta_1)\} = O(n/\epsilon' + \epsilon'^{-2}).$$

These together with (4.1)-(4.3) give the covariance matrix for the joint density function $T(\theta_1)$, $T(\theta_2)$, $T'(\theta_1)$ and $T'(\theta_2)$ as

$$\Sigma = \begin{bmatrix} n/2 + O(\epsilon'^{-1}) & A(\theta_1, \theta_2) & C(\theta_1, \theta_1) & C(\theta_2, \theta_1) \\ A(\theta_1, \theta_2) & n/2 + O(\epsilon'^{-1}) & C(\theta_1, \theta_2) & C(\theta_2, \theta_2) \\ C(\theta_1, \theta_1) & C(\theta_1, \theta_2) & n^3/6 + O(n^2/\epsilon') & B(\theta_1, \theta_2) \\ C(\theta_2, \theta_1) & C(\theta_2, \theta_2) & B(\theta_1, \theta_2) & n^3/6 + O(n^2/\epsilon') \end{bmatrix}. \tag{4.4}$$

This covariance matrix, for all $n \geq 4$, $0 < \theta_1$, $\theta_2 < \pi$ such that $\theta_1 \neq \theta_2$, is positive definite. Hence $|\Sigma| > 0$ and, if Σ_{ij} is cofactor of the (ij)th element of Σ, then $\Sigma_{33} > 0$, $\Sigma_{44} > 0$ and $\Sigma_{34} = \Sigma_{43}$. From [1, page 26] we have

$$p_{\theta_1, \theta_2}(0,0,x,y)$$
$$= (4\pi^2)^{-1} |\Sigma|^{-1/2} \exp[-\{\Sigma_{33}x^2 + \Sigma_{44}y^2 + (\Sigma_{34} + \Sigma_{43})xy\}/2|\Sigma|]. \tag{4.5}$$

Now let $q = (\Sigma_{33}/|\Sigma|)^{1/2}x$ and $s = (\Sigma_{44}/|\Sigma|)^{1/2}y$. Then from (4.5) we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |xy|\, p_{\theta_1, \theta_2}(0,0,x,y)\,dx\,dy$$
$$= (4\pi^2)^{-1} |\Sigma|_{33}^{-1}\Sigma_{44}^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |qs| \exp\{-(q^2 + {}^2 + 2\rho qs)/2\}\,dq\,ds \tag{4.6}$$

where $\rho = (\Sigma_{34} + \Sigma_{43})/2(\Sigma_{33}\Sigma_{44})^{1/2}$ and $0 \leq \rho^2 < 1$. The value of the integral in (4.6) can be obtained by a similar method to [1, page 211]. Let $u = (1-\rho^2)^{1/2}q$ and $v = (1-\rho^2)^{1/2}s$ then we have

$$I = \int_0^{\infty}\int_0^{\infty} \exp\{-(q^2 + s^2 + 2\rho qs)/2\}\,dq\,ds$$

$$= (1-\rho^2)^{-1} \int_0^{\infty}\int_0^{\infty} \exp\{-(u^2 + v^2 + 2\rho uv)/2(1-\rho^2)\}\,du\,dv$$

$$= \pi(1-\rho^2)^{-1/2}\{1/2 - (\pi)^{-1}\int_0^{\rho}(1-z^2)^{-1/2}dz\}$$

$$= (1-\rho^2)^{-1/2}\arccos\rho$$

$$= \phi\csc\phi \tag{4.7}$$

where $\rho = \cos\phi$. Use has been made of the fact that (see for example [1, page 27])

$$\int_0^{\infty}\int_0^{\infty} \exp\{-(u^2 + v^2 - 2\rho uv)/2(1-\rho^2)\}\,du\,dv$$

$$= \pi(1-\rho^2)^{1/2}/2 + (1-\rho^2)^{1/2}\int_0^{\rho}(1-z^2)^{-1/2}dz.$$

Therefore from (4.7) by differentiation we can obtain

$$\int_0^{\infty}\int_{-\infty}^{\infty} qs\exp\{-(q^2 + s^2 + 2\rho qs)\}\,dq\,ds$$

$$= -(dI)/(d\rho) = \csc^2\phi(1 - \phi\cot\phi). \tag{4.8}$$

Using (4.8) we can easily show that

$$\int_0^\infty \int_{-\infty}^0 qs\exp\{-(q^2+s^2+2\rho qs)\}dq\,ds$$

$$= \csc^2\phi\{1+(\pi-\phi)\cot\phi\}$$

which together with (4.8) evaluates the integral in (4.6) as

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \mid qs \mid \exp\{-(q^2+s^2+2\rho qs)/2\}dq\,ds$$

$$= 4\csc^2\phi\{1+(\pi/2-\phi)\cot\phi\}. \tag{4.9}$$

Now from (4.4) we can show

$$\Sigma_{44} = n^5/24 + O(n^4/\epsilon') = \Sigma_{33} \tag{4.10}$$

and

$$\Sigma_{34} = O(n^4/\epsilon') = \Sigma_{43}. \tag{4.11}$$

Also from (4.10) and (4.11) and with the above choice of ϵ', we can obtain

$$\rho = (\Sigma_{34}+\Sigma_{43})/2(\Sigma_{33}\Sigma_{44})^{1/2} = O(1/n\epsilon') \rightarrow 0 \text{ as } n\rightarrow\infty.$$

Therefore $\phi \rightarrow \pi/2$, for all sufficiently large n and hence from (4.9), we can see

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \mid qs \mid \exp\{-(q^2+s^2+2\rho qs)/2\}dq\,ds$$

$$= 4 + O(1/n\epsilon'). \tag{4.12}$$

Also from (4.1)-(4.4) we can write

$$\mid \Sigma \mid \;= \{n/2 + O(\epsilon'^{-1})\}^2\{n^3/6 + O(n^2/\epsilon')\}^2.$$

Therefore from this (4.6) and (4.12) the integrand that appears in (3.3) is asymptotically independent of θ_1 and θ_2 and since by the definition of $D(\epsilon)$, the area of the integration is $(\pi-2\epsilon')^2 - \epsilon(\pi-2\epsilon') + \epsilon^2 = \pi^2 + O(\epsilon+\epsilon')$, we have

$$E[N(\epsilon',\pi-\epsilon')\{N(\epsilon',\pi-\epsilon')-1\}] = n^2/3 + O(n/\epsilon' + n\epsilon + n\epsilon'). \tag{4.13}$$

We now evaluate the mathematical expectation of N^2 in the interval $(0,\epsilon')$. Similar to [2] or [3, page 1407] we apply Jensen's theorem on a random integral function of the complex variable z,

$$T(z,\omega) = \sum_{j=1}^n g_j(\omega)\cos jz.$$

Let $N(r)$ denote the number of real zeros of $T(z,\omega)$ in $z < r$. For any integer j from [3, page 1408] we have

$$\Pr\{N(\epsilon') > 3n\epsilon' + j\} < (2/\sqrt{n})e^{-j/2} + \exp(-j/2 - n^2e^j/2) < 3e^{-j/2}. \tag{4.14}$$

Let $n' = [3n\epsilon']$ be the smallest integer greater than or equal to $3n\epsilon'$ then since $N(\epsilon') \leq 2n$ is a nonnegative integer, from (4.14) and by dominated convergence, for sufficiently large n we have

$$EN^2(\epsilon') = \sum_{j>0} (2j-1)\Pr(N(\epsilon') \geq j)$$

$$= \sum_{0<j\leq n'} (2j-1)\Pr(N(\epsilon') \geq j) + \sum_{j=1}^{n} (2n'-1+2j)\Pr(N(\epsilon') \geq n'+j)$$

$$\leq \sum_{j=1}^{n'} (2j-1) + 3\sum_{j=1}^{n} (2n'-1+1+2j)e^{-j/2} = n'^2 + O(n')$$

$$= O(n^2\epsilon'^2). \tag{4.15}$$

The interval $(\pi-\epsilon', \pi)$ can also be treated in exactly the same way to give the same result. Now we can use the delicate result due to Wilkins [12] which states that $EN(0,\pi) = n/\sqrt{2} + O(1)$. From this and (4.13), (4.15) and since $\epsilon = \epsilon'$, we obtain

$$\operatorname{var}\{N(0,\pi)\} = E\{N(0,\epsilon') + N(\epsilon', \pi-\epsilon') + N(\pi-\epsilon', \pi)\}^2 - \{EN(0,\pi)\}^2$$

$$= n^2/3 + O(n^2\epsilon'^2 + n/\epsilon' + n^2\epsilon') - \{n/\sqrt{3} + O(1)\}^2$$

$$= O(n^2\epsilon'^2 + n/\epsilon' + n^2\epsilon'). \tag{4.16}$$

Use has been made of the fact that $EN(\epsilon', \pi-\epsilon') \sim EN(0,\epsilon') = O(n\epsilon')$, see [5, page 556] and therefore $E[N(0,\epsilon')N(\epsilon',\pi-\epsilon')] = nO(N(0,\epsilon')) = O(n^2\epsilon')$ and also from (4.15), $EN^2(0,\epsilon') \sim EN^2(\pi-\epsilon',\pi) = O(n^2\epsilon'^2)$. Finally from (4.16) and since $\epsilon' = n^{-1/2}$, we have the proof of Theorem 3.

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