A COMPUTATIONAL APPROACH TO PIVOT SELECTION IN THE LP RELAXATION OF SET PROBLEMS

F. DJANNATY AND B. ROSTAMY

Received 30 January 2006; Revised 17 July 2006; Accepted 10 October 2006

It has long been known to the researchers that choosing a variable having the most negative reduced cost as the entering variable is not the best choice in the simplex method as shown by Harris (1975). Thus, suitable modifications in the pivot selection criteria may enhance the algorithm. Previous efforts such as that by Dantzig and steepest-edge rules for pivot selection are based on finding a unified strategy for entering variable in all linear programming problems. In the present work, a number of strategies for pivot selection in the LP relaxation of the set problems are proposed which consider the specific knowledge of the problem. A significant reduction in the number of iterations is achieved for a set of randomly generated test problems.

1. Introduction

At the very inception of linear programming, Dantzig realized that the criterion of most negative reduced cost for selecting a new basic variable, chosen for computational ease, was not necessarily the best [5].

Many other techniques have subsequently been suggested such as "positive normalized" procedure of Dickson and Friderick [6]. Computational experiments by Wolf and Cutler [3] and kuhn and Quandt [7] showed that both the greatest change and particularly the normalized procedures were much superior to the criterion of most negative reduced cost. Since they were devised using the tableau form simplex method, they had to be discarded as impractical when the product form simplex method supersede it.

The first practical steepest-edge algorithm was developed by Harris [5] which was significantly superior to the standard simplex method. A practicable steepest-edge simplex algorithm developed by Goldfarb and Reid [7] proved to be better than reduced cost algorithm of Dantzig for updating weighted factors which are very similar to the formulas

Hindawi Publishing Corporation
Journal of Applied Mathematics and Decision Sciences
Volume 2006, Article ID 76187, Pages 1–11
DOI 10.1155/JAMDS/2006/76187

developed by kuhn and Quandt [7]. A number of other steepest-edge algorithms were presented by Forrest and Goldfarb [3] which involve both primal and dual simplexes.

2. Set problems

Set problems comprising set covering, set partitioning, and set packing have attracted attention for many years and have application in airline crew scheduling, bus crew scheduling, plant location, circuit switching, and information retrieval assembly line balancing [2].

Let $M = \{1,2,\dots,m\}$ be the set of m integer and let S denote a set of n subset of M. Thus

$$\begin{gathered} N = \{1,2,\dots,n\}, \\ S = \{s_1,s_2,\dots,s_n\} \quad \text{where } S_j \subseteq M,\ j \in N\ (i = 1,\dots,m,\ j = 1,\dots,n). \end{gathered} \tag{2.1}$$

Let

$$a_{ij} = \begin{cases} 1, & i \in S_j, \\ 0, & i \notin S_j. \end{cases} \tag{2.2}$$

The set covering problem (SCP) can be defined as follows:

$$\begin{gathered} \min \sum_{j=1}^{n} C_j x_j, \\ \text{s.t.} \sum a_{ij} x_j \ge 1 \quad (i = 1,\dots,m), \\ x_j \in \{0,1\} \quad (j = 1,\dots,n). \end{gathered} \tag{2.3}$$

The decision variable x_j indicates whether s_j is selected or not and c_j is the cost associated with selecting s_j. The problem can be interpreted as finding the minimum cost selecting of subsets of S.

If we replace "≥" by "=" in each of the constraints of the above model, the modified problem is called the set partitioning problem (SPP). If "≥" is replaced by "≤" and the objective function is to be maximized, the resulting model is the set packing problem (SPK).

Graph theoretic relaxation of problems is an alternative way of finding quick and sharp lower bounds for set problems [2].

3. Shortest route relaxation of the set problems

Shortest route relaxation of the set problems as described in [1] is as follows. Each column a_j is decomposed into K_j arcs, where each arc corresponds to a segment of ones. If a segment of ones covers row k to row $k+p$, then the associated arc runs from vertex k (row index) to vertex $k+p+1$. A set of columns which constitute the shortest route from vertex 1 to vertex $m+1$ defines a feasible solution to the SCP(SPP). Row $m+1$ which corresponds to vertex $m+1$ is identical to row m. The SCP can be written as

$$\begin{aligned} \min \sum_{j=1}^{n} C_j X_j, \\ \text{s.t.} \sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad (i = 1,\dots,m), \\ x_j \in \{0,1\} \quad (j = 1,\dots,n). \end{aligned} \tag{3.1}$$

Let $H'_j = \{i \mid a_{ij} = 1, a_{i-1,j} = 0,\ i \in \{1,2,\dots,m\}\}$ where $a_{0j} = 0$. Let $K_j = |h'_j|$ denote the number of segments of arcs in column a_j. Let SP define the cardinality of segment P in column a_j where $P = \{1,\dots,K\}$. Let S_j denote the index set of the segment cardinality for the column a_j, such that

$$S_j = \{S_1,\dots,S_{k_j}\}. \tag{3.2}$$

Let H'_j be reexpressed as $H'_j = \{i_1, i_2,\dots,i_{k_j}\}$. Introduce the vertex set V corresponding to the rows $i = 1,\dots,m,m+1$ such that

$$A_j = \{(v_{i_1}, v_{i_1+s_1}),\dots,(v_{i_k}, v_{i_{k_j}+s_{k_j}})\}, \quad j = 1,\dots,n. \tag{3.3}$$

Let the associated cost for each arc in the arc set A_j be defined as

$$d^j_{pq} = \frac{(q-p)c_j}{|H_j|} \quad \text{such that} \sum_{(v_p,v_q)\in A_j} d^j_{pq} = c_j. \tag{3.4}$$

Note that this is only one cost allocation strategy. A number of strategies for the shortest route relaxation of the set covering problem are proposed in [2]. The role of the row counts and column counts in upgrading the cost allocation strategy for set problems is emphasized.

Based on this work, allocating small cost to rows having small row counts in the cost allocation strategy enhances the shortest route relaxation of the set problems. We were motivated to apply a similar strategies involving row counts and column counts to the LP relaxation of the set problems. It turned out that some of these strategies can be applied to the LP relaxation of these problems after some modifications. In order to evaluate the proposed strategies a number of test problems were generated randomly whose details are discussed in Section 4.

4. Problem-specific knowledge

Consider the linear programming problem

$$\begin{aligned} \max Z = CX, \\ \text{s.t.}\ AX = b, \\ X \geq 0, \end{aligned} \tag{4.1}$$

where A is a matrix of order $m \times n$, b is a column vector $m \times 1$, and C is a row vector $1 \times n$. George B. Dantzig developed simplex method which deals with the linear programming problems. Simplex method is an iterative process where in each iteration the algorithm moves from one extreme point to an adjacent extreme point with a better objective function. This move involves selecting a nonbasic variable as entering variable, selecting a basic variable as leaving variable, and replacing it by the entering variable. Dantzig rule of pivot selection involves choosing the column of the most negative reduced cost. This criterion uses information obtained from the cost row (C), right-hand sides, and the pivot column. As only a small part of the problem information is utilized and problem specific knowledge is not considered, this criterion is not the best strategy for the pivot selection. Harris [5] has made a significant effort to use some information about technological coefficients of the problem and reported promising results. He proposes a general strategy for pivot selection; however, the problem specific knowledge is not fully utilized.

We will show that a pivot selection strategy may work well for a particular class of problems while it may not be suitable for another problem instance.

Linear programming is extensively used in solving integer programming problem and most successful approaches to IP problem are usually based on linear programming. In this paper, pivot selection in an important class of problems, namely, set problem is investigated.

5. Pivot selection strategies

Let **R** be the set of indices of the nonbasic variables in the linear programming problem, c_j the coefficient of the original variable x_j, in the initial tableau, $\bar{c}_j$ is the coefficient of x_j in row zero of other iterations (i.e., $\bar{c}_j = z_j - c_j$), and $R_r = \{j \in R \mid \overline{C}_j < 0\}$.

Strategy 1. The relation $Z = C_B B^{-1} b$, where B is the current basis, can justify the proportionality of the norm with $\overline{C}_j$ and b_i in the numerator. As the number of nonzeros in a column is restrictive and imposes some restriction on each equation where it has nonzero coefficient then there should be some inverse proportionality to h_j.

Let $j \in R_r$, then we define

$$\text{norm}(j) = \frac{(\overline{C}_j)^2 \times \sqrt{\sum_{i=1}^{m} a_{ij} b_i^2}}{\sqrt{1 + h_j^2}}, \tag{5.1}$$

where h_j is the number of nonzero entries in column j. A column having the largest norm is selected as the pivot column.

Strategy 2. The importance of row counts in enhancing the shortest route relaxation of the set covering problem is investigated in [1]. We were motivated to apply the same approach to the LP relaxation of set problems. One way of showing the restrictive effect of selecting pivot column on the linear system of equations $AX = b$ is by involving row count in the pivot selection. Row count of each row reflects the number of variables that are affected by choosing one of the nonbasic variables appearing in that row with a positive coefficient to enter the basis.

Let $j \in R_r$ and let f_i be the number of nonzero entries in row i, then for $(i = 1,\dots,m)$ we define

$$\text{norm}(j) = \frac{(\overline{C}_j)^2 \times \sqrt{\sum_{i=1}^{m} a_{ij} b_i^2}}{\sqrt{1 + \sum_{i=1}^{m} a_{ij} (\text{rowcount}(i))^2}}. \tag{5.2}$$

A column having the largest norm is selected as the pivot column.

Strategy 3. Let $j \in R_r$, then we define

$$\text{norm}(j) = \frac{\overline{C}_j}{\sqrt{1 + \sum_{i=1}^{m} a_{ij} (\text{rowcount}(i))^2}}. \tag{5.3}$$

A column having the largest norm is selected as the pivot column.

Strategy 4. Let $j \in R_r$, then we define

$$\text{norm}(j) = \frac{\overline{C}_j}{1 + h_j^2}. \tag{5.4}$$

A column having the largest norm is selected as the pivot column.

Strategy 5. Let $j \in R_r$, then we define

$$\text{norm}(j) = \frac{(\overline{C}_j)^2}{(1 + h_j^2) \times \left(\sqrt{1 + \sum_{i=1}^{m} a_{ij} (\text{rowcount}(i))^2}\right)}. \tag{5.5}$$

A column having the largest norm is selected as the pivot column.

Strategy 6. Let $j \in R_r$, then we define

$$\text{norm}(j) = \frac{\overline{C}_j}{\sqrt{1 + \sum_{i=1}^{m} a_{ij} e^{\text{rowcount}(i)}}}. \tag{5.6}$$

A column having the largest norm is selected as the pivot column.

Strategy 7. Let $j \in R_r$, then we define

$$\text{norm}(j) = \text{random}(\overline{C}_j). \tag{5.7}$$

As noted in [8] the effect of parameters C, B, and A is investigated.

Table 6.1. SCP problem.

SCP1, 200 × 200		
Run time	No. of iterations	Strategy
3.15	5195	Dantzig (P)
1.20	2399	Strategy 1
3.45	465	Strategy 3
0.24	450	Strategy 4
1.50	354	Strategy 5
3.37	450	Strategy 6
3.31	5358	Strategy 7

6. Computational results

6.1. Test problems. Avis and Chavtal [6] have generated a class of LP problems which are used as test problems. The general form of these problems is as follows:

$$\begin{aligned} &\max \sum_{j=1}^{n} X_j, \\ \text{s.t.} &\sum_{j=1}^{n} a_{ij}X_j \le 10^4 \quad (i = 1,\dots,m), \\ &X_j \ge 0 \quad (j = 1,\dots,n). \end{aligned} \tag{6.1}$$

The only parameters which have to be specified are a_{ij}'s which are taken as random integers in the interval [1,1000].

Inspired by the random model proposed by Avis and Chavtal, we used a similar model to create our test problems which can be described as follows:

$$\begin{aligned} &\min \sum_{j=1}^{n} C_jX_j, \\ \text{s.t.} &\sum_{j=1}^{n} a_{ij}X_j \ge 1 \quad (i = 1,\dots,m), \\ &0 \le X_j \le 1 \quad (j = 1,\dots,n), \end{aligned} \tag{6.2}$$

where a_{ij}'s are binary random integers, and C_j's are random integers, taken in the interval [1,100]. The smallest and largest problems considered are 10×10 and 500×700, respectively. The reason that a_{ij}'s are binary is that we are dealing with set problems.

As can be seen from Tables 6.1, 6.2, 6.3, and 6.4, all strategies except for Strategy 2 are better than Dantzig rule both in the sense of the number of iterations and in the sense of execution times. Strategy 5 has reduced the number of iterations by 40 times which causes a significant reduction in round-off errors. As it is expected, random strategy is the worst in the sense of the number of iterations.

Computational result for SPP problems. As can be seen in Tables 6.5, 6.6, 6.7, and 6.8 Strategy 1 is better than other strategies, both in the number of iterations and in the sense

Table 6.2. SCP problem.

SCP2, 250 × 300		
Run time	No. of iterations	Strategy
11.40	10196	Dantzig (P)
2.15	1995	Strategy 1
7.45	564	Strategy 3
0.49	524	Strategy 4
4.50	324	Strategy 5
8.12	524	Strategy 6
11.23	9976	Strategy 7

Table 6.3. SCP problem.

CP3, 350 × 500		
Run time	No. of iterations	Strategy
45.20	16124	Dantzig (P)
11.20	3947	Strategy 1
20.35	743	Strategy 3
2.20	701	Strategy 4
21.00	452	Strategy 5
24.30	701	Strategy 6
46.50	17112	Strategy 7

Table 6.4. SCP problem.

SCP4, 500 × 700		
Run time	No. of iterations	Strategy
66.15	23839	Dantzig (P)
34.30	7411	Strategy 1
41.23	1416	Strategy 3
8.27	1381	Strategy 4
39.44	981	Strategy 5
45.56	1381	Strategy 6
69.58	25912	Strategy 7

Table 6.5. SPP problem.

SPP1, 200 × 200		
Run time	No. of iterations	Strategy
0.30	1359	Dantzig (P)
0.15	372	Strategy 1
0.50	317	Strategy 2
0.52	352	Strategy 5
0.41	1590	Strategy 7

Table 6.6. SPP problem.

SPP2, 250 × 300		
Run time	No. of iterations	Strategy
1.14	1712	Dantzig (P)
0.37	637	Strategy 1
2.40	555	Strategy 2
4.48	1150	Strategy 5
1.27	1910	Strategy 7

Table 6.7. SPP problem.

SPP3, 350×500		
Run time	No. of iterations	Strategy
7.02	4187	Dantzig (P)
3.01	1605	Strategy 1
22.37	1594	Strategy 2
27.39	2011	Strategy 5
8.11	4980	Strategy 7

Table 6.8. SPP problem.

SPP4, 500 × 700		
Run time	No. of iterations	Strategy
42.15	24971	Dantzig (P)
16.47	7983	Strategy 1
89.14	7251	Strategy 2
102.44	11298	Strategy 5
53.13	27356	Strategy 7

Table 6.9. SPK problem.

SPK1, 200 × 200		
Run time	No. of iterations	Strategy
0.34	1558	Dantzig (P)
0.27	516	Strategy 1
1.30	560	Strategy 2
2.12	610	Strategy 5
0.42	1743	Strategy 7

of execution time. Strategy 7 is the worst because it is based on the random selection of the pivot columns.

Computational result for SPK problems. As can be seen in Tables 6.9, 6.10, 6.11, 6.12 Strategy 1 is better than all other strategies including Dantzig rule in the sense of iteration number and execution times.

Table 6.10. SPK problem.

SPK2, 250×300		
Run time	No. of iterations	Strategy
1.33	2374	Dantzig (P)
0.54	872	Strategy 1
2.51	911	Strategy 2
3.23	873	Strategy 5
1.58	2691	Strategy 7

Table 6.11. SPK problem.

SPK3, 350×500		
Run time	No. of iterations	Strategy
9.33	5077	Dantzig (P)
5.2	2808	Strategy 1
11.4	2914	Strategy 2
12.43	2863	Strategy 5
10.29	5492	Strategy 7

Table 6.12. SPK problem.

SPK4, 500×700		
Run time	No. of iterations	Strategy
47.11	21349	Dantzig (P)
21.34	5013	Strategy 1
44.28	6143	Strategy 2
50.35	5493	Strategy 5
52.22	23982	Strategy 7

7. Comparison with a variant of steepest edge

In this section we review the primal steepest-edge algorithm proposed in [4] by Goldfarb and Reid for solving the standard form linear programming problem:

$$\begin{aligned} &\text{minimize } c^T x, \\ &\text{subject to } Ax = b, \\ &x \geq 0, \end{aligned} \tag{$*$}$$

where A is an $m \times n$ matrix of rank m and $m < n$.

Consider a single step of the simplex method applied to $(*)$ and let B and N denote the submatrices of A corresponding to basic and nonbasic columns, respectively. To simplify

Table 7.1

SCP	Dantzig	St1	St2	St3	St4	St5	St6	St7	Ried
30×30	97	46	60	51	49	50	50	91	84
200×200	4601	1512	2911	394	259	611	324	3491	779
250×300	4102	1881	2974	566	519	413	521	7196	783
350×500	15914	4721	7119	814	769	601	764	15113	1162
500×700	21409	6013	11146	1410	1416	1201	1422	24917	1587

Table 7.2

SPP	Dantzig	St1	St2	St3	St4	St5	St6	St7	Ried
30×30	32	27	26	57	49	23	66	40	30
200×200	792	371	311	—	—	357	—	894	686
250×300	1712	637	555	—	—	1150	—	1816	1612
350×500	4187	1605	1594	—	—	2011	—	5041	7870
500×700	24971	7983	7251	—	—	11298	—	28432	10327

our exposition, we will henceforth assume that the first m columns of A and components of x are basic at the start of a step. This step is of the form

$$\overline{x} = x + \theta\eta_q, \quad \text{where } x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}. \tag{7.1}$$

The basic feasible solution at the start of the step, η_q, is one of the set of edge directions

$$\eta_j = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix} e_{j-m}, \quad j = m+1,\dots,n. \tag{7.2}$$

Emanating from the vertex x, and θ is the length of the step, e_i denotes the ith column of the identity matrix I. The edge direction η_q must be "downhill," that is, η_q must make an obtuse angle with the gradient c of objective function, or equivalently, the reduced cost $\overline{c}_q = c^T\eta_q$ must be negative. In the steepest-edge simplex algorithm, the edge η_q is chosen, such that

$$\frac{c^T\eta_q}{||\eta_q||} = \min_{j>m}\left\{\frac{c^T\eta_j}{||\eta_j||}\right\}. \tag{7.3}$$

Tables 7.1, 7.2, and 7.3 present a comparison of our strategies with the steepest edge of Goldfarb and Reid.

As previously mentioned Strategies 3 and 4 are not strong enough to be included in the above-mentioned tables. It can be concluded that the strategies mentioned in the paper are superior to this variant of the steepest edge.

Table 7.3

SPK	Dantzig	St1	St2	St3	St4	St5	St6	St7	Ried
30×30	21	19	26	77	60	25	74	28	28
200×200	1589	874	1167	—	—	703	—	1873	735
250×300	2374	827	911	—	—	873	—	2691	1074
350×500	5077	2808	2914	—	—	2863	—	5492	3548
500×700	21349	5013	6143	—	—	5493	—	23982	8119

8. Conclusion

A number of strategies were proposed for the pivot selection in the LP relaxation of the set problems. It is demonstrated that considering problem specific knowledge in pivoting in the LP relaxation of the set problems can enhance pivot selection in the simplex method for each instance of the set problems. one of the strategies works better than the others. Researchers in the future may come up with new strategies for particular instances of LP problems which considerably enhance pivot selection in the simplex method.

References

[1] F. Djannaty, *Network based heuristics for the set covering problem*, Ph.D. thesis, Department of Mathematics and Statistic, Brunel University, Brunel, 1997.

[2] ______, *Some useful cost allocation strategies for the shortest route relaxation of the set covering problem*, Electronic Journal of SADIO **22** (2002), no. 5, 12–23.

[3] J. J. Forrest and D. Goldfarb, *Steepest-edge simplex algorithms for linear programming*, Mathematical Programming **57** (1992), no. 3, 341–374.

[4] D. Goldfarb and J. K. Reid, *A practicable steepest-edge simplex algorithm*, Mathematical Programming **12** (1977), no. 3, 361–371.

[5] P. M. J. Harris, *Pivot selection methods of the Devex Lp code*, Computational Practice in Mathematical Programming, Mathematical Programming Studies, vol. 4, North-Holland, Amsterdam, 1975, pp. 30–57.

[6] B. L. Kaluzny, *Finite Pivot algorithms and feasibility*, Msc thesis, School of Computer Science , McGill University, Quebec, 2001.

[7] H. W. Kuhn and R. E. Quandt, *On upper bound for the number of iterations in solving linear Programs*, Operations Research **12** (1964), no. 1, 161–165.

[8] B. Rostamy, *Pivot selection strategies in simplex method*, Msc thesis, University of Kurdistan, Kurdistan, 2005.

F. Djannaty: Department of Mathematics, Faculty of Science, Kurdistan University,
P.O. Box 416, Sanandaj 66177, Iran
E-mail address: fdjanaty@uok.ac.ir

B. Rostamy: Department of Mathematics, Faculty of Science, Kurdistan University,
P.O. Box 416, Sanandaj 66177, Iran
E-mail address: brorzou_rostami@yahoo.com