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Research Article

On Intuitionistic Fuzzy Context-Free Languages

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Taking intuitionistic fuzzy sets as the structures of truth values, we propose the notions of intuitionistic fuzzy context-free grammars (IFCFGs, for short) and pushdown automata with final states (IFPDAs). Then we investigate algebraic characterization of intuitionistic fuzzy recognizable languages including decomposition form and representation theorem. By introducing the generalized subset construction method, we show that IFPDAs are equivalent to their simple form, called intuitionistic fuzzy simple pushdown automata (IF-SPDAs), and then prove that intuitionistic fuzzy recognizable step functions are the same as those accepted by IFPDAs. It follows that intuitionistic fuzzy pushdown automata with empty stack and IFPDAs are equivalent by classical automata theory. Additionally, we introduce the concepts of Chomsky normal form grammar (IFCNF) and Greibach normal form grammar (IFGNF) based on intuitionistic fuzzy sets. The results of our study indicate that intuitionistic fuzzy context-free languages generated by IFCFGs are equivalent to those generated by IFGNFs and IFCNFs, respectively, and they are also equivalent to intuitionistic fuzzy recognizable step functions. Then some operations on the family of intuitionistic fuzzy context-free languages are discussed. Finally, pumping lemma for intuitionistic fuzzy context-free languages is investigated.

1. Introduction

Intuitionistic fuzzy set (IFS) introduced by Atanassov [1–3], which emerges from the simultaneous consideration of the degrees of membership and nonmembership with a degree of hesitancy, has been found to be highly useful in dealing with problems with vagueness and uncertainty. The notion of vague set, proposed by Gau and Buehrer [4], is another generalization of fuzzy sets. However, Burillo and Bustince [5] showed that it is an equivalence of the IFS and studied intuitionistic fuzzy relations. Recently, IFS theory has supported a wealth of important applications in many fields such as fuzzy multiple attribute decision making, fuzzy pattern recognition, medical diagnosis, fuzzy control, and fuzzy optimization [6–10].

In classical theoretical computer science, it is well known that formal languages are very useful in the description of natural languages and programming languages. But they are not powerful in the processing of human languages. For this, Lee and Zadeh [11] introduced the notion of fuzzy languages and gave some characterizations, where fuzzy languages took values in the unit interval $[0, 1]$. Malik and Mordeson [12–14] studied algebraic properties of fuzzy languages. They stated that fuzzy regular languages can be characterized by fuzzy finite automata, fuzzy regular expressions, and fuzzy regular grammars. Meanwhile, as one of the generators of fuzzy languages, fuzzy automata have been used to solve meaningful issues such as the model of computing with words [15], clinical monitoring [16], neural networks [17], and pattern recognition [18]. Also, fuzzy grammars, automata, and languages tend to the improvement of lexical analysis and simulating fuzzy discrete event dynamical systems and hybrid systems [14, 19].

As is well known, quantum logic was proved by Birkhoff and Von Neumann as a logic of quantum mechanics and is currently understood as a logic with truth values taken from an orthomodular lattice. To study quantum computation, Ying [20, 21] first proposed automata theory based on quantum logic where quantum automata are defined to be orthomodular lattice-valued generalization of classical automata. More systematic exposition of this theory appeared in [22, 23]. Moore and Crutchfield [24] defined quantum version of pushdown automata and regular and context-free grammars. He showed that the corresponding languages generated by quantum grammars and recognized by quantum automata have satisfactory properties in analogy to their classical counterparts. A basic framework of grammar theory on quantum logic was established by Cheng and Wang [25]. They proved that the set of lattice-valued quantum regular languages generated by lattice-valued quantum regular grammars coincides with that of lattice-valued quantum languages recognized by lattice-valued quantum automata. Then some algebraic properties of automata based on quantum logic were discussed by Qiu [26, 27]. To enhance the processing ability of fuzzy automata, the membership grades were extended to many general algebraic structures. For example, by combining the ideas in [20–23] and the idea in Ying's another work on topology based on residuated lattice-valued logic [28], Qiu has primarily established automata theory based on complete residuated lattice-valued logic [29–31]. And Li and Pedrycz [32] studied automata theory with membership values in lattice-ordered monoids. They showed that lattice-valued finite automata have more power to recognize fuzzy languages than that of classical fuzzy finite automata. Recently, Li [33] studied automata theory with membership values in lattices, introduced the technique of extended subset construction to prove the equivalence between lattice-valued finite automata and lattice-valued deterministic finite automata, and then presented a minimization algorithm of lattice-valued deterministic finite automata. On the basis of breadth-first and depth-first ways, Jin and Li [34] established a fundamental framework of fuzzy grammars based on lattices, which provided a necessary tool for the analysis of fuzzy automata.

Fuzzy context-free languages, more powerful than fuzzy regular languages, have also been studied and can be characterized by fuzzy pushdown automata with two distinct ways and fuzzy context-free grammars, respectively [14, 35]. As a continuation of the work in [29–31], a fundamental framework of fuzzy pushdown automata theory based on complete residuated lattice-valued logic has been established in recent years by Xing et al. [36], and the work generalizes the previous fuzzy automata theory systematically studied by Mordeson and Malik to some extent. The pumping lemma for fuzzy context-free grammar theory in this setting was also investigated by Xing and Qiu [37].

Using the notions of IFSs and fuzzy finite automata, Jun [38, 39] presented the concept of intuitionistic fuzzy finite state machines as a generalization of fuzzy finite state machines, and Zhang and Li [40] discussed intuitionistic fuzzy recognizers, intuitionistic fuzzy finite automata, and intuitionistic fuzzy language. They showed that the languages recognized by intuitionistic fuzzy recognizers are regular, and the intuitionistic fuzzy languages recognized by the intuitionistic fuzzy finite automata and the intuitionistic fuzzy languages recognized by deterministic intuitionistic fuzzy finite automata are equivalent. Recently Chen et al. [41] utilized the intuitionistic fuzzy automata to deal with consumers' advertising involvement when considering the expression of an IFS characterized by a pair of membership degree and nonmembership degree is similar to human thinking logic with pros and cons. Due to pushdown automata being another kind of important computational models [15] and also motivated by the importance of grammars, languages and models theory [14], it stands to reason that we ought consider the notions of intuitionistic fuzzy pushdown automata, intuitionistic fuzzy context-free grammars, and fuzzy context-free languages because our discussion in this paper will provide a fundamental framework for studying intuitionistic fuzzy set theory on fuzzy pushdown automata and generators as well. How to characterize intuitionistic fuzzy context-free languages and its pumping lemma in this setting becomes open problems; however, there is no research on the algebraic characterization of intuitionistic fuzzy context-free languages. We will try to solve the problems in this paper. Additionally, some examples are given to illustrate the significance of the results. In particular, Example 35 presented in this paper will show that intuitionistic fuzzy pushdown automata have more power than fuzzy pushdown automata when comparing two distinct strings although the degrees of membership of these strings recognized by the underlying fuzzy pushdown automata are equal. Investigating intuitionistic fuzzy context-free languages will reduce the gap between the precision of formal languages and the imprecision of human languages.

The remaining parts of the paper are arranged as follows. Section 2 describes some basic concepts of IFSs. Section 3 gives the definitions of intuitionistic fuzzy pushdown automata with two distinct ways and their languages. It is investigated that, for any intuitionistic fuzzy pushdown automaton with final states (IFPDA, for short), there is a cover, which consists of a collection of classical pushdown automata, equivalent to the IFPDA. By introducing intuitionistic fuzzy recognizable step functions, it is shown that intuitionistic fuzzy pushdown automata with final states and empty stack are intuitionistic fuzzy recognizable step functions, respectively, and conversely any intuitionistic fuzzy recognizable step function can be recognized by an intuitionistic fuzzy pushdown automaton with final states or empty stack. It follows that intuitionistic fuzzy pushdown automata with final states and empty stack are equivalent. Section 4 studies intuitionistic fuzzy context-free grammars (IFCFGs) as a type of generator of intuitionistic fuzzy context-free languages (IFCFLs). The notions of intuitionistic fuzzy Chomsky normal form (IFCNF) and Greibach normal form (IFGNF) are proposed. The results of our study indicate that IFCFLs generated by IFCFGs are equivalent to those generated by IFGNFs and IFCNFs, respectively, and they are also equivalent to intuitionistic fuzzy recognizable step functions. The algebraic properties of IFCFLs are also discussed. Section 5 establishes pumping lemma for IFCFLs. Some examples are then given to illustrate the application of pumping lemma and the significance of IFCFLs. Finally, conclusions and directions for future work are presented in Section 6.

2. Basic Concepts

Definition 1 (see [40]). An intuitionistic fuzzy set A in a nonempty set X is an object having the form:

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}, \quad (1)$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (i.e., $\mu_A(x)$) and the degree of nonmembership ($\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and the two quantities satisfy the following inequalities:

$$0 \le \mu_A(x) + \nu_A(x) \le 1, \quad \forall x \in X. \quad (2)$$

For the sake of simplicity, we use the notation $A = (\mu_A, \nu_A)$ instead of $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$. An intuitionistic fuzzy set will be abbreviated as an IFS.

Let $\{A_i \mid i \in I\}$ be a family of IFSs in X. Then the infimum and supremum operations of IFSs are defined as follows:

$$\begin{aligned} \bigcap_{i\in I} A_i &= \left\{\left(x, \bigwedge_{i\in I}\mu_{A_i}(x), \bigvee_{i\in I}\nu_{A_i}(x)\right) \mid x \in X\right\}, \\ \bigcup_{i\in I} A_i &= \left\{\left(x, \bigvee_{i\in I}\mu_{A_i}(x), \bigwedge_{i\in I}\nu_{A_i}(x)\right) \mid x \in X\right\}, \end{aligned} \quad (3)$$

where $\bigvee$ and $\bigwedge$ denote supremum and infimum of real numbers in $[0,1]$, respectively.

For two IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$, we say $A = B$ if $\mu_A = \mu_B$ and $\nu_A = \nu_B$. In addition, if the IFS $A = (\mu_A, \nu_A)$ in X satisfies the condition that, for any $x \in X$, $\mu_A(x) + \nu_A(x) = 1$, then A reduces to a fuzzy set in X. The difference between intuitionistic fuzzy sets and fuzzy sets is whether the sum of the degrees of membership and nonmembership of an element to a set equals one.

An IFR in $X \times Y$ is an intuitionistic fuzzy subset of $X \times Y$; that is, it is an expression E given by

$$E = \{((x, y), \mu_E(x, y), \nu_E(x, y)) \mid x \in X, y \in Y\}, \quad (4)$$

where the mappings $\mu_E : X \times Y \to [0,1]$ and $\nu_E : X \times Y \to [0,1]$ satisfy

$$0 \le \mu_E(x, y) + \nu_E(x, y) \le 1, \quad \forall (x, y) \in X \times Y. \quad (5)$$

An IFBR over X is an IFS of $X \times X$. Let $P = (\mu_P, \nu_P)$ and $E = (\mu_E, \nu_E)$ be IFRs in $X \times Y$ and $Y \times Z$, respectively. Define the composition of IFRs, $P \circ E = (\mu_{P\circ E}, \nu_{P\circ E})$ in $X \times Z$, by

$$\begin{aligned} \mu_{P\circ E}(x, z) &= \bigvee_{y\in Y}(\mu_P(x, y) \wedge \mu_E(y, z)), \\ \nu_{P\circ E}(x, z) &= \bigwedge_{y\in Y}(\nu_P(x, y) \vee \nu_E(y, z)), \end{aligned} \quad (6)$$

for all $(x, z) \in X \times Z$.

Furthermore, if R is an IFBR over X, then its reflexive and transitive closure is $R^* = \bigcup_{n=0}^{\infty} R^n$, where $R^{n+1} = R^n \circ R$, $n \ge 0$, and $R^0 = id = (\mu_{id}, \nu_{id})$, that is,

$$\begin{aligned} \mu_{id}(u, v) &= \begin{cases} 1, & \text{if } u = v \\ 0, & \text{if } u \ne v, \end{cases} \\ \nu_{id}(u, v) &= \begin{cases} 0, & \text{if } u = v \\ 1, & \text{if } u \ne v, \end{cases} \end{aligned} \quad (7)$$

for all $(u, v) \in X \times X$.

Definition 2. Let $A = (\mu_A, \nu_A)$ be an IFS in X. Then the image set of A, denoted as $\mathrm{Im}(A)$, is given as

$$\mathrm{Im}(A) = \mathrm{Im}(\mu_A) \cup \mathrm{Im}(\nu_A), \quad (8)$$

where $\mathrm{Im}(\mu_A) = \{\mu_A(x) \mid x \in X\}$ and $\mathrm{Im}(\nu_A) = \{\nu_A(x) \mid x \in X\}$.

For any $\lambda, \theta \in [0,1]$, $\lambda + \theta \le 1$, the (λ, θ)-cut set of A is defined as

$$A_{(\lambda,\theta)} = \{x \in X \mid \mu_A(x) \ge \lambda, \nu_A(x) \le \theta\}. \quad (9)$$

And the support set of A, denoted as $\mathrm{supp}(A)$, is defined by

$$\mathrm{supp}(A) = \{x \in X \mid \mu_A(x) > 0, \nu_A(x) < 1\}. \quad (10)$$

If $\mathrm{supp}(A)$ is finite, then A is called a finite IFS in X.

3. Intuitionistic Fuzzy Pushdown Automata

It is well known that any language accepted by a pushdown automaton with final states can be accepted by a certain pushdown automaton with empty stack, and vice versa. As a natural generalization of pushdown automata, we give the notions of intuitionistic fuzzy pushdown automata with final states and empty stack, respectively, and then do research in the algebraic characterization of their intuitionistic fuzzy recognizable languages including decomposition form and representation theorem. Note that Σ^* is the free monoid generated from the set Σ with the operator of concatenation, where the empty string ε is identified with the identity of Σ. And the length of the string $\omega \in \Sigma^*$ is denoted by $|\omega|$. $N_k = \{1, \ldots, k\}$.

Definition 3. An intuitionistic fuzzy pushdown automaton with final states (IFPDA, for short) is a seven tuple $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$, where

(i) Q is a finite nonempty set of states;

(ii) Σ is a finite nonempty set of input symbols;

(iii) Γ is a finite nonempty set of stack symbols;

(iv) $\delta = (\mu_\delta, \nu_\delta)$ is a finite IFS in $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$;

(v) $Z_0 \in \Gamma$ is called the start stack symbol;

(vi) $I = (\mu_I, \nu_I)$ and $F = (\mu_F, \nu_F)$ are intuitionistic fuzzy subsets in Q, which are called the intuitionistic fuzzy subsets of initial and final states, respectively.

Definition 4. An intuitionistic fuzzy pushdown automaton with empty stack (IFPDA$^{\emptyset}$, for short) is a seven tuple $\mathcal{N} = (Q, \Sigma, \Gamma, \delta, I, Z_0, \emptyset)$, where $Q, \Sigma, \Gamma, \delta, I$ and Z_0 are the same as those in IFPDA $\mathcal{M}$, and $\emptyset$ represents an empty set.

Definition 5. Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$ be an IFPDA. Define an IFBR on $Q \times \Sigma^* \times \Gamma^*$, $\vdash_{\mathcal{M}} = (\mu_{\vdash_{\mathcal{M}}}, \nu_{\vdash_{\mathcal{M}}})$ in the form of

$$\mu_{\vdash_{\mathcal{M}}}((q,\omega,\beta),(p,u,\alpha)) = \begin{cases} \mu_\delta(q,\varepsilon,\text{head}(\beta),\ p,\alpha\setminus\text{tail}(\beta)), & \text{if } u=\omega,\ \text{tail}(\beta)\le\alpha \\ \mu_\delta(q,\text{head}(\omega),\text{head}(\beta),\ p,\alpha\setminus\text{tail}(\beta)), & \text{if } u=\text{tail}(\omega),\ \text{tail}(\beta)\le\alpha \\ 0, & \text{otherwise,} \end{cases}$$
$$\nu_{\vdash_{\mathcal{M}}}((q,\omega,\beta),(p,u,\alpha)) = \begin{cases} \nu_\delta(q,\varepsilon,\text{head}(\beta),p,\alpha\setminus\text{tail}(\beta)), & \text{if } u=\omega,\ \text{tail}(\beta)\le\alpha \\ \nu_\delta(q,\text{head}(\omega),\text{head}(\beta),p,\alpha\setminus\text{tail}(\beta)), & \text{if } u=\text{tail}(\omega),\ \text{tail}(\beta)\le\alpha \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

for any $(q,\omega,\beta),(p,u,\alpha) \in Q \times \Sigma^* \times \Gamma^*$. Here, for any non-empty string $u = x_1 \cdots x_n$, $n \ge 1$, $\text{head}(u) = x_1$, $\text{tail}(u) = x_2 \cdots x_n$, and $\text{tail}(u) \le u$. $\vdash^*_{\mathcal{M}}$ is the reflexive and transitive closure of $\vdash_{\mathcal{M}}$.

If no confusion, we denote $\vdash$ and $\vdash^*$ instead of $\vdash_{\mathcal{M}}$ and $\vdash^*_{\mathcal{M}}$, respectively.

Definition 6. Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$ be an IFPDA. Then we call $\mathcal{L}(\mathcal{M})$ an intuitionistic fuzzy language accepted by $\mathcal{M}$ with final states, where $\mathcal{L}(\mathcal{M}) = (\mu_{\mathcal{L}(\mathcal{M})}, \nu_{\mathcal{L}(\mathcal{M})})$, $\mu_{\mathcal{L}(\mathcal{M})}$, and $\nu_{\mathcal{L}(\mathcal{M})}$ are functions from Σ^* to the unit interval $[0,1]$, and

$\mu_{\mathcal{L}(\mathcal{M})}(\omega) = \bigvee\{\mu_I(q_0) \wedge \mu_{\vdash^*_{\mathcal{M}}}((q_0,\omega,z_0),(p,\varepsilon,r)) \wedge \mu_F(p) \mid q_0, p \in Q, r \in \Gamma^*\}$,

$\nu_{\mathcal{L}(\mathcal{M})}(\omega) = \bigwedge\{\nu_I(q_0) \vee \nu_{\vdash^*_{\mathcal{M}}}((q_0,\omega,z_0),(p,\varepsilon,r)) \vee \nu_F(p) \mid q_0, p \in Q, r \in \Gamma^*\}$

for any $\omega \in \Sigma^*$.

Definition 7. Let $\mathcal{N} = (Q, \Sigma, \Gamma, \delta, I, Z_0, \emptyset)$ be an IFPDA$^{\emptyset}$. Then we call $\mathcal{L}(\mathcal{N})$ an intuitionistic fuzzy language accepted by $\mathcal{N}$ with empty stack, where $\mathcal{L}(\mathcal{N}) = (\mu_{\mathcal{L}(\mathcal{N})}, \nu_{\mathcal{L}(\mathcal{N})})$, $\mu_{\mathcal{L}(\mathcal{N})}$ and $\nu_{\mathcal{L}(\mathcal{N})}$ are functions from Σ^* to the unit interval $[0,1]$, and

$\mu_{\mathcal{L}(\mathcal{N})}(\omega) = \bigvee\{\mu_I(q_0) \wedge \mu_{\vdash^*_{\mathcal{N}}}((q_0,\omega,z_0),(p,\varepsilon,\varepsilon)) \mid q_0, p \in Q\}$,

$\nu_{\mathcal{L}(\mathcal{N})}(\omega) = \bigwedge\{\nu_I(q_0) \vee \upsilon_{\vdash^*_{\mathcal{N}}}((q_0,\omega,z_0),(p,\varepsilon,\varepsilon)) \mid q_0, p \in Q\}$

for any $\omega \in \Sigma^*$.

Lemma 8 (see [33]). *Let l be a lattice and X a finite subset of l. Then the $\wedge$-semilattice of l generated by X, written as $X_\wedge$, is finite, and the $\vee$-semilattice of l generated by X, denoted as $X_\vee$, is also finite, where $X_\wedge = \{x_1 \wedge \cdots \wedge x_k : k \ge 1, x_1, \ldots, x_k \in X\} \cup \{1\}$, and $X_\vee = \{x_1 \vee \cdots \vee x_k : k \ge 1, x_1, \ldots, x_k \in X\} \cup \{0\}$.*

Proposition 9. *If f can be accepted by some IFPDA $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$, then f is an IFS in Σ^*, and the image set of f is finite.*

Proof. We have the following.

Claim 1 ($f = (\mu_f, \nu_f)$ is an IFS in Σ^*).

It suffices to show that $0 \le \mu_f(\omega) + \nu_f(\omega) \le 1$, for any $\omega = u_1 \cdots u_n$, $u_i \in \Sigma \cup \{\varepsilon\}$, $i = 1, \ldots, n$.

Clearly, $\mu_f(\omega) = \bigvee\{\mu_I(q_0) \wedge \mu_{\vdash^*_{\mathcal{M}}}((q_0,\omega,z_0),(p,\varepsilon,r)) \wedge \mu_F(p) \mid q_0, p \in Q, r \in \Gamma^*\} = \bigvee\{\mu_I(q_0) \wedge \mu_\vdash((q_0,\omega,z_0),(q_1,u_2\cdots u_n,z_1r_1)) \wedge \mu_\vdash((q_1,u_2\cdots u_n,z_1r_1),(q_2,u_3\cdots u_n,z_2r_2)) \wedge \cdots \wedge \mu_\vdash((q_{n-1},u_n,z_{n-1}r_{n-1}),(q_n,\varepsilon,r_n)) \wedge \mu_F(q_n) \mid (q_0,q_1,\ldots,q_n) \in Q^{n+1}, z_1,\ldots,z_{n-1} \in \Gamma, r_1,\ldots,r_n \in \Gamma^*\}$, and $\nu_f(\omega) = \bigwedge\{\nu_I(q_0) \vee \nu_{\vdash^*_{\mathcal{M}}}((q_0,\omega,z_0),(p,\varepsilon,r)) \vee \nu_F(p) \mid q_0, p \in Q, r \in \Gamma^*\} = \bigwedge\{\nu_I(q_0) \vee \nu_\vdash((q_0,\omega,z_0),(q_1,u_2\cdots u_n,z_1r_1)) \vee \nu_\vdash((q_1,u_2\cdots u_n,z_1r_1),(q_2,u_3\cdots u_n,z_2r_2)) \vee \cdots \vee \nu_\vdash((q_{n-1},u_n,z_{n-1}r_{n-1}),(q_n,\varepsilon,r_n)) \vee \nu_F(q_n) \mid (q_0,q_1,\ldots,q_n) \in Q^{n+1}, z_1,\ldots,z_{n-1} \in \Gamma, r_1,\ldots,r_n \in \Gamma^*\}$.

On the one hand, $0 \le \mu_f(\omega) + \nu_f(\omega)$; on the other hand, there exists a sequence $(q_0,q_1,\ldots,q_n) \in Q^{n+1}$, $z_1,\ldots,z_{n-1} \in \Gamma$, $r_1,\ldots,r_n \in \Gamma^*$ such that $\mu_f(\omega) = \mu_I(q_0) \wedge \mu_\vdash((q_0,\omega,z_0),(q_1,u_2\cdots u_n,z_1r_1)) \wedge \mu_\vdash((q_1,u_2\cdots u_n,z_1r_1),(q_2,u_3\cdots u_n,z_2r_2)) \wedge \cdots \wedge \mu_\vdash((q_{n-1},u_n,z_{n-1}r_{n-1}),(q_n,\varepsilon,r_n)) \wedge \mu_F(q_n)$. Hence $\nu_f(\omega) \le \nu_I(q_0) \vee \nu_\vdash((q_0,\omega,z_0),(q_1,u_2\cdots u_n,z_1r_1)) \vee \nu_\vdash((q_1,u_2\cdots u_n,z_1r_1),(q_2,u_3\cdots u_n,z_2r_2)) \vee \cdots \vee \nu_\vdash((q_{n-1},u_n,z_{n-1}r_{n-1}),(q_n,\varepsilon,r_n)) \vee \nu_F(q_n)$.

Therefore, $\mu_f(\omega) + \nu_f(\omega) \le (\mu_I(q_0) + \nu_I(q_0)) \vee (\mu_\vdash((q_0,\omega,z_0),(q_1,u_2\cdots u_n,z_1r_1)) + \nu_\vdash((q_0,\omega,z_0),(q_1,u_2\cdots u_n,z_1r_1))) \vee (\mu_\vdash((q_1,u_2\cdots u_n,z_1r_1),(q_2,u_3\cdots u_n,z_2r_2)) + \nu_\vdash((q_1,u_2\cdots u_n,z_1r_1),(q_2,u_3\cdots u_n,z_2r_2))) \vee \cdots \vee (\mu_\vdash((q_{n-1},u_n,z_{n-1}r_{n-1}),(q_n,\varepsilon,r_n)) + \nu_\vdash((q_{n-1},u_n,z_{n-1}r_{n-1}),(q_n,\varepsilon,r_n))) \vee (\mu_F(q_n) + \nu_F(q_n)) \le 1 \vee 1 \vee \cdots \vee 1 = 1$.

Claim 2 ($\text{Im}(f)$ is finite).

In fact, let $X = \text{Im}(\mu_I) \cup \text{Im}(\mu_\delta) \cup \text{Im}(\mu_F)$ and $Y = \text{Im}(\nu_I) \cup \text{Im}(\nu_\delta) \cup \text{Im}(\nu_F)$. Then $X_\wedge = \{x_1 \wedge \cdots \wedge x_k \mid k \ge 1, x_1, \ldots, x_k \in X\} \cup \{1\}$ and $X_\vee = \{x_1 \vee \cdots \vee x_k \mid k \ge 1, x_1, \ldots, x_k \in X\} \cup \{0\}$ are finite sets by Lemma 8. Since $\delta = (\mu_\delta, \nu_\delta)$ is a finite IFS, for any $\omega = u_1 \cdots u_n$, $u_i \in \Sigma \cup \{\varepsilon\}$, $i = 1, \ldots, n$, there exists a natural number $d \in N$ such that $\mu_f(\omega) = \bigvee\{\mu_I(q_0) \wedge \mu_\vdash((q_0,\omega,z_0),(q_1,u_2\cdots u_n,z_1r_1)) \wedge \mu_\vdash((q_1,u_2\cdots u_n,z_1r_1),(q_2,u_3\cdots u_n,z_2r_2)) \wedge \cdots \wedge \mu_\vdash((q_{n-1},u_n,z_{n-1}r_{n-1}),(q_n,\varepsilon,r_n)) \wedge \mu_F(q_n) \mid (q_0,q_1,\ldots,q_n) \in Q^{n+1}, z_1,\ldots,z_{n-1} \in \Gamma, r_1,\ldots,r_n \in \Gamma^*\} = (a_{10} \wedge \cdots \wedge a_{1,n-1} \wedge a_{1n}) \vee \cdots \vee (a_{d0} \wedge \cdots \wedge a_{d,n-1} \wedge a_{dn}))$, where $a_{ij} \in X$, $i = 1, \ldots, d$; $j = 0, \ldots, n$. By Lemma 8, $(X_\wedge)_\vee$ is also finite. Since $\mu_f(\omega) \in (X_\wedge)_\vee$ for any $\omega \in \Sigma^*$, $\text{Im}(\mu_f) \subseteq (X_\wedge)_\vee$. Hence $\text{Im}(\mu_f)$ is a finite subset of $[0,1]$.

Similarly, it follows that $\text{Im}(\nu_f)$ is also a finite subset of $[0,1]$.

Therefore, $\text{Im}(f) = \text{Im}(\mu_f) \cup \text{Im}(\nu_f)$ is finite.

If f can be accepted by some IFPDA$^{\emptyset}$ $\mathcal{N} = (Q, \Sigma, \Gamma, \delta, I, Z_0, \emptyset)$, then, by Definition 7, for any $\omega = u_1 \cdots u_n$, $u_i \in \Sigma \cup \{\varepsilon\}$, $i = 1, \ldots, n$, we have $\mu_f(\omega) = \bigvee\{\mu_I(q_0) \wedge$

$\mu_{\vdash^*_{\mathcal{N}}}((q_0, \omega, z_0), (p, \varepsilon, \varepsilon)) \mid q_0, p \in Q\} = \bigvee\{\mu_I(q_0) \wedge \mu_{\vdash}((q_0, \omega, z_0), (q_1, u_2 \cdots u_n, z_1 r_1)) \wedge \mu_{\vdash}((q_1, u_2 \cdots u_n, z_1 r_1), (q_2, u_3 \cdots u_n, z_2 r_2)) \wedge \cdots \wedge \mu_{\vdash}((q_{n-1}, u_n, z_{n-1} r_{n-1}), (q_n, \varepsilon, \varepsilon)) \mid (q_0, q_1, \ldots, q_n) \in Q^{n+1}, z_1, \ldots, z_{n-1} \in \Gamma, r_1, \ldots, r_{n-1} \in \Gamma^*\}$, and $\nu_f(\omega) = \bigwedge \{\nu_I(q_0) \vee \nu_{\vdash^*_{\mathcal{N}}}((q_0, \omega, z_0), (p, \varepsilon, \varepsilon)) \mid q_0, p \in Q\} = \bigwedge \{\nu_I(q_0) \vee \nu_{\vdash}((q_0, \omega, z_0), (q_1, u_2 \cdots u_n, z_1 r_1)) \vee \nu_{\vdash}((q_1, u_2 \cdots u_n, z_1 r_1), (q_2, u_3 \cdots u_n, z_2 r_2)) \vee \cdots \vee \nu_{\vdash}((q_{n-1}, u_n, z_{n-1} r_{n-1}), (q_n, \varepsilon, \varepsilon)) \mid (q_0, q_1, \ldots, q_n) \in Q^{n+1}, z_1, \ldots, z_{n-1} \in \Gamma, r_1, \ldots, r_{n-1} \in \Gamma^*\}$.

In a similar manner, it is concluded that the following must be true. □

Proposition 10. *If f can be accepted by some IFPDA$^{\emptyset}$ $\mathcal{N} = (Q, \Sigma, \Gamma, \delta, I, Z_0, \emptyset)$, then f is an IFS in Σ^*, and the image set of f is finite.*

Specially, the IFPDA $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$ will be abbreviated as $\mathcal{M}' = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, whenever $\text{Im}(I) \subseteq \{0, 1\}$ and $\text{supp}(I) = \{q_0\}$. Moreover, if $\text{Im}(I) \cup \text{Im}(F) \cup \text{Im}(\delta) \subseteq \{0, 1\}$ and $\text{supp}(I)$ has only one element, then the IFPDA is a classical PDA.

For two IFPDAs $\mathcal{M}_1$ and $\mathcal{M}_2$, we say that they are equivalent if they accept the same intuitionistic fuzzy language.

Proposition 11. *Let A be an IFS in a nonempty set Σ^*. Then the following statements are equivalent:*

(i) *A can be accepted by an IFPDA $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$;*

(ii) *A can be accepted by a certain IFPDA $\mathcal{M}' = (Q', \Sigma, \Gamma', \delta', q_0, X_0, F')$, where $q_0 \in Q'$.*

Proof. (i) implies (ii). Construct an IFPDA $\mathcal{M}' = (Q', \Sigma, \Gamma', \delta', I', X_0, F')$ as follows: $Q' = Q \cup \{q_0\}$, $\Gamma' = \Gamma \cup \{X_0\}$, where $q_0 \notin Q$, $X_0 \notin \Gamma$. Define an IFS I' in Q' by

$$\mu_{I'}(q) = \begin{cases} 1, & \text{if } q = q_0 \\ 0, & \text{if } q \neq q_0, \end{cases} \qquad \nu_{I'}(q) = \begin{cases} 0, & \text{if } q = q_0 \\ 1, & \text{if } q \neq q_0. \end{cases} \tag{12}$$

Define an IFS F' in Q' by

$$\mu_{F'}(q) = \begin{cases} 0, & \text{if } q = q_0 \\ \mu_F(q), & \text{if } q \neq q_0, \end{cases} \qquad \nu_{F'}(q) = \begin{cases} 1, & \text{if } q = q_0 \\ \nu_F(q), & \text{if } q \neq q_0. \end{cases} \tag{13}$$

Define an IFS δ' in $Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma' \times Q' \times \Gamma'^*$ by mappings $\mu_{\delta'}, \nu_{\delta'} : Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma' \times Q' \times \Gamma'^* \to [0, 1]$, $\mu_{\delta'}(q_0, \varepsilon, X_0, p, Z_0) = \mu_I(p)$, $\nu_{\delta'}(q_0, \varepsilon, X_0, p, Z_0) = \nu_I(p)$, $\mu_{\delta'}(q, \tau, z, p, \gamma) = \mu_\delta(q, \tau, z, p, \gamma)$, $\nu_{\delta'}(q, \tau, z, p, \gamma) = \nu_\delta(q, \tau, z, p, \gamma)$, where $q, p \in Q$, $\tau \in \Sigma \cup \{\varepsilon\}$, $z \in \Gamma$, $\gamma \in \Gamma^*$; otherwise, $\mu_{\delta'}(q_0, \tau, z, p, \gamma) = 0$ and $\nu_{\delta'}(q_0, \tau, z, p, \gamma) = 1$.

Then for any $\omega = u_1 \cdots u_n \in \Sigma^*$, $u_i \in \Sigma \cup \{\varepsilon\}$, $i = 1, \ldots, n$, we have $\mu_{\mathcal{L}(\mathcal{M}')}(\omega) = \bigvee\{\mu_{I'}(q) \wedge \mu_{\vdash_{\mathcal{M}'}}((q, \omega, X_0), (p_0, \omega, Z_0)) \wedge \mu_{\vdash_{\mathcal{M}'}}((p_0, \omega, Z_0), (q_1, u_2 \cdots u_n, z_1 r_1)) \wedge \mu_{\vdash_{\mathcal{M}'}}((q_1, u_2 \cdots u_n, z_1 r_1), (q_2, u_3 \cdots u_n, z_2 r_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathcal{M}'}}((q_{n-1}, u_n, z_{n-1} r_{n-1}), (q_n, \varepsilon, r_n)) \wedge \mu_{F'}(q_n) \mid q \in Q', (p_0, q_1, \ldots, q_n) \in Q'^{n+1}, z_1, \ldots, z_{n-1} \in \Gamma', r_1, \ldots, r_n \in \Gamma'^*\} = \bigvee\{1 \wedge \mu_I(p_0) \wedge \mu_{\vdash_{\mathcal{M}}}((p_0, \omega, Z_0), (q_1, u_2 \cdots u_n, z_1 r_1)) \wedge \mu_{\vdash_{\mathcal{M}}}((q_1, u_2 \cdots u_n, z_1 r_1), (q_2, u_3 \cdots u_n, z_2 r_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathcal{M}}}((q_{n-1}, u_n, z_{n-1} r_{n-1}), (q_n, \varepsilon, r_n)) \wedge \mu_F(q_n) \mid (p_0, q_1, \ldots, q_n) \in Q^{n+1}, z_1, \ldots, z_{n-1} \in \Gamma, r_1, \ldots, r_n \in \Gamma^*\} = \mu_{\mathcal{L}(\mathcal{M})}(\omega)$, and $\nu_{\mathcal{L}(\mathcal{M}')}(\omega) = \bigwedge\{\nu_{I'}(q) \vee \nu_{\vdash_{\mathcal{M}'}}((q, \omega, X_0), (p_0, \omega, Z_0)) \vee \nu_{\vdash_{\mathcal{M}'}}((p_0, \omega, Z_0), (q_1, u_2 \cdots u_n, z_1 r_1)) \vee \nu_{\vdash_{\mathcal{M}'}}((q_1, u_2 \cdots u_n, z_1 r_1), (q_2, u_3 \cdots u_n, z_2 r_2)) \vee \cdots \vee \nu_{\vdash_{\mathcal{M}'}}((q_{n-1}, u_n, z_{n-1} r_{n-1}), (q_n, \varepsilon, r_n)) \vee \nu_{F'}(q_n) \mid q \in Q', (p_0, q_1, \ldots, q_n) \in Q'^{n+1}, z_1, \ldots, z_{n-1} \in \Gamma', r_1, \ldots, r_n \in \Gamma'^*\} = \bigwedge\{1 \vee \nu_I(p_0) \vee \nu_{\vdash_{\mathcal{M}}}((p_0, \omega, Z_0), (q_1, u_2 \cdots u_n, z_1 r_1)) \vee \nu_{\vdash_{\mathcal{M}}}((q_1, u_2 \cdots u_n, z_1 r_1), (q_2, u_3 \cdots u_n, z_2 r_2)) \vee \cdots \vee \nu_{\vdash_{\mathcal{M}}}((q_{n-1}, u_n, z_{n-1} r_{n-1}), (q_n, \varepsilon, r_n)) \vee \nu_F(q_n) \mid (p_0, q_1, \ldots, q_n) \in Q^{n+1}, z_1, \ldots, z_{n-1} \in \Gamma, r_1, \ldots, r_n \in \Gamma^*\} = \nu_{\mathcal{L}(\mathcal{M})}(\omega)$. Therefore, $\mathcal{L}(\mathcal{M}') = \mathcal{L}(\mathcal{M})$.

From the construction, it is clearly that $\mathcal{M}'$ can be denoted as $\mathcal{M}' = (Q', \Sigma, \Gamma', \delta', q_0, X_0, F')$.

(ii) implies (i). Suppose the IFS A is accepted by the IFPDA $\mathcal{M}' = (Q', \Sigma, \Gamma', \delta', q_0, X_0, F')$. Then we construct an IFS I in Q' by

$$\mu_I(q) = \begin{cases} 1, & \text{if } q = q_0 \\ 0, & \text{if } q \neq q_0, \end{cases} \qquad \nu_I(q) = \begin{cases} 0, & \text{if } q = q_0 \\ 1, & \text{if } q \neq q_0. \end{cases} \tag{14}$$

It follows that the IFPDA $\mathcal{M} = (Q', \Sigma, \Gamma', \delta', I, X_0, F')$ accepts A.

Similarly, it is easily concluded that the following must be true. □

Proposition 12. *Let A be an IFS in a nonempty set Σ^*. Then the following statements are equivalent:*

(i) *A can be accepted by an IFPDA$^{\emptyset}$ $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, \emptyset)$;*

(ii) *There exists an IFPDA$^{\emptyset}$ $\mathcal{M}' = (Q', \Sigma, \Gamma', \delta', q_0, X_0, \emptyset)$ recognizing A, where $q_0 \in Q'$.*

There is especially a simple type of intuitionistic fuzzy pushdown automata, which is called intuitionistic fuzzy simple pushdown automata. The definition is given as follows.

Definition 13. An IFPDA $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is called an intuitionistic fuzzy simple pushdown automaton (IFSPDA) if the image set of δ is contained in the set $\{0, 1\}$.

Next any IFPDA is proven to be an equivalence of a certain IFSPDA by utilizing the generalized subset construction method. Noting that an IFS requires that the sum of the degrees of membership and nonmembership of an element to a set is no more than the natural number 1. So the proof technique is to some extent different from the technique of extended subset construction introduced by Li in [33], and it is not an easy task to conduct reasoning in the realm of the modified techniques.

Proposition 14. *Let $\mathscr{M}$ be an IFPDA. Then there exists an IFSPDA $\mathscr{M}'$ such that $\mathscr{L}(\mathscr{M}') = \mathscr{L}(\mathscr{M})$.*

Proof. Let $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be an IFPDA. Then we construct an IFSPDA $\mathscr{M}' = (Q', \Sigma, \Gamma, \delta', q_0', Z_0, F')$ as follows:

(i) $Q' = Q \times (L_1 - \{0\}) \times (L_2 - \{1\})$, where $L_1 = X_\wedge$, $L_2 = Y_\vee$, $X = \mathrm{Im}(\mu_\delta) \cup \mathrm{Im}(\mu_F)$ and $Y = \mathrm{Im}(\nu_\delta) \cup \mathrm{Im}(\nu_F)$;

(ii) $q_0' = (q_0, 1, 0) \in Q'$;

(iii) $\delta' = (\mu_{\delta'}, \nu_{\delta'})$ is an IFS in $Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q' \times \Gamma^*$, where the mappings $\mu_{\delta'}, \nu_{\delta'} : Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q' \times \Gamma^* \to \{0, 1\}$ are given as follows. For any $(q, a, b), (q', c, d) \in Q'$, $\tau \in \Sigma \cup \{\varepsilon\}$, $X \in \Gamma$ and $\gamma \in \Gamma^*$, $\mu_{\delta'}((q, a, b), \tau, X, (q', c, d), \gamma) = 1$, and $\nu_{\delta'}((q, a, b), \tau, X, (q', c, d), \gamma) = 0$ whenever there exist a' and b' such that $\mu_\delta(q, \tau, X, q', \gamma) = a' > 0$, $\nu_\delta(q, \tau, X, q', \gamma) = b' < 1$, $c = a \wedge a'$ and $d = b \vee b'$. Otherwise, $\mu_{\delta'}((q, a, b), \tau, X, (q', c, d), \gamma) = 0$ and $\nu_{\delta'}((q, a, b), \tau, X, (q', c, d), \gamma) = 1$;

(iv) $F' = (\mu_{F'}, \nu_{F'})$ is an IFS in Q'. For any $(q, a, b) \in Q'$,

$$\mu_{F'}((q, a, b)) = \begin{cases} a \wedge \mu_F(q), & \text{if } 0 \le a + b \le 1 \\ 0, & \text{if } a + b > 1, \end{cases}$$
$$\nu_{F'}((q, a, b)) = \begin{cases} b \vee \nu_F(q), & \text{if } 0 \le a + b \le 1 \\ 1, & \text{if } a + b > 1. \end{cases} \tag{15}$$

Now, it is claimed that, for any $\omega = \tau_1 \cdots \tau_n \in \Sigma^*$, $\tau_i \in \Sigma \cup \{\varepsilon\}$, $i \in \{1, \ldots, n\}$ and for any $(q_n, a_n, b_n) \in Q'$, $Z_n, \gamma_n \in \Gamma^*$, $\mu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_n, a_n, b_n), \varepsilon, Z_n\gamma_n)) = 1$ and $\nu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_n, a_n, b_n), \varepsilon, Z_n\gamma_n)) = 0$ whenever the following condition is satisfied.

(P1) There exist $q_1, \ldots, q_{n-1} \in Q$, $Z_1, \ldots, Z_{n-1} \in \Gamma$ and $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma^*$ such that $a_n = \mu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \wedge \mu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, Z_n\gamma_n))$ and $b_n = \nu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \vee \nu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \vee \cdots \vee \nu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, Z_n\gamma_n))$. Otherwise, $\mu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_n, a_n, b_n), \varepsilon, Z_n\gamma_n)) = 0$ and $\nu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_n, a_n, b_n), \varepsilon, Z_n\gamma_n)) = 1$.

It is proved by induction. In fact, if $|\omega| = 0$, then $\omega = \varepsilon$, $\mu_{\vdash^*_{\mathscr{M}'}}((q_0, \varepsilon, Z_0), (q_0, \varepsilon, Z_0)) = 1$ and $\nu_{\vdash^*_{\mathscr{M}}}((q_0, \varepsilon, Z_0), (q_0, \varepsilon, Z_0)) = 0$. Hence $\mu_{\vdash^*_{\mathscr{M}}}(((q_0, 1, 0), \varepsilon, Z_0), ((q_0, 1, 0), \varepsilon, Z_0)) = 1$ and $\nu_{\vdash^*_{\mathscr{M}'}}(((q_0, 1, 0), \varepsilon, Z_0), ((q_0, 1, 0), \varepsilon, Z_0)) = 0$.

Suppose the result still holds whenever $|\omega| \le n$, $n \in N$. If $|\omega| = n + 1$, $\omega = \tau_1 \cdots \tau_k\tau_{k+1} = x\tau_{k+1}$ and $\tau_{k+1} \in \Sigma$, then $|x| = n$ and $x = \tau_1 \cdots \tau_k$.

Next, for any $(q_{k+1}, a_{k+1}, b_{k+1}) \in Q'$, $Z_{k+1}, \gamma_{k+1} \in \Gamma^*$, whenever (P1) is satisfied; that is, there exists a sequence of states $q_1, \ldots, q_k \in Q$, $Z_1, \ldots, Z_k \in \Gamma$, $\gamma_1, \ldots, \gamma_k \in \Gamma^*$ such that

$\mu_{\vdash_{\mathscr{M}}}((q_i, \tau_{i+1} \cdots \tau_{k+1}, Z_i\gamma_i), (q_{i+1}, \tau_{i+2} \cdots \tau_{k+1}, Z_{i+1}\gamma_{i+1})) = c_{i+1} > 0$,

$\nu_{\vdash_{\mathscr{M}}}((q_i, \tau_{i+1} \cdots \tau_{k+1}, Z_i\gamma_i), (q_{i+1}, \tau_{i+2} \cdots \tau_{k+1}, Z_{i+1}\gamma_{i+1})) = d_{i+1} < 1$,

$\mu_\delta(q_k, \tau_{k+1}, Z_k, q_{k+1}, Z_{k+1}) = c_{k+1} > 0$,

$\nu_\delta(q_k, \tau_{k+1}, Z_k, q_{k+1}, Z_{k+1}) = d_{k+1} < 1$,

where $\gamma_0 = \varepsilon$, $i = 0, 1, \ldots, k - 1$.

Let $a_l = c_1 \wedge \cdots \wedge c_l$, $b_l = d_1 \vee \cdots \vee d_l$, $l \in \{1, \ldots, k + 1\}$. Then

$\mu_{\vdash_{\mathscr{M}'}}(((q_i, a_i, b_i), \tau_{i+1} \cdots \tau_{k+1}, Z_i\gamma_i), ((q_{i+1}, a_{i+1}, b_{i+1}), \tau_{i+2} \cdots \tau_{k+1}, Z_{i+1}\gamma_{i+1})) = 1$,

$\nu_{\vdash_{\mathscr{M}'}}(((q_i, a_i, b_i), \tau_{i+1} \cdots \tau_{k+1}, Z_i\gamma_i), ((q_{i+1}, a_{i+1}, b_{i+1}), \tau_{i+2} \cdots \tau_{k+1}, Z_{i+1}\gamma_{i+1})) = 0$,

$\mu_{\vdash_{\mathscr{M}'}}(((q_k, a_k, b_k), \tau_{k+1}, Z_k\gamma_k), ((q_{k+1}, a_{k+1}, b_{k+1}), \varepsilon, Z_{k+1}\gamma_{k+1})) = 1$, and

$\nu_{\vdash_{\mathscr{M}'}}(((q_k, a_k, b_k), \tau_{k+1}, Z_k\gamma_k), ((q_{k+1}, a_{k+1}, b_{k+1}), \varepsilon, Z_{k+1}\gamma_{k+1})) = 0$,

where $a_0 = 1$, $b_0 = 0$, $\gamma_0 = \varepsilon$, $i = 0, 1, \ldots, k - 1$.

By assumption, $\mu_{\vdash^*_{\mathscr{M}'}}((q_0', x, Z_0), ((q_k, a_k, b_k), \varepsilon, Z_k\gamma_k)) = 1$, $\nu_{\vdash^*_{\mathscr{M}'}}((q_0', x, Z_0), ((q_k, a_k, b_k), \varepsilon, Z_k\gamma_k)) = 0$, and so $\mu_{\vdash^*_{\mathscr{M}'}}((q_0', x\tau_{k+1}, Z_0), ((q_k, a_k, b_k), \tau_{k+1}, Z_k\gamma_k)) = 1$, $\nu_{\vdash^*_{\mathscr{M}'}}((q_0', x\tau_{k+1}, Z_0), ((q_k, a_k, b_k), \tau_{k+1}, Z_k\gamma_k)) = 0$.

Therefore, $\mu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_{k+1}, a_{k+1}, b_{k+1}), \varepsilon, Z_{k+1}\gamma_{k+1})) = \bigvee\{\mu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_k, a_k, b_k), \tau_{k+1}, Z_k\gamma_k)) \wedge \mu_{\vdash_{\mathscr{M}'}}(((q_k, a_k, b_k), \tau_{k+1}, Z_k\gamma_k), ((q_{k+1}, a_{k+1}, b_{k+1}), \varepsilon, Z_{k+1}\gamma_{k+1})) \mid (q_k, a_k, b_k) \in Q', Z_k \in \Gamma, \gamma_k \in \Gamma^*\} = 1 \wedge 1 = 1$. $\nu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_{k+1}, a_{k+1}, b_{k+1}), \varepsilon, Z_{k+1}\gamma_{k+1})) = \bigwedge\{\nu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_k, a_k, b_k), \tau_{k+1}, Z_k\gamma_k)) \vee \nu_{\vdash_{\mathscr{M}'}}(((q_k, a_k, b_k), \tau_{k+1}, Z_k\gamma_k), ((q_{k+1}, a_{k+1}, b_{k+1}), \varepsilon, Z_{k+1}\gamma_{k+1})) \mid (q_k, a_k, b_k) \in Q', Z_k \in \Gamma, \gamma_k \in \Gamma^*\} = 0 \vee 0 = 0$.

For any $(q_{k+1}, a_{k+1}, b_{k+1}) \in Q'$, $Z_{k+1}, \gamma_{k+1} \in \Gamma^*$, if (P1) is not satisfied, then it follows that

$$\mu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_{k+1}, a_{k+1}, b_{k+1}), \varepsilon, Z_{k+1}\gamma_{k+1})) = 0,$$
$$\nu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_{k+1}, a_{k+1}, b_{k+1}), \varepsilon, Z_{k+1}\gamma_{k+1})) = 1.$$

Hence, for any $\omega = \tau_1 \cdots \tau_n \in \Sigma^*$, $\tau_i \in \Sigma \cup \{\varepsilon\}$, $i \in \{1, \ldots, n\}$, we have $\mu_{\mathscr{L}(\mathscr{M}')}(\omega) = \bigvee\{\mu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_n, a_n, b_n), \varepsilon, \gamma)) \wedge \mu_{F'}((q_n, a_n, b_n)) \mid (q_n, a_n, b_n) \in Q', \gamma \in \Gamma^*\} = \bigvee\{a_n \wedge \mu_F(q_n) \mid q_1, \ldots, q_{n-1} \in Q, Z_1, \ldots, Z_{n-1} \in \Gamma, \gamma_1, \ldots, \gamma_{n-1} \in \Gamma^*, a_n = \mu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \wedge \mu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \gamma)), b_n = \nu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \vee \nu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \vee \cdots \vee \nu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \gamma))\} = \bigvee\{\mu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \wedge \mu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)(q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \gamma)) \wedge \mu_F(q_n) \mid q_1, \ldots, q_{n-1} \in Q, Z_1, \ldots, Z_{n-1} \in \Gamma, \gamma_1, \ldots, \gamma_{n-1} \in \Gamma^*\} = \mu_{\mathscr{L}(\mathscr{M})}(\omega)$, $\nu_{\mathscr{L}(\mathscr{M}')}(\omega) = \bigwedge\{\nu_{\vdash^*_{\mathscr{M}'}}((q_0', \omega, Z_0), ((q_n, a_n, b_n), \varepsilon, \gamma)) \vee \nu_{F'}((q_n, a_n,

$b_n)) \mid (q_n, a_n, b_n) \in Q', \gamma \in \Gamma^*\} = \bigwedge\{b_n \vee \nu_F(q_n) \mid q_1, \ldots, q_{n-1} \in Q, Z_1, \ldots, Z_{n-1} \in \Gamma, \gamma_1, \ldots, \gamma_{n-1} \in \Gamma^*, a_n = \mu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \wedge \mu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \gamma)), b_n = \nu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \vee \nu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \vee \cdots \vee \nu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \gamma))\} = \bigwedge\{\nu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \vee \nu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \vee \cdots \vee \nu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \gamma)) \vee \nu_F(q_n) \mid q_1, \ldots, q_{n-1} \in Q, Z_1, \ldots, Z_{n-1} \in \Gamma, \gamma_1, \ldots, \gamma_{n-1} \in \Gamma^*\} = \nu_{\mathscr{L}(\mathscr{M})}(\omega)$.

Therefore, $\mathscr{L}(\mathscr{M}') = \mathscr{L}(\mathscr{M})$. □

Clearly, an IFPDA is a generalization of a classical pushdown automaton (PDA). Next, it will be shown that any IFPDA can be characterized by a collection of pushdown automata. To describe the behavior of a pushdown automaton $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, we need to introduce the concept of instantaneous description. An instantaneous description is a three-tuple $(q, \omega, \gamma) \in Q \times \Sigma^* \times \Gamma^*$, which means that the automaton is in the state q and has unexpended input ω and stack contents γ. An instantaneous description represents the configuration of a pushdown automaton at a given instant. To introduce the transition in a pushdown automaton in terms of instantaneous descriptions, we define $\succ_{\mathscr{M}}$ as a binary relation on $Q \times \Sigma^* \times \Gamma^*$. We say $(q, a\omega, Z\gamma) \succ_{\mathscr{M}} (p, \omega, \sigma\gamma)$ if $\delta(q, a, Z)$ contains (p, σ), where $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\omega \in \Sigma^*$, $Z \in \Gamma$, and $\gamma, \sigma \in \Gamma^*$. Furthermore, we define $\succ^*_{\mathscr{M}}$ as the reflexive and transitive closure of $\succ_{\mathscr{M}}$. Then the language accepted by $\mathscr{M}$ with final states is defined as

$$\mathscr{L}(\mathscr{M}) = \{\omega \in \Sigma^* \mid (q_0, \omega, Z_0) \succ^*_{\mathscr{M}} (p, \varepsilon, \gamma), p \in F, \gamma \in \Gamma^*\}. \tag{16}$$

Definition 15. A collection of classical pushdown automata with final states

$$S = \{\mathscr{M}_{ab} \mid \mathscr{M}_{ab} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F_{ab}), \\ 0 \le a + b \le 1, a, b \in [0,1]\} \tag{17}$$

is called a cover if the following conditions hold:

(i) $a_1 \le a_2$ and $b_2 \le b_1$ imply $F_{a_2 b_2} \subseteq F_{a_1 b_1}$;

(ii) $F_{01} = Q$.

For a cover S, its recognized intuitionistic fuzzy language $f_S = (\mu_{f_S}, \nu_{f_S})$ in Σ^* is given by

$\mu_{f_S}(\omega) = \bigvee\{a \in [0,1] \mid \mathscr{M}_{ab}$ accepts $\omega, \mathscr{M}_{ab} \in S\}$,

$\nu_{f_S}(\omega) = \bigwedge\{b \in [0,1] \mid \mathscr{M}_{ab}$ accepts $\omega, \mathscr{M}_{ab} \in S\}$, for all $\omega \in \Sigma^*$.

Theorem 16. *Let f be an IFS in Σ^*. Then f can be accepted by an IFPDA if and only if f can be recognized by a cover S.*

Proof. If f can be accepted by an IFPDA, then there exists an IFSPDA $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ such that $\mathscr{M}$ accepts f by Proposition 14. Next we construct a cover

$$S = \{\mathscr{M}_{ab} \mid \mathscr{M}_{ab} = (Q, \Sigma, \Gamma, \delta', q_0, Z_0, F_{ab}), \\ 0 \le a + b \le 1, a, b \in [0,1]\}, \tag{18}$$

where $F_{ab} = \{q \in Q \mid \mu_F(q) \ge a, \ \nu_F(q) \le b\}$; the mapping $\delta' : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$ is given by

$\delta'(q, \tau, Z) = \{(p, \gamma) \mid \mu_\delta(q, \tau, Z, p, \gamma) = 1, p \in Q, \gamma \in \Gamma^*\}$, for all $(q, \tau, Z) \in Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma$.

Clearly, the cover S is well defined.

Next, we will show that f can be recognized by the cover S. In fact, we have

$(q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma)$ if and only if $\mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1$,

for all $a, b \in [0,1]$ with $a + b \le 1$, for all $\omega \in \Sigma^*, \gamma \in \Gamma^*$. $\mu_{f_S}(\omega) = \bigvee\{a \in [0,1] \mid \mathscr{M}_{ab}$ accepts $\omega, \mathscr{M}_{ab} \in S\} = \bigvee\{a \in [0,1] \mid q \in F_{ab}, \ (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma), \ \gamma \in \Gamma^*\} = \bigvee\{a \in [0,1] \mid \mu_F(q) \ge a, \ \nu_F(q) \le b, (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma), \ \gamma \in \Gamma^*\} = \bigvee\{a \in [0,1] \mid \mu_F(q) \ge a, (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma), \ q \in Q, \gamma \in \Gamma^*\} = \bigvee\{\mu_F(q) \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1, \ q \in Q, \ \gamma \in \Gamma^*\} = \mu_f(\omega) = \mu_{\mathscr{L}(\mathscr{M})}(\omega)$, $\nu_{f_S}(\omega) = \bigwedge\{b \in [0,1] \mid \mathscr{M}_{ab}$ accepts $\omega, \mathscr{M}_{ab} \in S\} = \bigwedge\{b \in [0,1] \mid q \in F_{ab}, (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma), \ \gamma \in \Gamma^*\} = \bigwedge\{b \in [0,1] \mid \mu_F(q) \ge a, \ \nu_F(q) \le b, (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma), \ \gamma \in \Gamma^*\} = \bigwedge\{b \in [0,1] \mid \nu_F(q) \le b, (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma), \ q \in Q, \gamma \in \Gamma^*\} = \bigwedge\{\nu_F(q) \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1, \ q \in Q, \ \gamma \in \Gamma^*\} = \nu_f(\omega) = \nu_{\mathscr{L}(\mathscr{M})}(\omega)$.

Therefore, $f_S = \mathscr{L}(\mathscr{M}) = f$.

Conversely, suppose f can be recognized by a cover

$$S = \{\mathscr{M}_{ab} \mid \mathscr{M}_{ab} = (Q, \Sigma, \Gamma, \delta', q_0, Z_0, F_{ab}), \\ 0 \le a + b \le 1, a, b \in [0,1]\}. \tag{19}$$

Then we construct an IFSPDA $\mathscr{M} = (Q, \Sigma, \Gamma, \eta, q_0, Z_0, F)$, where $\eta = (\mu_\eta, \nu_\eta)$ is an IFS in $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$, and the mappings $\mu_\eta, \nu_\eta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^* \to \{0,1\}$ are defined as

$$\mu_\eta(q, \tau, Z, p, \gamma) = \begin{cases} 1, & \text{if } (p, \gamma) \in \delta'(q, \tau, Z) \\ 0, & \text{otherwise,} \end{cases} \\ \nu_\eta(q, \tau, Z, p, \gamma) = \begin{cases} 0, & \text{if } (p, \gamma) \in \delta'(q, \tau, Z) \\ 1, & \text{otherwise,} \end{cases} \tag{20}$$

for any $(q, \tau, Z, p, \gamma) \in Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$.

$F = (\mu_F, \nu_F)$ is an IFS in Q, where $\mu_F(q) = \bigvee\{a \in [0,1] \mid q \in F_{ab}\}$ and $\nu_F(q) = \bigwedge\{b \in [0,1] q \in F_{ab}\}$, for all $q \in Q$. Then $\mathscr{L}(\mathscr{M}) = f_S$. In fact, for any $\omega \in \Sigma^*$, $\mu_{\mathscr{L}(\mathscr{M})}(\omega) = \bigvee\{\mu_F(q) \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1, \ q \in Q, \ \gamma \in \Gamma^*\} = \bigvee\{\bigvee\{a \in [0,1] \mid q \in F_{ab}\} \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1, \ q \in Q, \ \gamma \in \Gamma^*\} = \bigvee\{a \in [0,1] \mid q \in F_{ab}, \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1, \ q \in Q, \ \gamma \in \Gamma^*\} = \bigvee\{a \in [0,1] \mid q \in F_{ab}, (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma), \ \gamma \in \Gamma^*\} = \bigvee\{a \in [0,1] \mid \mathscr{M}_{ab}$ accepts $\omega, \mathscr{M}_{ab} \in S\} = \mu_{f_S}(\omega)$, $\nu_{\mathscr{L}(\mathscr{M})}(\omega) = \bigwedge\{\nu_F(q) \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1, \ q \in Q, \ \gamma \in \Gamma^*\} = \bigwedge\{\bigwedge\{b \in [0,1] \mid q \in F_{ab}\} \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1, \ q \in Q, \ \gamma \in \Gamma^*\} = \bigwedge\{b \in [0,1] \mid q \in F_{ab},$

$(q_0, \omega, Z_0) \succ^*_{\mathscr{M}_{ab}} (q, \varepsilon, \gamma),\ \gamma \in \Gamma^*\} = \bigwedge\{b \in [0,1] \mid \mathscr{M}_{ab}$ accepts $\omega, \mathscr{M}_{ab} \in S\} = \nu_{f_S}(\omega)$.

Therefore, the IFSPDA $\mathscr{M}$ accepts f. □

Theorem 16 shows that every IFPDA is equivalent to a certain cover; however, the cover may have infinite classical pushdown automata elements. Is there a finite cover who is equivalent to the IFPDA? To solve the problem, we introduce the notion of an intuitionistic fuzzy recognizable step function as follows.

Definition 17. An IFS A over Σ^* is called an intuitionistic fuzzy recognizable step function if there are a finite natural number $n \in N$, recognizable context-free languages $\mathscr{L}_1, \dots, \mathscr{L}_n \subseteq \Sigma^*$, and $(a_i, b_i) \in (0,1] \times [0,1)$ with $0 \le a_i + b_i \le 1$ for $i = 1, \dots, n$ such that

$$A = (\mu_A, \nu_A) = \coprod_{i=1}^{n} (a_i, b_i) \cdot \mathbf{1}_{\mathscr{L}_i}, \qquad (\star)$$

where $\mathbf{1}_{\mathscr{L}_i} = (\mu_{\mathbf{1}_{\mathscr{L}_i}}, \nu_{\mathbf{1}_{\mathscr{L}_i}})$ represents the intuitionistic characterized function of $\mathscr{L}_i$, $i = 1, \dots, n$, that is,

$$\mu_{\mathbf{1}_{\mathscr{L}_i}}(\omega) = \begin{cases} 1, & \text{if } \omega \in \mathscr{L}_i \\ 0, & \text{if } \omega \notin \mathscr{L}_i, \end{cases}$$
$$\nu_{\mathbf{1}_{\mathscr{L}_i}}(\omega) = \begin{cases} 0, & \text{if } \omega \in \mathscr{L}_i \\ 1, & \text{if } \omega \notin \mathscr{L}_i. \end{cases} \qquad (21)$$

And the equation $(\star)$ means that the following equations hold:

$$\mu_A(\omega) = \bigvee_{i=1}^{n} a_i \wedge \mu_{\mathbf{1}_{\mathscr{L}_i}}(\omega), \quad \nu_A(\omega) = \bigwedge_{i=1}^{n} b_i \vee \nu_{\mathbf{1}_{\mathscr{L}_i}}(\omega), \quad \forall \omega \in \Sigma^*. \qquad (22)$$

Noting that the family of all the intuitionistic fuzzy recognizable step functions over Σ^* is denoted by $\mathrm{Step}^C(\Sigma)$.

Proposition 18. *Let $\mathscr{M}_1' = (Q_1, \Sigma, \Gamma_1, \delta_1, q_{01}, Z_{01}, F_1)$ be an IFPDA. Then the language recognized by $\mathscr{M}_1'$ is an intuitionistic fuzzy recognizable step function over Σ^*, that is, $\mathscr{L}(\mathscr{M}_1') \in \mathrm{Step}^C(\Sigma)$.*

Proof. By Proposition 14, there is an IFSPDA $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ equivalent to $\mathscr{M}_1'$.

Let $R = \{(\mu_F(q), \nu_F(q)) \mid q \in Q\} \setminus \{(0,1)\} = \{(a_i, b_i) \mid i \in N_k\}$, $N_k = \{1, \dots, k\}$. Put $F_i = \{q \in Q \mid \mu_F(q) = a_i, \nu_F(q) = b_i\}$, for all $i \in N_k$. Then we construct a PDA $\mathscr{M}_i = (Q, \Sigma, \Gamma, \delta', q_0, Z_0, F_i)$, where the mapping $\delta' : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$ is defined by

$\delta'(q, \tau, X) = \{(p, \gamma) \mid \mu_\delta(q, \tau, X, p, \gamma) = 1, p \in Q, \gamma \in \Gamma^*\}$, for all $(q, \tau, X) \in Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma$.

Then $\mathscr{L}(\mathscr{M}_i) = \mathscr{L}_i = \{\omega \in \Sigma^* \mid (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_i} (q, \varepsilon, \gamma), q \in F_i, \gamma \in \Gamma^*\}$.

Therefore, for any $\omega \in \Sigma^*$, we have $\mu_{\mathscr{L}(\mathscr{M}_1')}(\omega) = \mu_{\mathscr{L}(\mathscr{M})}(\omega) = \bigvee\{\mu_F(q) \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 1,\ q \in Q,\ \gamma \in \Gamma^*\} = \bigvee\{a_i \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0),\ (q, \varepsilon, \gamma)) = 1, \mu_F(q) = a_i, \nu_F(q) = b_i, i \in N_k, \gamma \in \Gamma^*\} = \bigvee\{a_i \mid (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_i} (q, \varepsilon, \gamma), q \in F_i, i \in N_k, \gamma \in \Gamma^*\} = \bigvee\{a_i \mid \omega \in \mathscr{L}(\mathscr{M}_i), i \in N_k\} = \bigvee_{i \in N_k} a_i \wedge \mu_{\mathbf{1}_{\mathscr{L}(\mathscr{M}_i)}}(\omega)$, and $\nu_{\mathscr{L}(\mathscr{M}_1')}(\omega) = \nu_{\mathscr{L}(\mathscr{M})}(\omega) = \bigwedge\{\nu_F(q) \mid \nu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) = 0,\ q \in Q,\ \gamma \in \Gamma^*\} = \bigwedge\{b_i \mid \mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0),\ (q, \varepsilon, \gamma)) = 1, \mu_F(q) = a_i, \nu_F(q) = b_i, i \in N_k, \gamma \in \Gamma^*\} = \bigwedge\{b_i \mid (q_0, \omega, Z_0) \succ^*_{\mathscr{M}_i} (q, \varepsilon, \gamma), q \in F_i, i \in N_k, \gamma \in \Gamma^*\} = \bigwedge\{b_i \mid \omega \in \mathscr{L}(\mathscr{M}_i), i \in N_k\} = \bigwedge_{i \in N_k} b_i \vee \nu_{\mathbf{1}_{\mathscr{L}(\mathscr{M}_i)}}(\omega)$.

So $\mathscr{L}(\mathscr{M}) = \coprod_{i \in N_k} (a_i, b_i) \cdot \mathbf{1}_{\mathscr{L}_i}$. □

Proposition 18 shows that the set of the languages recognized by all the IFPDAs is a subset of $\mathrm{Step}^C(\Sigma)$. In fact, $\mathrm{Step}^C(\Sigma)$ is also a subset of the set of the languages recognized by all the IFPDAs. We will prove the decomposition form in the following.

Theorem 19. *Let A be an IFS over Σ^*. Then the following statements are equivalent:*

(i) *$A \in \mathrm{Step}^C(\Sigma)$;*

(ii) *there is an IFPDA $\mathscr{M}$ such that $A = \mathscr{L}(\mathscr{M})$;*

(iii) *there is an IFSPDA $\mathscr{M}$ such that $A = \mathscr{L}(\mathscr{M})$.*

Proof. (i) implies (iii). Suppose $A \in \mathrm{Step}^C(\Sigma)$. Then there is a finite natural number $n \in N$, recognizable context-free languages $\mathscr{L}_1, \dots, \mathscr{L}_n \subseteq \Sigma^*$, and $(a_i, b_i) \in (0,1] \times [0,1)$ with $0 \le a_i + b_i \le 1$ for $i = 1, \dots, n$ such that $A = (\mu_A, \nu_A) = \coprod_{i=1}^{n} (a_i, b_i) \cdot \mathbf{1}_{\mathscr{L}_i}$. Let $\mathscr{L}_i$ be recognized by a PDA $\mathscr{M}_i = (Q_i, \Sigma, \Gamma_i, \delta_i, q_{0i}, Z_{0i}, F_i)$, and $Q_i \cap Q_j = \emptyset$ whenever $i \neq j$, $i, j \in N_n$.

Next, construct an IFSPDA $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ as follows: $Q = \bigcup_{i=1}^{n} Q_i \cup \{q_0\}$, $\Gamma = \bigcup_{i=1}^{n} \Gamma_i \cup \{Z_0\}$, where $q_0 \notin \bigcup_{i=1}^{n} Q_i$, $Z_0 \notin \bigcup_{i=1}^{n} \Gamma_i$. $\delta = (\mu_\delta, \nu_\delta)$ is an IFS over $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$, where the mappings $\mu_\delta, \nu_\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^* \to \{0,1\}$ are defined by $\mu_\delta(q_0, \varepsilon, Z_0, p, \gamma) = 1$ and $\nu_\delta(q_0, \varepsilon, Z_0, p, \gamma) = 0$ if $(p, \gamma) = (q_{0i}, Z_{0i})$, $i \in N_n$; $\mu_\delta(q, \tau, Z, p, \gamma) = \mu_{\delta_i}(q, \tau, Z, p, \gamma)$ and $\nu_\delta(q, \tau, Z, p, \gamma) = \nu_{\delta_i}(q, \tau, Z, p, \gamma)$ if $p, q \in Q_i$, $Z \in \Gamma_i, \gamma \in \Gamma_i^*$, $\tau \in \Sigma \cup \{\varepsilon\}$, $i \in N_n$. Otherwise, $\mu_\delta(q, \tau, Z, p, \gamma) = 0$ and $\nu_\delta(q, \tau, Z, p, \gamma) = 1$.

$F = (\mu_F, \nu_F)$ is an IFS in Q, where

$$\mu_F(q_0) = \bigvee \{a_i \in [0,1] \mid q_{0i} \in F_i, i \in N_n\},$$
$$\nu_F(q_0) = \bigwedge \{b_i \in [0,1] \mid q_{0i} \in F_i, i \in N_n\},$$
$$\mu_F(q) = \begin{cases} a_i, & \text{if } q \in F_i \\ 0, & \text{if } q \notin F_i \cup \{q_0\}, \end{cases} \qquad (23)$$
$$\nu_F(q) = \begin{cases} b_i, & \text{if } q \in F_i \\ 1, & \text{if } q \notin F_i \cup \{q_0\}. \end{cases}$$

Therefore, for any $\omega \in \Sigma^*$, we have

$\mu_{\mathscr{L}(\mathscr{M})}(\omega) = \bigvee\{\mu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) \wedge \mu_F(q) \mid q \in Q, \gamma \in \Gamma^*\}$,

$\nu_{\mathscr{L}(\mathscr{M})}(\omega) = \bigwedge\{\nu_{\vdash^*_{\mathscr{M}}}((q_0, \omega, Z_0), (q, \varepsilon, \gamma)) \vee \nu_F(q) \mid q \in Q, \gamma \in \Gamma^*\}$.

If $\omega = \varepsilon$, then $\mu_{\mathscr{L}(\mathscr{M})}(\varepsilon) = (\mu_{\vdash^*_{\mathscr{M}}}((q_0,\varepsilon,Z_0),(q_0,\varepsilon,Z_0)) \wedge \mu_F(q_0)) \vee (\bigvee\{\mu_{\vdash^*_{\mathscr{M}}}((q_0,\varepsilon,Z_0),(q_{0i},\varepsilon,Z_{0i})) \wedge \mu_F(q_{0i}) \mid q_{0i} \in Q_i, Z_{0i} \in \Gamma_i, i \in N_n\}) = \mu_F(q_0) \vee (\bigvee\{\mu_F(q_{0i}) \mid q_{0i} \in Q_i, Z_{0i} \in \Gamma_i, i \in N_n\}) = \bigvee\{a_i \in [0,1] \mid q_{0i} \in F_i, i \in N_n\} = \bigvee_{i=1}^n a_i \wedge \mu_{\mathbf{1}_{\mathscr{L}_i}}(\varepsilon)$; $\nu_{\mathscr{L}(\mathscr{M})}(\varepsilon) = (\nu_{\vdash^*_{\mathscr{M}}}((q_0,\varepsilon,Z_0),(q_0,\varepsilon,Z_0)) \vee \nu_F(q_0)) \wedge (\bigwedge\{\nu_{\vdash^*_{\mathscr{M}}}((q_0,\varepsilon,Z_0),(q_{0i},\varepsilon,Z_{0i})) \vee \nu_F(q_{0i}) \mid q_{0i} \in Q_i, Z_{0i} \in \Gamma_i, i \in N_n\}) = \nu_F(q_0) \wedge (\bigwedge\{\nu_F(q_{0i}) \mid q_{0i} \in Q_i, Z_{0i} \in \Gamma_i, i \in N_n\}) = \bigwedge\{b_i \in [0,1] \mid q_{0i} \in F_i, i \in N_n\} = \bigwedge_{i=1}^n b_i \vee \nu_{\mathbf{1}_{\mathscr{L}_i}}(\varepsilon)$. If $\omega \in \Sigma^* \setminus \{\varepsilon\}$, then $\mu_{\mathscr{L}(\mathscr{M})}(\omega) = \bigvee\{\mu_{\vdash^*_{\mathscr{M}}}((q_{0i},\omega,Z_{0i}), (q,\varepsilon,\gamma)) \wedge \mu_F(q) \mid q_{0i} \in Q_i, \gamma \in \Gamma^*, q \in Q, i \in N_n\} = \bigvee\{\mu_F(q) \mid q_{0i} \in Q_i, \gamma \in \Gamma_i^*, q \in Q, i \in N_n, \mu_{\vdash^*_{\mathscr{M}}}((q_{0i},\omega,Z_{0i}),(q,\varepsilon,\gamma)) = 1\} = \bigvee\{a_i \mid q \in F_i, i \in N_n, \mu_{\vdash^*_{\mathscr{M}}}((q_{0i},\omega,Z_{0i}),(q,\varepsilon,\gamma)) = 1, \gamma \in \Gamma_i^*\} = \bigvee\{a_i \mid \omega \in \mathscr{L}_i, i \in N_n\} = \bigvee_{i=1}^n a_i \wedge \mu_{\mathbf{1}_{\mathscr{L}_i}}(\omega)$, $\nu_{\mathscr{L}(\mathscr{M})}(\omega) = \bigwedge\{\nu_{\vdash^*_{\mathscr{M}}}((q_{0i},\omega,Z_{0i}), (q,\varepsilon,\gamma)) \vee \nu_F(q) \mid q_{0i} \in Q_i, \gamma \in \Gamma^*, q \in Q, i \in N_n\} = \bigwedge\{\nu_F(q) \mid q_{0i} \in Q_i, \gamma \in \Gamma_i^*, q \in Q, i \in N_n, \mu_{\vdash^*_{\mathscr{M}}}((q_{0i},\omega,Z_{0i}),(q,\varepsilon,\gamma)) = 1\} = \bigwedge\{b_i \mid q \in F_i, i \in N_n, \mu_{\vdash^*_{\mathscr{M}}}((q_{0i},\omega,Z_{0i}),(q,\varepsilon,\gamma)) = 1, \gamma \in \Gamma_i^*\} = \bigwedge\{b_i \mid \omega \in \mathscr{L}_i, i \in N_n\} = \bigwedge_{i=1}^n a_i \vee \nu_{\mathbf{1}_{\mathscr{L}_i}}(\omega)$.

(iii) implies (ii): obviously.

(ii) implies (i): it is concluded by Proposition 18. □

Next, we will discuss the characterization of IFPDA$^{\emptyset}$.

Proposition 20. *Let $\mathscr{M} = (Q,\Sigma,\Gamma,\delta,q_0,Z_0,\emptyset)$ be an IFPDA$^{\emptyset}$. Then there is a special IFPDA$^{\emptyset}$ $\mathscr{M}' = (Q',\Sigma,\Gamma',\delta',q_0',X_0,\emptyset)$ equivalent to $\mathscr{M}$.*

Proof. Given $\mathscr{M}$, we construct an IFPDA$^{\emptyset}$ $\mathscr{M}' = (Q',\Sigma,\Gamma',\delta',q_0',X_0,\emptyset)$ as follows:

(1) $Q' = Q_2 \times (L_1 \setminus \{0\}) \times (L_2 \setminus \{1\})$, where $Q_2 = Q \cup \{p_0\}$, $\Gamma' = \Gamma \cup \{X_0\}$, $L_1 = X_\wedge$, $L_2 = Y_\vee$, $X = \mathrm{Im}(\mu_\delta)$, $Y = \mathrm{Im}(\nu_\delta)$, and $q_0' = (p_0,1,0)$;

(2) $\delta' = (\mu_{\delta'},\nu_{\delta'})$ is an IFS in $Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma' \times Q' \times \Gamma'^*$, where the mappings $\mu_{\delta'},\nu_{\delta'} : Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma' \times Q' \times \Gamma'^* \to [0,1]$ are defined by

(i) $\mu_{\delta'}((p_0,1,0),\varepsilon,X_0,(q_0,1,0),Z_0X_0) = 1$, $\nu_{\delta'}((p_0,1,0),\varepsilon,X_0,(q_0,1,0),Z_0X_0) = 0$;

(ii) $\mu_{\delta'}((q,a,b),\tau,X,(q',a \wedge a',b \vee b'),\gamma) = 1$ and $\nu_{\delta'}((q,a,b),\tau,X,(q',a \wedge a',b \vee b'),\gamma) = 0$ whenever $\mu_\delta(q,\tau,X,q',\gamma) = a' > 0$ and $\nu_\delta(q,\tau,X,q',\gamma) = b' < 1$, for all $(q,a,b) \in Q'$, $\tau \in \Sigma \cup \{\varepsilon\}$, $X \in \Gamma$, $\gamma \in \Gamma^*$;

(iii) if $0 \le a+b \le 1$, then $\mu_{\delta'}((q,a,b),\varepsilon,X_0,(q,a,b),\varepsilon) = a$ and $\nu_{\delta'}((q,a,b),\varepsilon,X_0,(q,a,b),\varepsilon) = b$. If $a+b > 1$, then $\mu_{\delta'}((q,a,b),\varepsilon,X_0,(q,a,b),\varepsilon) = 0$ and $\nu_{\delta'}((q,a,b),\varepsilon,X_0,(q,a,b),\varepsilon) = 1$;

(iv) In other cases, $\mu_{\delta'}((q,a,b),\tau,X,(p,c,d),\gamma) = 0$ and $\nu_{\delta'}((q,a,b),\tau,X,(p,c,d),\gamma) = 1$.

Obviously, $\delta' = (\mu_{\delta'},\nu_{\delta'})$ is a finite IFS in $Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma' \times Q' \times \Gamma'^*$.

Next, we show $\mathscr{L}(\mathscr{M}') = \mathscr{L}(\mathscr{M})$.

Firstly let us show that, for any $\omega = \tau_1 \cdots \tau_n \in \Sigma^*$, $\tau_i \in \Sigma \cup \{\varepsilon\}$, $i \in \{1,\ldots,n\}$, $q_n \in Q$,

$$\mu_{\vdash^*_{\mathscr{M}'}}\left(\left(q_0',\omega,X_0\right),\left(\left(q_n,a_n,b_n\right),\varepsilon,X_0\right)\right) = \begin{cases} 1, & \text{if (P2) is satisfied} \\ 0, & \text{otherwise,} \end{cases}$$
$$\nu_{\vdash^*_{\mathscr{M}'}}\left(\left(q_0',\omega,X_0\right),\left(\left(q_n,a_n,b_n\right),\varepsilon,X_0\right)\right) = \begin{cases} 0, & \text{if (P2) is satisfied} \\ 1, & \text{otherwise,} \end{cases} \tag{24}$$

where the condition (P2) is the following:

(P2) there exist $q_1,\ldots,q_{n-1} \in Q$, $Z_1,\ldots,Z_{n-1} \in \Gamma$, $\gamma_1,\ldots,\gamma_{n-1} \in \Gamma^*$, s.t. $a_n = \mu_{\vdash_{\mathscr{M}}}((q_0,\omega,Z_0),(q_1,\tau_2\cdots\tau_n,Z_1\gamma_1)) \wedge \mu_{\vdash_{\mathscr{M}}}((q_1,\tau_2\cdots\tau_n,Z_1\gamma_1),(q_2,\tau_3\cdots\tau_n,Z_2\gamma_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}}}((q_{n-1},\tau_n,Z_{n-1}\gamma_{n-1}),(q_n,\varepsilon,\varepsilon)) > 0$ and $b_n = \nu_{\vdash_{\mathscr{M}}}((q_0,\omega,Z_0),(q_1,\tau_2\cdots\tau_n,Z_1\gamma_1)) \vee \nu_{\vdash_{\mathscr{M}}}((q_1,\tau_2\cdots\tau_n,Z_1\gamma_1),(q_2,\tau_3\cdots\tau_n,Z_2\gamma_2)) \vee \cdots \vee \nu_{\vdash_{\mathscr{M}}}((q_{n-1},\tau_n,Z_{n-1}\gamma_{n-1}),(q_n,\varepsilon,\varepsilon)) < 1$.

In fact, if (P2) is satisfied, then let $\mu_{\vdash_{\mathscr{M}}}((q_i,\tau_{i+1}\cdots\tau_n,Z_i\gamma_i),(q_{i+1},\tau_{i+2}\cdots\tau_n,Z_{i+1}\gamma_{i+1})) = c_i$, $\nu_{\vdash_{\mathscr{M}}}((q_i,\tau_{i+1}\cdots\tau_n,Z_i\gamma_i),(q_{i+1},\tau_{i+2}\cdots\tau_n,Z_{i+1}\gamma_{i+1})) = d_i$, $i = 0,1,\ldots,n-2$, where $\gamma_0 = \varepsilon$. We have

$\mu_{\vdash_{\mathscr{M}'}}(((q_0,1,0),\omega,Z_0X_0),((q_1,c_0,d_0),\tau_2\cdots\tau_n,Z_1\gamma_1X_0)) = 1$,

$\nu_{\vdash_{\mathscr{M}'}}(((q_0,1,0),\omega,Z_0X_0),((q_1,c_0,d_0),\tau_2\cdots\tau_n,Z_1\gamma_1X_0)) = 0$,

$\mu_{\vdash_{\mathscr{M}'}}(((q_1,c_0,d_0),\tau_2\cdots\tau_n,Z_1\gamma_1X_0),((q_2,c_0 \wedge c_1,d_0 \vee d_1),\tau_3\cdots\tau_n,Z_2\gamma_2X_0)) = 1$,

$\nu_{\vdash_{\mathscr{M}'}}(((q_1,c_0,d_0),\tau_2\cdots\tau_n,Z_1\gamma_1X_0),((q_2,c_0 \wedge c_1,d_0 \vee d_1),\tau_3\cdots\tau_n,Z_2\gamma_2X_0)) = 0$,

$\vdots$

$\mu_{\vdash_{\mathscr{M}'}}(((q_{n-1},c_0 \wedge c_1 \wedge \cdots \wedge c_{n-2},d_0 \vee d_1 \vee \cdots \vee d_{n-2}),\tau_n,Z_{n-1}\gamma_{n-1}X_0),((q_n,a_n,b_n),\varepsilon,X_0)) = 1$,

$\nu_{\vdash_{\mathscr{M}'}}(((q_{n-1},c_0 \wedge c_1 \wedge \cdots \wedge c_{n-2},d_0 \vee d_1 \vee \cdots \vee d_{n-2}),\tau_n,Z_{n-1}\gamma_{n-1}X_0),((q_n,a_n,b_n),\varepsilon,X_0)) = 0$.

Hence $\mu_{\vdash^*_{\mathscr{M}'}}((q_0',\omega,X_0),((q_n,a_n,b_n),\varepsilon,X_0) \ge \mu_{\vdash_{\mathscr{M}'}}(((q_0,1,0),\omega,Z_0X_0),((q_1,c_0,d_0),\tau_2\cdots\tau_n,Z_1\gamma_1X_0)) \wedge \mu_{\vdash_{\mathscr{M}'}}(((q_1,c_0,d_0),\tau_2\cdots\tau_n,Z_1\gamma_1X_0),((q_2,c_0 \wedge c_1,d_0 \vee d_1),\tau_3\cdots\tau_n,Z_2\gamma_2X_0)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}'}}(((q_{n-1},c_0 \wedge c_1 \wedge \cdots \wedge c_{n-2},d_0 \vee d_1 \vee \cdots \vee d_{n-2}),\tau_n,Z_{n-1}\gamma_{n-1}X_0),((q_n,a_n,b_n),\varepsilon,X_0)) = 1$, $\nu_{\vdash^*_{\mathscr{M}'}}((q_0',\omega,X_0),((q_n,a_n,b_n),\varepsilon,X_0)) \le \nu_{\vdash_{\mathscr{M}'}}(((q_0,1,0),\omega,Z_0X_0),((q_1,c_0,d_0),\tau_2\cdots\tau_n,Z_1\gamma_1X_0)) \vee \nu_{\vdash_{\mathscr{M}'}}(((q_1,c_0,d_0),\tau_2\cdots\tau_n,Z_1\gamma_1X_0),((q_2,c_0 \wedge c_1,d_0 \vee d_1),\tau_3\cdots\tau_n,Z_2\gamma_2X_0)) \vee \cdots \vee \nu_{\vdash_{\mathscr{M}'}}(((q_{n-1},c_0 \wedge c_1 \wedge \cdots \wedge c_{n-2},d_0 \vee d_1 \vee \cdots \vee d_{n-2}),\tau_n,Z_{n-1}\gamma_{n-1}X_0),((q_n,a_n,b_n),\varepsilon,X_0)) = 0$.

Since $\mu_{\vdash^*_{\mathscr{M}'}}((q_0',\omega,X_0),((q_n,a_n,b_n),\varepsilon,X_0)) \le 1$ and $\nu_{\vdash^*_{\mathscr{M}'}}((q_0',\omega,X_0),((q_n,a_n,b_n),\varepsilon,X_0)) \ge 0$, $\mu_{\vdash^*_{\mathscr{M}'}}((q_0',\omega,X_0),$

$((q_n, a_n, b_n), \varepsilon, X_0)) = 1$ and $\nu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q_n, a_n, b_n), \varepsilon, X_0)) = 0$.

If (P2) is not satisfied, then we assume $\mu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q_n, a_n, b_n), \varepsilon, X_0)) > 0$.

So there at least exist $q_1, \ldots, q_{n-1} \in Q$, $Z_1, \ldots, Z_{n-1} \in \Gamma$, $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma^*$, s.t. $\mu_{\vdash_{\mathscr{M}'}}((q_0, 1, 0), \omega, Z_0X_0), ((q_1, c_0, d_0), \tau_2 \cdots \tau_n, Z_1\gamma_1X_0)) \wedge \mu_{\vdash_{\mathscr{M}'}}(((q_1, c_0, d_0), \tau_2 \cdots \tau_n, Z_1\gamma_1X_0), ((q_2, c_0 \wedge c_1, d_0 \vee d_1), \tau_3 \cdots \tau_n, Z_2\gamma_2X_0)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}'}}(((q_{n-1}, c_0 \wedge c_1 \wedge \cdots \wedge c_{n-2}, d_0 \vee d_1 \vee \cdots \vee d_{n-2}), \tau_n, Z_{n-1}\gamma_{n-1}X_0), ((q_n, a_n, b_n), \varepsilon, X_0)) > 0$.

Hence $a_n = \mu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \wedge \mu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \varepsilon)) > 0$ and $b_n = \nu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \vee \nu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \vee \cdots \vee \nu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \varepsilon)) < 1$.

It contradicts with the assumption. So $\mu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q_n, a_n, b_n), \varepsilon, X_0) = 0$ if (P2) is not satisfied. In a similar way, it is easily concluded that $\nu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q_n, a_n, b_n), \varepsilon, X_0) = 1$ if (P2) is not satisfied.

Secondly, for any $\omega = \tau_1 \cdots \tau_n \in \Sigma^*$, $\tau_i \in \Sigma \cup \{\varepsilon\}$, $i \in \{1, \ldots, n\}$, $\mu_{\mathscr{L}(\mathscr{M}')}(\omega) = \bigvee\{\mu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q_n, a_n, b_n), \varepsilon, X_0)) \wedge \mu_{\vdash_{\mathscr{M}'}}((q_n, a_n, b_n), \varepsilon, X_0), ((q_n, a_n, b_n), \varepsilon, \varepsilon)) \mid (q_n, a_n, b_n) \in Q'\} = \bigvee\{\mu_{\vdash_{\mathscr{M}}}((q_0, \omega, Z_0), (q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1)) \wedge \mu_{\vdash_{\mathscr{M}}}((q_1, \tau_2 \cdots \tau_n, Z_1\gamma_1), (q_2, \tau_3 \cdots \tau_n, Z_2\gamma_2)) \wedge \cdots \wedge \mu_{\vdash_{\mathscr{M}}}((q_{n-1}, \tau_n, Z_{n-1}\gamma_{n-1}), (q_n, \varepsilon, \varepsilon)) \mid q_1, \ldots, q_n \in Q, Z_1, \ldots, Z_{n-1} \in \Gamma, \gamma_1, \ldots, \gamma_{n-1} \in \Gamma^*\} = \mu_{\mathscr{L}(\mathscr{M})}(\omega)$.

Similarly, $\nu_{\mathscr{L}(\mathscr{M}')}(\omega) = \nu_{\mathscr{L}(\mathscr{M})}(\omega)$.

Hence $\mathscr{L}(\mathscr{M}') = \mathscr{L}(\mathscr{M})$. □

Remark 21. Proposition 20 presents an equivalence of an IFPDA$^\emptyset$. In particular, due to the underlying truth-valued domain being an IFS, the proof technique used in Proposition 20 is to some extent different from the technique of extended subset construction in [33]. Moreover, Proposition 20 plays an important role in proving the fact that any language recognized by an IFPDA$^\emptyset$ is an intuitionistic fuzzy recognizable step function.

Proposition 22. *Let $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \emptyset)$ be an IFPDA$^\emptyset$. Then the language recognized by $\mathscr{M}$ is an intuitionistic fuzzy recognizable step function over Σ^*, that is, $\mathscr{L}(\mathscr{M}) \in Step^C(\Sigma)$.*

Proof. Let $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \emptyset)$ be an IFPDA$^\emptyset$. Then there is a special IFPDA$^\emptyset$ $\mathscr{M}' = (Q', \Sigma, \Gamma', \delta', q'_0, X_0, \emptyset)$ constructed by Proposition 20, which is equivalent to $\mathscr{M}$. For any $(a, b) \in (L_1 \setminus \{0\}) \times (L_2 \setminus \{1\})$ with $0 < a + b \leq 1$, construct a classical PDA with empty stack $\mathscr{M}_{ab} = (Q', \Sigma, \Gamma', \delta''_{ab}, q'_0, X_0, \emptyset)$, where $Q', \Sigma, \Gamma', q'_0$, and X_0 are the same as those in $\mathscr{M}'$, and the function $\delta''_{ab} : Q' \times \Sigma \times \Gamma' \to 2^{Q' \times \Gamma'^*}$ is defined by

(i) $\delta''_{ab}((p_0, 1, 0), \varepsilon, X_0) = \{((q_0, 1, 0), Z_0X_0)\}$;

(ii) $\delta''_{ab}((q, c, d), \tau, X) = \{((q', c \wedge c_1, d \vee d_1), \gamma) \mid c_1 = \mu_\delta(q, \tau, X, q', \gamma) > 0, d_1 = \nu_\delta(q, \tau, X, q', \gamma) < 1, q' \in Q, \gamma \in \Gamma^*\}$;

(iii) $\delta''_{ab}((q, a, b), \varepsilon, X_0) = \{((q, a, b), \varepsilon)\}$;

(iv) $\delta''_{ab}((q, c, d), \tau, X) = \emptyset$ in other cases.

Then for any $\omega \in \Sigma^*$, we have $\mu_{\mathscr{L}(\mathscr{M})}(\omega) = \mu_{\mathscr{L}(\mathscr{M}')}(\omega) = \bigvee\{\mu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q_n, a_n, b_n), \varepsilon, X_0)) \wedge \mu_{\vdash_{\mathscr{M}'}}((q_n, a_n, b_n), \varepsilon, X_0), ((q_n, a_n, b_n), \varepsilon, \varepsilon)) \mid (q_n, a_n, b_n) \in Q'\} = \bigvee\{a_n \mid (q_n, a_n, b_n) \in Q', 0 < a_n + b_n \leq 1, \mu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q_n, a_n, b_n), \varepsilon, X_0)) = 1\} = \bigvee\{a_n \mid (q'_0, \omega, X_0) \succ^*_{\mathscr{M}_{ab}} ((q_n, a_n, b_n), \varepsilon, \varepsilon), q_n \in Q\} = \bigvee_{a \in L_1 \setminus \{0\}} a \wedge \mu_{1_{\mathscr{L}(\mathscr{M}_{ab})}}(\omega)$, and $\nu_{\mathscr{L}(\mathscr{M})}(\omega) = \nu_{\mathscr{L}(\mathscr{M}')}(\omega) = \bigwedge\{\nu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q, a, b), \varepsilon, X_0)) \vee \nu_{\vdash_{\mathscr{M}'}}((q, a, b), \varepsilon, X_0), ((q, a, b), \varepsilon, \varepsilon)) \mid (q, a, b) \in Q'\} = \bigwedge\{b \mid (q, a, b) \in Q', 0 < a + b \leq 1, \mu_{\vdash^*_{\mathscr{M}'}}((q'_0, \omega, X_0), ((q, a, b), \varepsilon, X_0)) = 1\} = \bigwedge\{b \mid (q'_0, \omega, X_0) \succ^*_{\mathscr{M}_{ab}} ((q, a, b), \varepsilon, X_0), (q, a, b) \in Q'\} = \bigwedge_{b \in L_2 \setminus \{1\}} b \vee \nu_{1_{\mathscr{L}(\mathscr{M}_{ab})}}(\omega)$.

Therefore $\mathscr{L}(\mathscr{M}) \in \mathrm{Step}^C(\Sigma)$. □

Theorem 23. *Let A be an IFS over Σ^*. Then the following statements are equivalent:*

(1) *$A \in Step^C(\Sigma)$;*

(2) *A is accepted by a certain IFPDA$^\emptyset$ $\mathscr{M}$.*

Proof. (i) implies (ii). Suppose $A \in \mathrm{Step}^C(\Sigma)$. Then there are a finite natural number $n \in N$, recognizable context-free languages $\mathscr{L}_1, \ldots, \mathscr{L}_n \subseteq \Sigma^*$, and $(a_i, b_i) \in (0, 1] \times [0, 1)$ with $0 \leq a_i + b_i \leq 1$ for $i = 1, \ldots, n$ such that $A = (\mu_A, \nu_A) = \coprod_{i=1}^n (a_i, b_i) \cdot \mathbf{1}_{\mathscr{L}_i}$. Let $\mathscr{L}_i$ be recognized by PDA $\mathscr{M}_i = (Q_i, \Sigma, \Gamma_i, \delta_i, q_{0i}, Z_{0i}, \emptyset)$, and $Q_i \cap Q_j = \emptyset$ whenever $i \neq j$, $i, j \in N_n$.

Next, construct an IFPDA$^\emptyset$ $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, \emptyset)$ as follows: $Q = \bigcup_{i=1}^n Q_i$, $\Gamma = \bigcup_{i=1}^n \Gamma_i \cup \{Z_0\}$, and $I = (\mu_I, \nu_I)$ are an IFS over Q, defined by $\mu_I(q_{0i}) = a_i$, $\nu_I(q_{0i}) = b_i$, $i \in N_n$; otherwise $\mu_I(q) = 0$ and $\nu_I(q) = 1$. $\delta = (\mu_\delta, \nu_\delta)$ is an IFS over $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$, where the mappings $\mu_\delta, \nu_\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^* \to \{0, 1\}$ are defined by $\mu_\delta(q_{0i}, \varepsilon, Z_0, q_{0i}, Z_{0i}) = 1$ and $\nu_\delta(q_{0i}, \varepsilon, Z_0, q_{0i}, Z_{0i}) = 0$, $i \in N_n$; $\mu_\delta(q, \tau, Z, p, \gamma) = 1$ and $\nu_\delta(q, \tau, Z, p, \gamma) = 0$ if $p, q \in Q_i$, $Z \in \Gamma_i, \gamma \in \Gamma_i^*$, $\tau \in \Sigma$, $i \in N_n$ and $(p, \gamma) \in \delta(q, \tau, Z)$. Otherwise, $\mu_\delta(q, \tau, Z, p, \gamma) = 0$ and $\nu_\delta(q, \tau, Z, p, \gamma) = 1$.

Then $A = \mathscr{L}(\mathscr{M}) = \coprod_{i=1}^n (a_i, b_i) \cdot \mathbf{1}_{\mathscr{L}_i}$.

(ii) implies (i). It is concluded by Proposition 22. □

One can see that IFPDAs and IFPDAs$^\emptyset$ are equivalent to a type of intuitionistic fuzzy recognizable step functions over a set, respectively, by Theorems 19 and 23. Therefore, IFPDAs and IFPDAs$^\emptyset$ are equivalent in the sense that they accept or recognize the same classes of intuitionistic fuzzy languages. That is to say, the following statement is true.

Corollary 24. *For any IFPDA $\mathscr{M}$, there is an IFPDA$^\emptyset$ $\mathscr{M}'$ equivalent to $\mathscr{M}$. For any IFPDA$^\emptyset$ $\mathscr{M}'$, there is an IFPDA $\mathscr{M}$ equivalent to $\mathscr{M}'$.*

4. Intuitionistic Fuzzy Context-Free Grammars

As a type of generator of intuitionistic fuzzy context-free languages, the notion of intuitionistic fuzzy context-free grammars is introduced in the section. Then the relationship between intuitionistic fuzzy context-free grammars, IFPDAs, IFPDAs$^{\emptyset}$, and intuitionistic fuzzy recognizable step functions is discussed. The algebraic properties of intuitionistic fuzzy context-free languages are investigated finally.

Definition 25. An intuitionistic fuzzy grammar is a system $G = (N, T, P, I)$, where

(i) N is a finite nonempty alphabet of variables;

(ii) T is a finite nonempty alphabet of terminals and $T \cap N = \emptyset$;

(iii) I is an intuitionistic fuzzy set over N;

(iv) P is a finite collection of productions over $T \cup N$, and $P = \{x \to y \mid x \in (N \cup T)^* N (N \cup T)^*, y \in (N \cup T)^*, \mu_\rho(x \to y) > 0, \nu_\rho(x \to y) < 1\}$,

where $\rho = (\mu_\rho, \nu_\rho)$ is an IFS over $(N \cup T)^* \times (N \cup T)^*$, $\mu_\rho(x, y)$, and $\nu_\rho(x, y)$ mean the membership degree and the nonmembership degree that x will be replaced by y, respectively, denoted by $\mu_\rho(x, y) = \mu_\rho(x \to y)$, $\nu_\rho(x, y) = \nu_\rho(x \to y)$.

For $\alpha, \beta \in (N \cup T)^*$, if $x \to y \in P$, then $\alpha y \beta$ is said to be directly derivable from $\alpha x \beta$, denoted by $\alpha x \beta \Rightarrow \alpha y \beta$, and define $\mu_\rho(\alpha x \beta \Rightarrow \alpha y \beta) = \mu_\rho(x \to y)$, $\nu_\rho(\alpha x \beta \Rightarrow \alpha y \beta) = \nu_\rho(x \to y)$.

If $\alpha_1, \ldots, \alpha_m$ are strings in $(N \cup T)^*$ and $\alpha_1 \to \alpha_2, \ldots, \alpha_{m-1} \to \alpha_m \in P$, then α_1 is said to derive α_m in G, or, equivalently, α_m is derivable from α_1 in G. This is expressed by $\alpha_1 \Rightarrow_G^* \alpha_m$ or simply $\alpha_1 \Rightarrow^* \alpha_m$. The expression $\alpha_1 \to \alpha_2 \to \cdots \to \alpha_m$ is referred to as a derivation chain from α_1 to α_m.

An intuitionistic fuzzy grammar G generates an intuitionistic fuzzy language $\mathscr{L}(G) = (\mu_G, \nu_G)$ in the following manner. For any $\theta = \omega_n \in T^*$, $n \geq 1$, $\mu_G(\theta) = \bigvee\{\mu_I(\omega_0) \wedge \mu_\rho(\omega_0 \Rightarrow \omega_1) \wedge \cdots \wedge \mu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid \omega_0 \in N, \omega_1, \ldots, \omega_{n-1} \in (N \cup T)^*\}$, and $\nu_G(\theta) = \bigwedge\{\nu_I(\omega_0) \vee \nu_\rho(\omega_0 \Rightarrow \omega_1) \vee \cdots \vee \nu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid \omega_0 \in N, \omega_1, \ldots, \omega_{n-1} \in (N \cup T)^*\}$.

$\mu_G(\theta)$ and $\nu_G(\theta)$ express the membership and nonmembership degree of θ in the language generated by grammar G, respectively. Obviously, $\mathscr{L}(G) = (\mu_G, \nu_G)$ is well defined. In fact, for any $\theta = \omega_n \in T^*$, $n \geq 1$, there is the strongest derivation from ω_0 to ω_n, that is, $\omega_0 \Rightarrow \omega_1' \Rightarrow \cdots \Rightarrow \omega_{n-1}' \Rightarrow \omega_n$, such that $\mu_G(\theta) = \mu_I(\omega_0) \wedge \mu_\rho(\omega_0 \Rightarrow \omega_1') \wedge \cdots \wedge \mu_\rho(\omega_{n-1}' \Rightarrow \omega_n)$. So $\nu_G(\theta) \leq \nu_I(\omega_0) \vee \nu_\rho(\omega_0 \Rightarrow \omega_1') \vee \cdots \vee \nu_\rho(\omega_{n-1}' \Rightarrow \omega_n)$, and $\mu_G(\theta) + \nu_G(\theta) \leq \mu_G(\theta) + \nu_I(\omega_0) \vee \nu_\rho(\omega_0 \Rightarrow \omega_1') \vee \cdots \vee \nu_\rho(\omega_{n-1}' \Rightarrow \omega_n) = (\mu_G(\theta) + \nu_I(\omega_0)) \vee (\mu_G(\theta) + \nu_\rho(\omega_0 \Rightarrow \omega_1')) \vee \cdots \vee (\mu_G(\theta) + \nu_\rho(\omega_{n-1}' \Rightarrow \omega_n)) \leq (\mu_I(\omega_0) + \nu_I(\omega_0)) \vee (\mu_\rho(\omega_0 \Rightarrow \omega_1') + \nu_\rho(\omega_0 \Rightarrow \omega_1')) \vee \cdots \vee (\mu_\rho(\omega_{n-1}' \Rightarrow \omega_n) + \nu_\rho(\omega_{n-1}' \Rightarrow \omega_n)) \leq 1$.

For any intuitionistic fuzzy grammars G_1 and G_2, if $\mathscr{L}(G_1) = \mathscr{L}(G_2)$ in the sense of equality of intuitionistic fuzzy sets, then the grammars G_1 and G_2 are said to be equivalent.

For any intuitionistic fuzzy grammar $G = (N, T, P, I)$, if $\mathrm{Im}(I) = \mathrm{Im}(\mu_I) \cup \mathrm{Im}(\nu_I) = \{0, 1\}$ and $\mathrm{supp}(I) = \{S\}$, then G is also written as $G = (N, T, P, S)$.

Proposition 26. *Let A be an IFS over T^*. Then the following statements are equivalent:*

(i) *A is generated by a certain intuitionistic fuzzy grammar $G = (N, T, P, I)$;*

(ii) *A is generated by a certain intuitionistic fuzzy grammar $G = (N', T', P', S)$.*

Proof. (i) implies (ii). Let A be generated by an intuitionistic fuzzy grammar $G = (N, T, P, I)$. Then we construct an intuitionistic fuzzy grammar $G' = (N', T', P', S)$ as follows: $N' = N \cup \{S\}$, $S \notin N$; $T' = T$, $P' = P \cup P_1$, where $P_1 = \{S \to q \mid q \in \mathrm{supp}(I), \mu_\rho(S \to q) = \mu_I(q), \nu_\rho(S \to q) = \nu_I(q)\}$.

Next we show that $\mathscr{L}(G') = \mathscr{L}(G)$. In fact, $G' = (N', T', P', I')$, where I' is an IFS over N', $\mu_{I'}(S) = 1$, $\nu_{I'}(S) = 0$; $\mu_{I'}(q) = 0$ and $\nu_{I'}(q) = 1$ when $q \in N$.

For any $\theta = \omega_n \in T^*$, $n \geq 1$, $\mu_{G'}(\theta) = \bigvee\{\mu_{I'}(\omega_0) \wedge \mu_\rho(\omega_0 \Rightarrow \omega_1) \wedge \cdots \wedge \mu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid \omega_0 \in N', \omega_1, \ldots, \omega_{n-1} \in (N' \cup T')^*\} = \bigvee\{\mu_\rho(S \Rightarrow \omega_1) \wedge \mu_\rho(\omega_1 \Rightarrow \omega_2) \wedge \cdots \wedge \mu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid \omega_1, \ldots, \omega_{n-1} \in (N' \cup T')^*\} = \bigvee\{\mu_\rho(S \Rightarrow q) \wedge \mu_\rho(q \Rightarrow \omega_2) \wedge \cdots \wedge \mu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid q \in N, \omega_2, \ldots, \omega_{n-1} \in (N \cup T)^*\} = \bigvee\{\mu_I(q) \wedge \mu_\rho(q \Rightarrow \omega_2) \wedge \cdots \wedge \mu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid q \in N, \omega_2, \ldots, \omega_{n-1} \in (N \cup T)^*\} = \mu_G(\theta)$ and $\nu_{G'}(\theta) = \bigwedge\{\nu_{I'}(\omega_0) \vee \nu_\rho(\omega_0 \Rightarrow \omega_1) \vee \cdots \vee \nu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid \omega_0 \in N', \omega_1, \ldots, \omega_{n-1} \in (N' \cup T')^*\} = \bigwedge\{\nu_\rho(S \Rightarrow \omega_1) \vee \nu_\rho(\omega_1 \Rightarrow \omega_2) \vee \cdots \vee \nu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid \omega_1, \ldots, \omega_{n-1} \in (N' \cup T')^*\} = \bigwedge\{\nu_\rho(S \Rightarrow q) \vee \nu_\rho(q \Rightarrow \omega_2) \vee \cdots \vee \nu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid q \in N, \omega_2, \ldots, \omega_{n-1} \in (N \cup T)^*\} = \bigwedge\{\nu_I(q) \vee \nu_\rho(q \Rightarrow \omega_2) \vee \cdots \vee \nu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid q \in N, \omega_2, \ldots, \omega_{n-1} \in (N \cup T)^*\} = \nu_G(\theta)$.

Hence $\mathscr{L}(G') = \mathscr{L}(G)$.

(ii) implies (i), obviously. □

Definition 27. (1) An intuitionistic fuzzy grammar $G = (N, T, P, I)$ is called context-free (IFCFG, for short) if it has only productions of the form $A \to \omega \in P$ with $A \in N$ and $\omega \in (N \cup T)^*$. And the language $\mathscr{L}(G)$, generated by the IFCFG G, is said to be an intuitionistic fuzzy context-free language (IFCFL).

(2) An IFCFG $G = (N, T, P, S)$ is called an intuitionistic fuzzy Chomsky normal form (IFCNF) if it has only productions of the form $A \to BC \in P$ or $A \to a \in P$ or $S \to \varepsilon$, where $A, B, C \in N$, $B \neq S$, $C \neq S$ and $a \in T$. (3) An IFCFG $G = (N, T, P, S)$ is called an intuitionistic fuzzy Greibach normal form (IFGNF) if all the productions are of the form $A \to ax \in P$ or $S \to \varepsilon$, where $A \in N$, $a \in T$, and $x \in (N \setminus \{S\})^*$. (4) An IFCFG $G = (N, T, P, I)$ is called a simple-typed intuitionistic fuzzy context-free grammar (IFSCFG) if

$$P = \left\{A \longrightarrow x \mid A \in N, x \in (N \cup T)^*, \mu_\rho (A \longrightarrow x) = 1\right\}. \tag{25}$$

Proposition 28. *Let $G = (N, T, P, I)$ be an IFCFG and $\mathscr{L}(G) = (\mu_G, \nu_G)$ be the intuitionistic fuzzy context-free language generated by G. Then the image set of $\mathscr{L}(G)$ is a finite subset of the unit interval* $[0, 1]$.

Proof. Let $G = (N, T, P, I)$ be an IFCFG. Then $\mathrm{Im}(\mu_\rho)$ and $\mathrm{Im}(\nu_\rho)$ are finite. For any $\theta = \omega_n \in T^*$, $n \geq 1$, there exist natural numbers d, l such that $\mu_G(\theta) = \bigvee\{\mu_I(S) \wedge \mu_\rho(S \Rightarrow \omega_1) \wedge \mu_\rho(\omega_1 \Rightarrow \omega_2) \wedge \cdots \wedge \mu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid S \in N, \omega_1, \ldots, \omega_{n-1} \in (N \cup T)^*\} = (a_{10} \wedge \cdots \wedge a_{1n-1} \wedge a_{1n}) \vee \cdots \vee (a_{d0} \wedge \cdots \wedge a_{dn-1} \wedge a_{dn})$ and $\nu_G(\theta) = \bigwedge\{\nu_I(S) \vee \nu_\rho(S \Rightarrow \omega_1) \vee \nu_\rho(\omega_1 \Rightarrow \omega_2) \vee \cdots \vee \nu_\rho(\omega_{n-1} \Rightarrow \omega_n) \mid S \in N, \omega_1, \ldots, \omega_{n-1} \in (N \cup T)^*\} = (b_{10} \vee \cdots \vee b_{1n-1} \vee b_{1n}) \wedge \cdots \wedge (b_{l0} \vee \cdots \vee b_{ln-1} \vee b_{ln})$, where $a_{ik} \in \mathrm{Im}(\mu_I) \cup \mathrm{Im}(\mu_\rho)$, $b_{jk} \in \mathrm{Im}(\nu_I) \cup \mathrm{Im}(\nu_\rho)$, $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, l\}$, $k \in \{0, 1, \ldots, n\}$. Let $X = \mathrm{Im}(\mu_I) \cup \mathrm{Im}(\mu_\rho)$ and $Y = \mathrm{Im}(\nu_I) \cup \mathrm{Im}(\nu_\rho)$. Then X and Y are finite subsets of the interval $[0, 1]$. $(X_\wedge)_\vee$ and $(Y_\vee)_\wedge$ are also finite by Lemma 8. Since $\mu_G(\theta) \in (X_\wedge)_\vee$ and $\nu_G(\theta) \in (Y_\vee)_\wedge$ for any $\theta \in T^*$, we have $\mathrm{Im}(\mu_G) \subseteq (X_\wedge)_\vee$ and $\mathrm{Im}(\nu_G) \subseteq (Y_\vee)_\wedge$. Hence $\mathrm{Im}(\mathscr{L}(G)) = \mathrm{Im}(\mu_G) \cup \mathrm{Im}(\nu_G)$ is finite. □

Proposition 29. *Let $G = (N, T, P, S)$ be an IFCFG. Then $\mathscr{L}(G) \in Step^C(T)$.*

Proof. Let $X = \mathrm{Im}(\mu_\rho)$ and $Y = \mathrm{Im}(\nu_\rho)$. Then X and Y have finite elements because P is a finite collection of productions over $T \cup N$. Suppose $L_1 = X_\wedge$ and $L_2 = Y_\vee$. Then L_1 and L_2 are finite by Lemma 8. For any $(a, b) \in (L_1 \setminus \{0\}) \times (L_2 \setminus \{1\})$, $0 \le a + b \le 1$, we construct a classical context-free grammar $G_{ab} = (N', T, P'_{ab}, S')$ as follows:

$N' = N \times (L_1 \setminus \{0\}) \times (L_2 \setminus \{1\})$, $S' = (S, 1, 0) \in N'$, P'_{ab} consists of the form:

(1) $(A, a_1, b_1) \rightarrow D_1 \cdots D_k$ whenever $\mu_\rho(A \rightarrow \tau_1 \cdots \tau_k) > 0$ and $\nu_\rho(A \rightarrow \tau_1 \cdots \tau_k) < 1$, where

$$D_i = \begin{cases} (\tau_i, a_2, b_2), & \text{if } \tau_i \in N \\ \tau_i, & \text{if } \tau_i \in T \end{cases} \quad (26)$$

$i = 1, \ldots, k$; $a_2 = a_1 \wedge \mu_\rho(A \rightarrow \tau_1 \cdots \tau_k)$ and $b_2 = b_1 \vee \nu_\rho(A \rightarrow \tau_1 \cdots \tau_k)$;

(2) $(A, a_1, b_1) \rightarrow x$ whenever $a \le a_1 \wedge \mu_\rho(A \rightarrow x)$ and $b \ge b_1 \vee \nu_\rho(A \rightarrow x)$, for all $A \in N, x \in (N \cup T)^*$.

Then $\mathscr{L}(G_{ab}) = \{\omega \in T^* \mid S' \Rightarrow^*_{G_{ab}} \omega\} = \{\omega \in T^* \mid a \le \mu_\rho(S \Rightarrow u_1) \wedge \mu_\rho(u_1 \Rightarrow u_2) \wedge \cdots \wedge \mu_\rho(u_{n-1} \Rightarrow \omega), b \ge \nu_\rho(S \Rightarrow u_1) \vee \nu_\rho(u_1 \Rightarrow u_2) \vee \cdots \vee \nu_\rho(u_{n-1} \Rightarrow \omega),\ u_1, \ldots, u_{n-1} \in (N \cup T)^*\}$.

Next it suffices to show that $\mathscr{L}(G) = \coprod (a, b) \cdot \mathbf{1}_{\mathscr{L}(G_{ab})}$, that is, $\mu_G(\omega) = \bigvee_{a \in L_1 \setminus \{0\}} a \wedge \mu_{1_{\mathscr{L}(G_{ab})}}(\omega)$ and $\nu_G(\omega) = \bigwedge_{b \in L_2 \setminus \{1\}} b \vee \nu_{1_{\mathscr{L}(G_{ab})}}(\omega)$, for all $\omega \in T^*$.

Suppose $\mu_G(\omega) = a_k > 0$. Then there exist $u'_1, \ldots, u'_{n-1} \in (N \cup T)^*$ such that $a_k = \mu_\rho(S \Rightarrow u'_1) \wedge \mu_\rho(u'_1 \Rightarrow u'_2) \wedge \cdots \wedge \mu_\rho(u'_{n-1} \Rightarrow \omega)$. Put $c = \nu_\rho(S \Rightarrow u'_1) \vee \nu_\rho(u'_1 \Rightarrow u'_2) \vee \cdots \vee \nu_\rho(u'_{n-1} \Rightarrow \omega)$. Then $c < 1$ and $\omega \in \mathscr{L}(G_{kc})$. Therefore, $\mu_G(\omega) \le \bigvee_{a \in L_1 \setminus \{0\}} a \wedge \mu_{1_{\mathscr{L}(G_{ab})}}(\omega)$ and $\nu_G(\omega) \ge \bigwedge_{b \in L_2 \setminus \{1\}} b \vee \nu_{1_{\mathscr{L}(G_{ab})}}(\omega)$ for any $\omega \in T^*$. In addition, $\bigvee_{a \in L_1 \setminus \{0\}} a \wedge \mu_{1_{\mathscr{L}(G_{ab})}}(\omega) = \bigvee\{a \in L_1 \setminus \{0\} \mid \omega \in \mathscr{L}(G_{ab})\} \le \bigvee\{\mu_\rho(S \Rightarrow u_1) \wedge \mu_\rho(u_1 \Rightarrow u_2) \wedge \cdots \wedge \mu_\rho(u_{n-1} \Rightarrow \omega) \mid \omega \in \mathscr{L}(G_{ab}),\ u_1, \ldots, u_{n-1} \in (N \cup T)^*\} \le \bigvee\{\mu_\rho(S \Rightarrow u_1) \wedge \mu_\rho(u_1 \Rightarrow u_2) \wedge \cdots \wedge \mu_\rho(u_{n-1} \Rightarrow \omega) \mid u_1, \ldots, u_{n-1} \in (N \cup T)^*\} = \mu_G(\omega)$, $\nu_G(\omega) = \bigwedge\{\nu_\rho(S \Rightarrow u_1) \vee \nu_\rho(u_1 \Rightarrow u_2) \vee \cdots \vee \nu_\rho(u_{n-1} \Rightarrow \omega) \mid u_1, \ldots, u_{n-1} \in (N \cup T)^*\} \le \bigwedge\{b \in L_2 \setminus \{1\} \mid b \ge \nu_\rho(S \Rightarrow u_1) \vee \nu_\rho(u_1 \Rightarrow u_2) \vee \cdots \vee \nu_\rho(u_{n-1} \Rightarrow \omega), u_1, \ldots, u_{n-1} \in (N \cup T)^*\} \le \bigwedge\{b \in L_2 \setminus \{1\} \mid b \ge \nu_\rho(S \Rightarrow u_1) \vee \nu_\rho(u_1 \Rightarrow u_2) \vee \cdots \vee \nu_\rho(u_{n-1} \Rightarrow \omega), a \le \mu_\rho(S \Rightarrow u_1) \wedge \mu_\rho(u_1 \Rightarrow u_2) \wedge \cdots \wedge \mu_\rho(u_{n-1} \Rightarrow \omega), u_1, \ldots, u_{n-1} \in (N \cup T)^*\} = \bigwedge\{b \in L_2 \setminus \{1\} \mid \omega \in \mathscr{L}(G_{ab})\} = \bigwedge_{b \in L_2 \setminus \{1\}} b \vee \nu_{1_{\mathscr{L}(G_{ab})}}(\omega)$.

It is concluded that $\mu_G(\omega) = \bigvee_{a \in L_1 \setminus \{0\}} a \wedge \mu_{1_{\mathscr{L}(G_{ab})}}(\omega)$ and $\nu_G(\omega) = \bigwedge_{b \in L_2 \setminus \{1\}} b \vee \nu_{1_{\mathscr{L}(G_{ab})}}(\omega)$, for all $\omega \in T^*$. □

Proposition 29 states that any language recognized by an IFCFG is an intuitionistic fuzzy recognizable step function, where the proof technique is constructive. Next we will show that the set consisting of all the intuitionistic fuzzy recognizable step functions coincides with that of languages recognized by IFCFGs.

Theorem 30. *Let A be an IFS over T^*. Then the following statements are equivalent:*

(1) *$A \in Step^C(T)$;*

(2) *There is an IFCFG G such that $A = \mathscr{L}(G)$;*

(3) *There is an IFSCFG G such that $A = \mathscr{L}(G)$.*

Proof. (1) implies (3). Let $A = \coprod_{i=1}^k (a_i, b_i) \cdot 1_{\mathscr{L}_i}$, where $a_i, b_i \in [0, 1]$, $a_i + b_i \le 1$, $i \in N_k$, and $\mathscr{L}_1, \ldots, \mathscr{L}_k \subset T^*$ are classical context-free languages. Suppose $\mathscr{L}_i$ is generated by a context-free grammar $G_i = (N_i, T, P_i, S_{0i})$ and $N_i \cap N_j = \emptyset$ whenever $i \ne j$. Then we construct an IFSCFG $G = (N, T, P, I)$ as follows: $N = \bigcup_{i=1}^k N_i$, $P = \bigcup_{i=1}^k P_i$, $I = (\mu_I, \nu_I)$ is an IFS over N, where the mappings $\mu_I, \nu_I : N \rightarrow [0, 1]$ are defined by $\mu_I(S_{0i}) = a_i$, $\nu_I(S_{0i}) = b_i$; $\mu_I(q) = 0$ and $\nu_I(q) = 1$ whenever $q \in N \setminus \{S_{0i} \mid i = 1, \ldots, k\}$.

Next, we show $A = \mathscr{L}(G) = \coprod_{i=1}^k (a_i, b_i) \cdot 1_{\mathscr{L}_i}$. In fact, for any $\omega = u_n \in T^*$, $\mu_G(\omega) = \bigvee\{\mu_I(S) \wedge \mu_\rho(S \Rightarrow u_1) \wedge \mu_\rho(u_1 \Rightarrow u_2) \wedge \cdots \wedge \mu_\rho(u_{n-1} \Rightarrow \omega) \mid S \in N, u_1, \ldots, u_{n-1} \in (N \cup T)^*\} = \bigvee\{\mu_I(S_{0i}) \mid S_{0i} \Rightarrow^*_{G_i} \omega, S_{0i} \in N, i \in N_k\} = \bigvee\{a_i \mid \omega \in \mathscr{L}_i, i \in N_k\} = \bigvee_{i=1}^k a_i \wedge \mu_{1_{\mathscr{L}_i}}(\omega)$ and $\nu_G(\omega) = \bigwedge\{\nu_I(S) \vee \nu_\rho(S \Rightarrow u_1) \vee \nu_\rho(u_1 \Rightarrow u_2) \vee \cdots \vee \nu_\rho(u_{n-1} \Rightarrow \omega) \mid S \in N, u_1, \ldots, u_{n-1} \in (N \cup T)^*\} = \bigwedge\{\nu_I(S_{0i}) \mid S_{0i} \Rightarrow^*_{G_i} \omega, S_{0i} \in N, i \in N_k\} = \bigwedge\{b_i \mid \omega \in \mathscr{L}_i, i \in N_k\} = \bigwedge_{i=1}^k b_i \vee \nu_{1_{\mathscr{L}_i}}(\omega)$.

Hence $\mathscr{L}(G) = (\mu_G, \nu_G) = \coprod_{i=1}^k (a_i, b_i) \cdot 1_{\mathscr{L}_i} = A$.

(3) implies (2). It is true obviously since an IFSCFG is a special IFCFG by Definition 27.

(2) implies (1). It is straightforward by Propositions 26 and 29. □

Theorem 31. *Let A be an IFS over Σ^*. Then the following statements are equivalent:*

(1) *$A \in Step^C(\Sigma)$;*

(2) *there is an IFPDA $\mathscr{M}$ such that $A = \mathscr{L}(\mathscr{M})$;*

(3) *there is an IFPDA$^\emptyset$ $\mathscr{M}$ such that $A = \mathscr{L}(\mathscr{M})$;*

(4) *there is an IFCFG G such that $A = \mathscr{L}(G)$;*

(5) *there is an IFCNF G such that $A = \mathscr{L}(G)$;*

(6) *there is an IFGNF G such that $A = \mathscr{L}(G)$.*

Proof. (1) implies (5). Let $A \in \mathrm{Step}^C(\Sigma)$. Then suppose $A = \coprod_{i=1}^{k}(a_i, b_i)\cdot 1_{\mathscr{L}_i}$, where $a_i, b_i \in [0,1]$, $a_i + b_i \le 1$, $i \in N_k$, and $\mathscr{L}_1, \ldots, \mathscr{L}_k \subset \Sigma^*$ are classical context-free languages.

If $\mathrm{supp}(A) = \{\varepsilon\}$, then we construct an IFCNF $G = (N, \Sigma, P, S)$ as follows: $N = \{S\}$, $P = \{S \to \varepsilon \mid \mu_\rho(S \to \varepsilon) = \mu_A(\varepsilon), \nu_\rho(S \to \varepsilon) = \mu_A(\varepsilon)\}$. Clearly, $\mathscr{L}(G) = A$.

If $\mathrm{supp}(A)\setminus\{\varepsilon\} \neq \emptyset$, then there is a Chomsky normal form grammar $G_i = (N_i, \Sigma, P_i, S_{0i})$ such that $\mathscr{L}(G_i) = \mathscr{L}_i \setminus \{\varepsilon\}$, for any $\mathscr{L}_i$ with $\mathscr{L}_i \setminus \{\varepsilon\} \neq \emptyset$, $i = 1, \ldots, k$. Let $N_i \cap N_j = \emptyset$ whenever $i \neq j$. Then we construct an IFSCFG $G = (N, \Sigma, P, I)$ according to the method constructed by Theorem 30, where $N = \bigcup_{i=1}^{k} N_i$, $P = \bigcup_{i=1}^{k} P_i$, $I = (\mu_I, \nu_I)$ is an IFS over N, and the mappings $\mu_I, \nu_I : N \to [0,1]$ are defined by $\mu_I(S_{0i}) = a_i$, $\nu_I(S_{0i}) = b_i$; $\mu_I(q) = 0$ and $\nu_I(q) = 1$ whenever $q \in N \setminus \{S_{0i} \mid i = 1, \ldots, k\}$. Next, we construct an IFCNF $G' = (N', \Sigma, P', S)$ as follows:

(i) $N' = N \cup \{S\}$, $S \notin N$;

(ii) $P' = P \cup P''$, where P'' has the productions in the form of

$(E1)$ $S \to \varepsilon$ with $\mu_\rho(S \to \varepsilon) = \mu_A(\varepsilon), \nu_\rho(S \to \varepsilon) = \mu_A(\varepsilon)$ whenever $\varepsilon \in \mathrm{supp}(A)$;

$(E2)$ $S \to BC$ with $\mu_\rho(S \to BC) = \mu_I(S_{0i})$ and $\nu_\rho(S \to BC) = \nu_I(S_{0i})$ whenever $S_{0i} \in \mathrm{supp}(I)$ and $S_{0i} \to BC \in P_i$, $i = 1, \ldots, k$;

$(E3)$ $S \to \alpha$ with $\mu_\rho(S \to \alpha) = \bigvee\{\mu_I(S_{0i}) \mid S_{0i} \in \mathrm{supp}(I), S_{0i} \to \alpha \in P_i, i = 1, \ldots, k\}$ and $\nu_\rho(S \to \alpha) = \bigwedge\{\nu_I(S_{0i}) \mid S_{0i} \in \mathrm{supp}(I), S_{0i} \to \alpha \in P_i, i = 1, \ldots, k\}$ whenever $S_{0i} \in \mathrm{supp}(I)$ and $S_{0i} \to \alpha \in P_i, i = 1, \ldots, k$.

Then we have $\mathscr{L}(G') = (\mu_{G'}, \nu_{G'}) = A$. In fact, for any $\omega \in \Sigma^*$, if $\omega = \varepsilon$, then $\mu_{G'}(\varepsilon) = \mu_\rho(S \to \varepsilon) = \mu_A(\varepsilon)$ and $\nu_{G'}(\varepsilon) = \nu_\rho(S \to \varepsilon) = \nu_A(\varepsilon)$; if $\omega \neq \varepsilon$, then $\mu_{G'}(\omega) = \bigvee\{\mu_\rho(S \Rightarrow_{G'} u_1) \wedge \mu_\rho(u_1 \Rightarrow_{G'} u_2) \wedge \cdots \wedge \mu_\rho(u_{n-1} \Rightarrow_{G'} \omega) \mid u_1, u_2, \ldots, u_{n-1} \in (N' \cup \Sigma)^*\} = \bigvee\{\mu_I(S_{0i}) \wedge \mu_\rho(S_{0i} \Rightarrow_{G_i} u_1) \wedge \mu_\rho(u_1 \Rightarrow_{G_i} u_2) \wedge \cdots \wedge \mu_\rho(u_{n-1} \Rightarrow_{G_i} \omega) \mid u_1, u_2, \ldots, u_{n-1} \in (N \cup \Sigma)^*, i \in N_k\} = \bigvee\{a_i \mid \omega \in \mathscr{L}(G_i), i \in N_k\} = \bigvee_{i=1}^{k} a_i \wedge \mu_{1_{\mathscr{L}_i}}(\omega)$ and $\nu_{G'}(\omega) = \bigwedge\{\nu_\rho(S \Rightarrow_{G'} u_1) \vee \nu_\rho(u_1 \Rightarrow_{G'} u_2) \vee \cdots \vee \nu_\rho(u_{n-1} \Rightarrow_{G'} \omega) \mid u_1, u_2, \ldots, u_{n-1} \in (N' \cup \Sigma)^*\} = \bigwedge\{\nu_I(S_{0i}) \vee \nu_\rho(S_{0i} \Rightarrow_{G_i} u_1) \vee \nu_\rho(u_1 \Rightarrow_{G_i} u_2) \vee \cdots \vee \nu_\rho(u_{n-1} \Rightarrow_{G_i} \omega) \mid u_1, u_2, \ldots, u_{n-1} \in (N \cup \Sigma)^*, i \in N_k\} = \bigwedge\{a_i \mid \omega \in \mathscr{L}(G_i), i \in N_k\} = \bigwedge_{i=1}^{k} a_i \vee \mu_{1_{\mathscr{L}_i}}(\omega)$.

(1) implies (6), similarly. The proof is omitted.

(5) implies (4), (6) implies (4), obviously, since IFCNF and IFGNF are special IFCFGs respectively.

(1), (2), (3), and (4) are equivalent mutually by Theorems 19, 23, and 30.

Theorem 31 states that IFCFLs, the set of intuitionistic fuzzy languages recognized by IFPDA and the set of intuitionistic fuzzy recognizable step functions, coincide with each other. Next we discuss some operations on the family of IFCFLs.

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs over Σ^*, $\lambda, \theta \in [0,1]$, and $0 \le \lambda + \theta \le 1$. Then the operations of union, scalar product, reversal, concatenation and Kleene closure are defined, respectively, by

(i) $A \cup B = (\mu_{A\cup B}, \nu_{A\cup B})$, $\mu_{A\cup B}(\omega) = \mu_A(\omega) \vee \mu_B(\omega)$, $\nu_{A\cup B}(\omega) = \nu_A(\omega) \wedge \nu_B(\omega)$;

(ii) $(\lambda, \theta)A = (\lambda \wedge \mu_A, \theta \vee \nu_A)$, $(\lambda \wedge \mu_A)(\omega) = \lambda \wedge \mu_A(\omega)$, $(\theta \vee \nu_A)(\omega) = \theta \vee \nu_A(\omega)$;

(iii) $A^{-1} = (\mu_{A^{-1}}, \nu_{A^{-1}})$, $\mu_{A^{-1}}(\omega) = \mu_A(\omega^{-1})$, $\nu_{A^{-1}}(\omega) = \nu_A(\omega^{-1})$;

(iv) $AB = (\mu_{AB}, \nu_{AB})$, $\mu_{AB}(\omega) = \bigvee\{\mu_A(\omega_1) \wedge \mu_B(\omega_2) \mid \omega_1\omega_2 = \omega\}$, $\nu_{AB}(\omega) = \bigwedge\{\nu_A(\omega_1) \vee \nu_B(\omega_2) \mid \omega_1\omega_2 = \omega\}$;

(v) $A^* = (\mu_{A^*}, \nu_{A^*})$, $\mu_{A^*}(\omega) = \bigvee\{\mu_A(\omega_1) \wedge \cdots \wedge \mu_A(\omega_k) : k \ge 1, \omega = \omega_1 \cdots \omega_k\}$, $\nu_{A^*}(\omega) = \bigwedge\{\nu_A(\omega_1) \vee \cdots \vee \nu_A(\omega_k) : k \ge 1, \omega = \omega_1 \cdots \omega_k\}$ for any $\omega \in \Sigma^*$, where ω^{-1} represents the reversal of ω, that is, if $\omega = \omega_1 \cdots \omega_k$, then $\omega^{-1} = \omega_k \cdots \omega_1$, for all $\omega \in \Sigma^*$. □

Theorem 32. (1) *The family $\mathrm{Step}^C(\Sigma)$ is closed under the operations of union, scalar product, reversal, concatenation, and Kleene closure. That is, $A \cup B$, $(\lambda, \theta)A$, A^{-1}, AB, $A^* \in \mathrm{Step}^C(\Sigma)$, for any $A, B \in \mathrm{Step}^C(\Sigma)$, $\lambda, \theta \in [0,1]$, $0 \le \lambda + \theta \le 1$.*

(2) Let $h : \Sigma_1^* \to \Sigma_2^*$ be a homomorphism. If $A \in \mathrm{Step}^C(\Sigma_2)$, then $h^{-1}(A) = A \circ h \in \mathrm{Step}^C(\Sigma_1)$. (3) Let $h : \Sigma_1^* \to \Sigma_2^*$ be a homomorphism. If h satisfies, for $\tau \in \Sigma$, $h(\tau) \neq \varepsilon$, and $g = (\mu_g, \nu_g) \in \mathrm{Step}^C(\Sigma_1)$, then $h(g) = (\mu_{h(g)}, \nu_{h(g)}) \in \mathrm{Step}^C(\Sigma_2)$, where $\mu_{h(g)}(\omega) = \bigvee\{\mu_g(\alpha) \mid h(\alpha) = \omega, \alpha \in \Sigma_1^*\}$, $\nu_{h(g)}(\omega) = \bigwedge\{\nu_g(\alpha) \mid h(\alpha) = \omega, \alpha \in \Sigma_1^*\}$ for any $\omega \in \Sigma_2^*$.

Proof. (1) Let $A, B \in \mathrm{Step}^C(\Sigma)$. By Definition 17, we can assume $A = (\mu_A, \nu_A) = \coprod_{i=1}^{k}(a_i, b_i)\cdot 1_{\mathscr{L}_i}$, $B = (\mu_B, \nu_B) = \coprod_{j=1}^{n}(c_j, d_j)\cdot 1_{\mathscr{M}_j}$, where all $\mathscr{L}_i$ and $\mathscr{M}_j$ are classical context-free languages, $0 \le a_i + b_i \le 1$, $0 \le c_j + d_j \le 1$, $a_i, b_i, c_j, d_j \in [0,1]$, $i \in N_k$, and $j \in N_n$. With respect to the union, we have $A \cup B \in \mathrm{Step}^C(\Sigma)$. That is, $A \cup B = (\mu_{A\cup B}, \nu_{A\cup B}) = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$, $\mu_{A\cup B}(\omega) = \mu_A(\omega) \vee \mu_B(\omega) = (\bigvee_{i=1}^{k} a_i \wedge \mu_{1_{\mathscr{L}_i}}(\omega)) \vee (\bigvee_{j=1}^{n} c_j \wedge \mu_{1_{\mathscr{M}_j}}(\omega))$, $\nu_{A\cup B}(\omega) = \nu_A(\omega) \wedge \nu_B(\omega) = (\bigwedge_{i=1}^{k} b_i \vee \nu_{1_{\mathscr{L}_i}}(\omega)) \wedge (\bigwedge_{j=1}^{n} d_j \vee \nu_{1_{\mathscr{M}_j}}(\omega))$, for all $\omega \in \Sigma^*$.

With respect to the scalar product, for each $(\lambda, \theta) \in [0,1]\times[0,1]$, $0 \le \lambda + \theta \le 1$, we have $(\lambda, \theta)A = (\lambda \wedge \mu_A, \theta \vee \nu_A)$, $(\lambda \wedge \mu_A)(\omega) = \lambda \wedge \mu_A(\omega) = \lambda \wedge (\bigvee_{i=1}^{k} a_i \wedge \mu_{1_{\mathscr{L}_i}}(\omega)) = \bigvee_{i=1}^{k}((\lambda \wedge a_i) \wedge \mu_{1_{\mathscr{L}_i}}(\omega))$, $(\theta \vee \nu_A)(\omega) = \theta \vee \nu_A(\omega) = \theta \vee (\bigwedge_{i=1}^{k} b_i \vee \nu_{1_{\mathscr{L}_i}}(\omega)) = \bigwedge_{i=1}^{k}(\theta \vee b_i) \vee \nu_{1_{\mathscr{L}_i}}(\omega)$, for all $\omega \in \Sigma^*$. By Definition 17, $(\lambda, \theta)A \in \mathrm{Step}^C(\Sigma)$. For the reversal operation, $\mathscr{L}_i^{-1} = \{\omega^{-1} \in \Sigma^* \mid \omega \in \mathscr{L}_i\}$, and $\mathscr{L}_i^{-1}$ is a context-free language because $\mathscr{L}_i$ is a context-free language, $i \in N_k$. For any $\omega \in \Sigma^*$, $\mu_{A^{-1}}(\omega) = \mu_A(\omega^{-1}) = \bigvee_{i=1}^{k} a_i \wedge \mu_{1_{\mathscr{L}_i}}(\omega^{-1}) = \bigvee_{i=1}^{k} a_i \wedge \mu_{1_{\mathscr{L}_i^{-1}}}(\omega)$, $\nu_{A^{-1}}(\omega) = \nu_A(\omega^{-1}) = \bigwedge_{i=1}^{k} b_i \vee \nu_{1_{\mathscr{L}_i}}(\omega^{-1}) = \bigwedge_{i=1}^{k} b_i \vee \nu_{1_{\mathscr{L}_i^{-1}}}(\omega)$, that is $A^{-1} \in \mathrm{Step}^C(\Sigma)$. For the operation of concatenation, since $\mu_{AB}(\omega) = \bigvee_{i=1}^{k}\bigvee_{j=1}^{n}(a_i \wedge c_j) \wedge \mu_{1_{\mathscr{L}_i\mathscr{M}_j}}(\omega)$, $\nu_{AB}(\omega) = \bigwedge_{i=1}^{k}\bigwedge_{j=1}^{n}(b_i \vee d_j) \vee \nu_{1_{\mathscr{L}_i\mathscr{M}_j}}(\omega)$, for all $\omega \in \Sigma^*$, where the elements of the family set $\{\mathscr{L}_i\mathscr{M}_j \mid i \in N_k, j \in N_n\}$ are also context-free languages since $\mathscr{L}_i$ and $\mathscr{M}_j$ are context-free languages. Hence $AB \in \mathrm{Step}^C(\Sigma)$.

For the Kleene closure, $A^* = (\mu_{A^*}, \nu_{A^*})$ is defined by $\mu_{A^*}(\omega) = \bigvee\{\mu_A(\omega_1) \wedge \cdots \wedge \mu_A(\omega_k) : k \geq 1, \omega = \omega_1 \cdots \omega_k\}$, $\nu_{A^*}(\omega) = \bigwedge\{\nu_A(\omega_1) \vee \cdots \vee \nu_A(\omega_k) : k \geq 1, \omega = \omega_1 \cdots \omega_k\}$ for any $\omega \in \Sigma^*$. Since $A \in \mathrm{Step}^C(\Sigma)$, we assume that the IFSPDA $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ accepts A by Theorem 19. Let $R = \{(\mu_F(q), \nu_F(q)) \mid q \in Q\} \setminus \{(0,1)\} = \{(a_i, b_i) \mid i \in N_k\}$. Then $A = \coprod_{i=1}^k (a_i, b_i) \cdot 1_{\mathscr{L}_i}$, where $\mathscr{L}_i$ is accepted by a PDA $\mathscr{M}_i = (Q, \Sigma, \Gamma, \delta', q_0, Z_0, F_i)$, the mapping $\delta' : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$ is defined by $\delta'(q, \tau, X) = \{(p, \gamma) \mid \mu_\delta(q, \tau, X, p, \gamma) = 1, p \in Q, \gamma \in \Gamma^*\}$, for all $(q, \tau, X) \in Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma$, and $F_i = \{q \in Q \mid \mu_F(q) = a_i, \nu_F(q) = b_i\}$, for all $i \in N_k$.

For any nonempty subset J of the set $\{1, 2, \ldots, k\}$, we can assume that $J = \{i_1, \ldots, i_s\}$. Let $r_J = a_{i_1} \wedge \cdots \wedge a_{i_s}$, $t_J = b_{i_1} \vee \cdots \vee b_{i_s}$, and $\mathscr{L}(J) = \bigcup_{p_1 \cdots p_s} \mathscr{L}_{p_1}^+ \mathscr{L}_{p_2}^+ \mathscr{L}_{p_1}^* \mathscr{L}_{p_3}^+ (\mathscr{L}_{p_1} \cup \mathscr{L}_{p_2})^* \cdots \mathscr{L}_{p_{s-1}}^+ (\mathscr{L}_{p_1} \cup \cdots \cup \mathscr{L}_{p_{s-2}})^* \mathscr{L}_{p_s}^+ (\mathscr{L}_{p_1} \cup \cdots \cup \mathscr{L}_{p_s})^*$, where $p_1 \cdots p_s$ is a permutation of $\{i_1, \ldots, i_s\}$, and $\mathscr{L}(J)$ takes unions under all permutations of $\{i_1, \ldots, i_s\}$. Hence $\mathscr{L}(J)$ is a context-free language. It is easily verified that $A^* = (\coprod_{\emptyset \neq J \subseteq N_k} (r_J, t_J) \cdot 1_{\mathscr{L}(J)}) \cup ((\mu_A(\varepsilon), \nu_A(\varepsilon)) \cdot 1_{\{\varepsilon\}})$. Therefore, $A^* \in \mathrm{Step}^C(\Sigma)$.

(2) If $A \in \mathrm{Step}^C(\Sigma_2)$, then $A = \coprod_{i=1}^k (a_i, b_i) \cdot 1_{\mathscr{L}_i}$, where a_i, $b_i \in [0,1]$, $a_i + b_i \leq 1$, $i \in N_k$, and $\mathscr{L}_1, \ldots, \mathscr{L}_k \subset \Sigma_2^*$ are classical context-free languages. Since $h : \Sigma_1^* \to \Sigma_2^*$ is a homomorphism, $\mu_{h^{-1}(A)}(\omega) = \mu_{A \circ h}(\omega) = \mu_A(h(\omega)) = \bigvee_{i=1}^k a_i \wedge \mu_{1_{\mathscr{L}_i}}(h(\omega)) = \bigvee_{i=1}^k a_i \wedge \mu_{1_{h^{-1}(\mathscr{L}_i)}}(\omega)$, and $\nu_{h^{-1}(A)}(\omega) = \nu_{A \circ h}(\omega) = \nu_A(h(\omega)) = \bigwedge_{i=1}^k b_i \vee \mu_{1_{\mathscr{L}_i}}(h(\omega)) = \bigwedge_{i=1}^k b_i \vee \nu_{1_{h^{-1}(\mathscr{L}_i)}}(\omega)$, where $\omega \in \Sigma_1^*, h^{-1}(\mathscr{L}_i) = \{\omega_1 \in \Sigma_1^* \mid h(\omega_1) \in \mathscr{L}_i\}, i \in N_k$, and the elements of the set $\{h^{-1}(\mathscr{L}_i) \mid i \in N_k\}$ are classical context-free languages. Hence $h^{-1}(A) = A \circ h = \coprod_{i=1}^k (a_i, b_i) \cdot 1_{h^{-1}(\mathscr{L}_i)}$. And so $h^{-1}(A) \in \mathrm{Step}^C(\Sigma_1)$. (3) If $h(\tau) \neq \varepsilon$, for all $\tau \in \Sigma$, and $g = (\mu_g, \nu_g) \in \mathrm{Step}^C(\Sigma_1)$, then $\mu_g(\omega) = \bigvee_{i=1}^k a_i \wedge \mu_{1_{\mathscr{L}_i}}(\omega)$, $\nu_g(\omega) = \bigwedge_{i=1}^k b_i \vee \mu_{1_{\mathscr{L}_i}}(\omega)$, for any $\omega \in \Sigma_1$, where $a_i, b_i \in [0,1]$, $a_i + b_i \leq 1, i \in N_k, \mathscr{L}_1, \ldots, \mathscr{L}_k \subset \Sigma_1^*$ are classical context-free languages. Since $h : \Sigma_1^* \to \Sigma_2^*$ is a homomorphism, $h(\mathscr{L}_i) = \{h(\omega) \in \Sigma_2^* \mid \omega \in \mathscr{L}_i\}$ is also a classical context-free language, $i \in N_k$. For any $x \in \Sigma_2^*$, $\mu_{h(g)}(x) = \bigvee\{\mu_g(\alpha) \mid h(\alpha) = x\} = \bigvee\{\bigvee_{i=1}^k a_i \wedge \mu_{1_{\mathscr{L}_i}}(\alpha) \mid h(\alpha) = x\} = \bigvee_{i=1}^k a_i \wedge \mu_{1_{h(\mathscr{L}_i)}}(x)$, $\nu_{h(g)}(x) = \bigwedge\{\nu_g(\alpha) \mid h(\alpha) = x\} = \bigwedge\{\bigwedge_{i=1}^k b_i \vee \nu_{1_{\mathscr{L}_i}}(\alpha) \mid h(\alpha) = x\} = \bigwedge_{i=1}^k b_i \vee \nu_{1_{h(\mathscr{L}_i)}}(x)$. Hence $h(g) = (\mu_{h(g)}, \nu_{h(g)}) \in \mathrm{Step}^C(\Sigma_2)$. □

5. Pumping Lemma for IFCFLs

In this section, we mainly discuss the pumping lemma for IFCFLs, which will become a powerful tool for proving a certain intuitionistic fuzzy language noncontext-free.

Theorem 33. *Let $A = (\mu_A, \nu_A)$ be an IFCFL over Σ^*. Then there exists a finite natural number n such that for any $z \in \Sigma^*$ with $n \leq |z|$, there have $u, v, w, x, y, u_1, v_1, w_1, x_1, y_1 \in \Sigma^*$ such that $z = uvwxy = u_1v_1w_1x_1y_1$, $|vwx| \leq n$, $|v_1w_1x_1| \leq n$, $|vx| \geq 1$, $|v_1x_1| \geq 1$, and $\mu_A(uv^iwx^iy) \geq \mu_A(uvwxy)$, $\nu_A(u_1v_1^iw_1x_1^iy_1) \leq \nu_A(u_1v_1w_1x_1y_1)$, for all $i \geq 0$.*

Proof. Let $A = (\mu_A, \nu_A)$ be an IFCFL over Σ^*. Then there is an IFCNF $G = (N, T, P, S)$ who accepts A. According to Proposition 29, $\mathscr{L}(G) = \coprod (a,b) \cdot \mathbf{1}_{\mathscr{L}(G_{ab})}$, where $(a,b) \in (X_\wedge \setminus \{0\}) \times (Y_\vee \setminus \{1\})$, $X = \mathrm{Im}(\mu_\rho)$, $Y = \mathrm{Im}(\nu_\rho)$, $0 \leq a + b \leq 1$ and the classical context-free grammar $G_{ab} = (N', T, P'_{ab}, S')$ is shown in the proof process of of Proposition 29. Let G have m variables. That means, $|N| = m$. Choose $n = 2^m$. Next, suppose $|z| \geq n$, $\mu_G(z) = a_0 > 0$ and $\nu_G(z) = b_0 < 1$. Then there exist $b_k \in Y_\vee \setminus \{1\}$ and $a_l \in X_\wedge \setminus \{0\}$ with $0 < a_0 + b_k \leq 1$ and $0 < a_l + b_0 \leq 1$ such that $z \in \mathscr{L}(G_{a_0b_k}) \cap \mathscr{L}(G_{a_lb_0})$. By pumping lemma for context-free languages, there are $u, v, w, x, y \in \Sigma^*$ satisfying $z = uvwxy$, $|vwx| \leq n$ and $|vx| \geq 1$ such that $uv^iwx^iy \in \mathscr{L}(G_{a_0b_k})$, for all $i \geq 0$. Then $\mu_A(uv^iwx^iy) \geq \mu_A(uvwxy)$ for any $i \geq 0$ since $\mu_A(uv^iwx^iy) = \bigvee_{a \in X_\wedge \setminus \{0\}} a \wedge 1_{\mathscr{L}(G_{ab})}(uv^iwx^iy) \geq a_0$.

Similarly, there are $u_1, v_1, w_1, x_1, y_1 \in \Sigma^*$ satisfying $z = u_1v_1w_1x_1y_1$, $|v_1w_1x_1| \leq n$ and $|v_1x_1| \geq 1$ such that $u_1v_1^iw_1x_1^iy_1 \in \mathscr{L}(G_{a_lb_0})$, for all $i \geq 0$. Then $\nu_A(u_1v_1^iw_1x_1^iy_1) \leq \nu_A(u_1v_1w_1x_1y_1)$ for any $i \geq 0$ since $\nu_A(u_1v_1^iw_1x_1^iy_1) = \bigwedge_{b \in Y_\vee \setminus \{1\}} b \vee 1_{\mathscr{L}(G_{ab})}(u_1v_1^iw_1x_1^iy_1) \leq b_0$. □

Next, let us look at an example to negate an intuitionistic fuzzy language to be an IFCFL.

Example 34. Let $A = (\mu_A, \nu_A)$ be an IFS over T^*. The mappings $\mu_A, \nu_A : T^* \to [0,1]$ are defined by

$$\mu_A(z) = \begin{cases} 0.5, & \text{if } z = a^ib^jc^k \ (i < j < k), \\ 0, & \text{otherwise}, \end{cases}$$
$$\nu_A(z) = \begin{cases} 0.3, & \text{if } z = a^ib^jc^k \ (i < j < k), \\ 1, & \text{otherwise}, \end{cases} \tag{27}$$

where i, j, and k are natural numbers.

Suppose A is an IFCFL. Then there exists a certain IFCNF G such that $\mathscr{L}(G) = A$. For constant n, put $z = a^nb^{n+1}c^{n+2}$. Hence, $\mu_A(z) = \mu_{\mathscr{L}(G)}(z) = 0.5$ and $\nu_A(z) = \nu_{\mathscr{L}(G)}(z) = 0.3$. Let $z = uvwxy$, where $|vwx| \leq n$ and $|vx| \geq 1$. If vwx does not have c's, then uv^3wx^3y has at least $n + 2a$'s or b's; if vwx has at least a c, then it has not an a since $|vwx| \leq n$. And so uwy has na's, but no more than $2n + 2b$'s and c's in total, that is, $|uwy| \leq n + 2n + 2$. Therefore, it is impossible that uwy has more b's than a's and also has more c's than b's. By calculation, we have $\mu_A(uwy) = 0$ and $\nu_A(uwy) = 1$. No matter how z is broken into $uvwxy$, we have a contradiction with Theorem 33. Therefore, A is not an IFCFL.

The following example will show that intuitionistic fuzzy pushdown automata have more power than fuzzy pushdown automata when comparing two distinct strings although the degrees of membership of these strings recognized by the underlying fuzzy pushdown automata are equal.

Example 35. Let $\Sigma = \{0, 1\}$. Then $L = \{\omega 1 \omega^{-1} \mid \omega \in \Sigma^*\} \subseteq \Sigma^*$ is clearly a context-free language but not a regular language by classical automata theory, where ω^{-1} represents the reversal of the string ω. Given an IFPDA $\mathscr{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$ and a fuzzy pushdown automaton $\mathscr{N} = (Q, \Sigma, \Gamma, \eta, \sigma_0, Z_0, \sigma_1)$. Put $Q = \{q_0, q_1, q_2\}$, $\Gamma = \{Z_0, 0, 1\}$, an IFS $\delta = (\mu_\delta, \nu_\delta)$ in $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$ is defined by

$\mu_\delta(q_0, 0, Z_0, q_0, 0Z_0) = 0.7$, $\nu_\delta(q_0, 0, Z_0, q_0, 0Z_0) = 0.2$,

$\mu_\delta(q_0, 1, Z_0, q_0, 1Z_0) = 0.6$, $\nu_\delta(q_0, 1, Z_0, q_0, 1Z_0) = 0.3$,

$\mu_\delta(q_0, 0, 0, q_0, 00) = 0.3$, $\nu_\delta(q_0, 0, 0, q_0, 00) = 0.6$,

$\mu_\delta(q_0, 0, 1, q_0, 01) = 0.3$, $\nu_\delta(q_0, 0, 1, q_0, 01) = 0.5$,

$\mu_\delta(q_0, 1, 0, q_0, 10) = 0.5$, $\nu_\delta(q_0, 1, 0, q_0, 10) = 0.4$,

$\mu_\delta(q_1, 0, 0, q_1, \varepsilon) = 0.6$, $\nu_\delta(q_1, 0, 0, q_1, \varepsilon) = 0.3$,

$\mu_\delta(q_1, 1, 1, q_1, \varepsilon) = 0.5$, $\nu_\delta(q_1, 1, 1, q_1, \varepsilon) = 0.35$,

$\mu_\delta(q_0, 1, Z_0, q_1, Z_0) = 1$, $\nu_\delta(q_0, 1, Z_0, q_1, Z_0) = 0$,

$\mu_\delta(q_0, 1, 0, q_1, 0) = 1$, $\nu_\delta(q_0, 1, 0, q_1, 0) = 0$,

$\mu_\delta(q_0, 1, 1, q_1, 1) = 1$, $\nu_\delta(q_0, 1, 1, q_1, 1) = 0$,

$\mu_\delta(q_1, \varepsilon, Z_0, q_2, Z_0) = 1$, $\nu_\delta(q_1, \varepsilon, Z_0, q_2, Z_0) = 0$.

Otherwise $\mu_\delta(q, \tau, Z, p, \gamma) = 0$ and $\nu_\delta(q, \tau, Z, p, \gamma) = 1$ for $(q, \tau, Z, p, \gamma) \in Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$.

The IFSs $I = (\mu_I, \nu_I)$ and $F = (\mu_F, \nu_F)$ in Q are defined by $\mu_I(q_0) = 1$, $\nu_I(q_0) = 0$, $\mu_I(q_1) = \mu_I(q_2) = 0$, $\nu_I(q_1) = \mu_I(q_2) = 1$, $\mu_F(q_2) = 1$, $\nu_F(q_2) = 0$, $\mu_F(q_0) = \mu_F(q_1) = 0$ and $\nu_F(q_0) = \mu_F(q_1) = 1$.

And set $\eta = \mu_\delta$, $\sigma_0 = \mu_I$, and $\sigma_1 = \mu_F$.

By computing with the strings, 010, 111, 01110, 10101, 0011100, and 1011101$\in \Sigma^*$, we have

$\mu_{\mathscr{L}(\mathscr{M})}(010) = f_{\mathscr{N}}(010) = 0.6$, $\nu_{\mathscr{L}(\mathscr{M})}(010) = 0.3$,

$\mu_{\mathscr{L}(\mathscr{M})}(111) = f_{\mathscr{N}}(111) = 0.5$, $\nu_{\mathscr{L}(\mathscr{M})}(111) = 0.35$,

$\mu_{\mathscr{L}(\mathscr{M})}(01110) = f_{\mathscr{N}}(01110) = 0.5$, $\nu_{\mathscr{L}(\mathscr{M})}(01110) = 0.4$,

$\mu_{\mathscr{L}(\mathscr{M})}(10101) = f_{\mathscr{N}}(10101) = 0.3$, $\nu_{\mathscr{L}(\mathscr{M})}(10101) = 0.5$,

$\mu_{\mathscr{L}(\mathscr{M})}(0011100) = f_{\mathscr{N}}(0011100) = 0.3$, $\nu_{\mathscr{L}(\mathscr{M})}(0011100) = 0.6$,

$\mu_{\mathscr{L}(\mathscr{M})}(1011101) = f_{\mathscr{N}}(1011101) = 0.3$, $\nu_{\mathscr{L}(\mathscr{M})}(1011101) = 0.5$.

This implies that 111 is better than 01110 because the degree of nonmembership of $\nu_{\mathscr{L}(\mathscr{M})}(111)$ is smaller than the $\nu_{\mathscr{L}(\mathscr{M})}(01110)$'s although the degrees of membership of the fuzzy context-free languages $f_{\mathscr{N}}(01110)$ and $f_{\mathscr{N}}(111)$ are equal. Comparing the above five strings, 010 is the best and 0011100 is the worst.

6. Conclusions

Taking intuitionistic fuzzy sets as the structures of truth values, we have investigated intuitionistic fuzzy context-free languages and established pumping lemma for the underlying languages. Firstly, the notions of intuitionistic fuzzy pushdown automata (IFPDAs) and their recognizable languages are introduced and discussed in detail. Using the generalized subset construction method, we show that IFPDAs are equivalent to IFSPDAs and then prove that intuitionistic fuzzy step functions are the same as those accepted by IFPDAs. Furthermore, we have presented algebraic characterization of intuitionistic fuzzy recognizable languages including decomposition form and representation theorem. It follows that the languages accepted by IFPDAs are equivalent to those accepted by IFPDAs$^\emptyset$ by classical automata theory. Secondly, we have introduced the notions of IFCFGs, IFCNFs, and IFGNFs. It is shown that they are equivalent in the sense that they generate the same classes of intuitionistic fuzzy context-free languages (IFCFLs). In particular, IFCFGs are proven to be an equivalence of IFPDAs as well. Then some operations on the family of IFCFLs are discussed. Finally pumping lemma for IFCFLs has been established. Thus, together with [38–40], we have more systematically established intuitionistic fuzzy automata theory as a generalization of fuzzy automata theory.

As mentioned in Section 1, IFS and fuzzy automata theory have supported a wealth of important applications in many fields. The next step is to consider the potential application of IFPDAs and IFCFLs such as in model checking and clinical monitoring. Additionally, many related researches in theories, such as IFPDAs based on the composition of t-norm and t-conorm and the minimal algorithm of IFPDAs, will be studied in the future.

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References

[1] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[2] K. T. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, vol. 35, Physica, Heidelberg, Germany, 1999.

[3] K. Atanassov, "Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade's paper: terminological difficulties in fuzzy set theory: the case of intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 156, no. 3, pp. 496–499, 2005.

[4] W. L. Gau and D. J. Buehrer, "Vague sets," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 23, no. 2, pp. 610–614, 1993.

[5] P. Burillo and H. Bustince, "Vague sets are intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 79, no. 3, pp. 403–405, 1996.

[6] Z. Chen and W. Yang, "A new multiple criteria decision making method based on intuitionistic fuzzy information," *Expert Systems with Applications*, vol. 39, no. 4, pp. 4328–4334, 2012.

[7] V. Khatibi and G. A. Montazer, "Intuitionistic fuzzy set versus fuzzy set application in medical pattern recognition," *Artificial Intelligence in Medicine*, vol. 47, no. 1, pp. 43–52, 2009.

[8] Z. Pei and L. Zheng, "A novel approach to multi-attribute decision making based on intuitionistic fuzzy sets," *Expert Systems with Applications*, vol. 39, no. 3, pp. 2560–2566, 2012.

[9] Q. S. Zhang, H. X. Yao, and Z. H. Zhang, "Some similarity measures of interval-valued intuitionistic fuzzy sets and application to pattern recognition," *International Journal of Applied Mechanics and Materials*, vol. 44–47, pp. 3888–3892, 2011.

[10] Q. S. Zhang, S. Y. Jiang, B. G. Jia, and S. H. Luo, "Some information measures for interval-valued intuitionistic fuzzy sets," *Information Sciences*, vol. 180, no. 12, pp. 5130–5145, 2010.

[11] E. T. Lee and L. A. Zadeh, "Note on fuzzy languages," *Information Sciences*, vol. 1, no. 4, pp. 421–434, 1969.

[12] D. S. Malik and J. N. Mordeson, "On fuzzy regular languages," *Information Sciences*, vol. 88, pp. 263–273, 1994.

[13] D. S. Malik and J. N. Mordeson, *Fuzzy Discrete Structures*, Physica, New York, NY, USA, 2000.

[14] J. N. Mordeson and D. S. Malik, *Fuzzy Automata and Languages: Theory and Applications*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2002.

[15] M. S. Ying, "A formal model of computing with words," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 5, pp. 640–652, 2002.

[16] F. Steimann and K.-P. Adlassnig, "Clinical monitoring with fuzzy automata," *Fuzzy Sets and Systems*, vol. 61, no. 1, pp. 37–42, 1994.

[17] C. L. Giles, C. W. Omlin, and K. K. Thornber, "Equivalence in knowledge representation: automata, recurrent neural networks, and dynamical fuzzy systems," *Proceedings of the IEEE*, vol. 87, no. 9, pp. 1623–1640, 1999.

[18] G. F. DePalma and S. S. Yau, "Fractionally fuzzy grammars with application to pattern recognition," in *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, L. A. Zadeh, K. S. Fu, K. Tanaka, and M. Shimura, Eds., pp. 329–351, Academic Press, New York, NY, USA, 1975.

[19] D. W. Qiu, "Supervisory control of fuzzy discrete event systems: a formal approach," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 35, no. 1, pp. 72–88, 2005.

[20] M. S. Ying, "Automata theory based on quantum logic. I," *International Journal of Theoretical Physics*, vol. 39, no. 4, pp. 985–995, 2000.

[21] M. S. Ying, "Automata theory based on quantum logic. II," *International Journal of Theoretical Physics*, vol. 39, no. 11, pp. 2545–2557, 2000.

[22] M. S. Ying, "A theory of computation based on quantum logic. I," *Theoretical Computer Science*, vol. 344, no. 2-3, pp. 134–207, 2005.

[23] M. S. Ying, "Quantum logic and automata theory," in *Handbook of Quantum Logic and Quantum Structures*, K. Engesser, D. M. Gabbay, and D. Lehmann, Eds., pp. 619–754, Elsevier, Amsterdam, The Netherlands, 2007.

[24] C. Moore and J. P. Crutchfield, "Quantum automata and quantum grammars," *Theoretical Computer Science*, vol. 237, no. 1-2, pp. 275–306, 2000.

[25] W. Cheng and J. Wang, "Grammar theory based on quantum logic," *International Journal of Theoretical Physics*, vol. 42, no. 8, pp. 1677–1691, 2003.

[26] D. W. Qiu, "Automata theory based on quantum logic: some characterizations," *Information and Computation*, vol. 190, no. 2, pp. 179–195, 2004.

[27] D. W. Qiu, "Notes on automata theory based on quantum logic," *Science in China*, vol. 50, no. 2, pp. 154–169, 2007.

[28] M. S. Ying, "Fuzzifying topology based on complete residuated lattice-valued logic. I," *Fuzzy Sets and Systems*, vol. 56, no. 3, pp. 337–373, 1993.

[29] D. W. Qiu, "Automata theory based on complete residuated lattice-valued logic," *Science in China*, vol. 44, no. 6, pp. 419–429, 2001.

[30] D. W. Qiu, "Automata theory based on complete residuated lattice-valued logic. II," *Science in China*, vol. 45, no. 6, pp. 442–452, 2002.

[31] D. W. Qiu, "Pumping lemma in automata theory based on complete residuated lattice-valued logic: a note," *Fuzzy Sets and Systems*, vol. 157, no. 15, pp. 2128–2138, 2006.

[32] Y. M. Li and W. Pedrycz, "Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids," *Fuzzy Sets and Systems*, vol. 156, no. 1, pp. 68–92, 2005.

[33] Y. M. Li, "Finite automata theory with membership values in lattices," *Information Sciences*, vol. 181, no. 5, pp. 1003–1017, 2011.

[34] J. Jin and Q. Li, "Fuzzy grammar theory based on lattices," *Soft Computing*, vol. 16, no. 8, pp. 1415–1426, 2012.

[35] H. Y. Xing, "Fuzzy pushdown automata," *Fuzzy Sets and Systems*, vol. 158, no. 13, pp. 1437–1449, 2007.

[36] H. Y. Xing, D. W. Qiu, and F. C. Liu, "Automata theory based on complete residuated lattice-valued logic: pushdown automata," *Fuzzy Sets and Systems*, vol. 160, no. 8, pp. 1125–1140, 2009.

[37] H. Y. Xing and D. W. Qiu, "Pumping lemma in context-free grammar theory based on complete residuated lattice-valued logic," *Fuzzy Sets and Systems*, vol. 160, no. 8, pp. 1141–1151, 2009.

[38] Y. B. Jun, "Intuitionistic fuzzy finite state machines," *Journal of Applied Mathematics & Computing*, vol. 17, no. 1-2, pp. 109–120, 2005.

[39] Y. B. Jun, "Intuitionistic fuzzy finite switchboard state machines," *Journal of Applied Mathematics & Computing*, vol. 20, no. 1-2, pp. 315–325, 2006.

[40] X. W. Zhang and Y. M. Li, "Intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata," *Soft Computing*, vol. 13, no. 6, pp. 611–616, 2009.

[41] T. Y. Chen, H. P. Wang, and J. C. Wang, "Fuzzy automata based on Atanassov fuzzy sets and applications on consumers' advertising involvement," *African Journal of Business Management*, vol. 6, no. 3, pp. 865–880, 2012.