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Research Article

The Almost Sure Local Central Limit Theorem for the Negatively Associated Sequences

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In this paper, the almost sure central limit theorem is established for sequences of negatively associated random variables: $\lim_{n\to\infty}(1/\log n)\sum_{k=1}^{n}(I(a_k \le S_k < b_k)/k)P(a_k \le S_k < b_k) = 1$, almost surely. This is the local almost sure central limit theorem for negatively associated sequences similar to results by Csáki et al. (1993). The results extend those on almost sure local central limit theorems from the i.i.d. case to the stationary negatively associated sequences.

1. Introduction

Definition 1. Random variables $X_1, X_2, \ldots, X_n$, $n \ge 2$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\left(f_1\left(X_i, i \in A_1\right), f_2\left(X_j, j \in A_2\right)\right) \le 0, \qquad (1)$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable) such that this covariance exists. A sequence of random variables $\{X_i, i \ge 1\}$ is said to be NA if every finite subfamily of $\{X_i, i \ge 1\}$ is NA.

Obviously, if $\{X_i, i \ge 1\}$ is a sequence of NA random variables and $\{f_i, i \ge 1\}$ is a sequence of nondecreasing (or nonincreasing) functions, then $\{f_i(X_i), i \ge 1\}$ is also a sequence of NA random variables.

This definition was introduced by the Joag-Dev and Proschan [1]. Statistical test depends greatly on sampling, and the random sampling without replacement from a finite population is NA, but it is not independent. NA sampling has wide applications such as those in multivariate statistical analysis and reliability theory. Because of the wide applications of NA sampling, the notions of NA random variables have received more and more attention recently. We refer to Joag-Dev and Proschan [1] for fundamental properties, Shao [2] for the moment inequalities, and Wu and Jiang [3] for Chover's law of the iterated logarithm.

Assume that $\{X_n, n \ge 1\}$ is a strictly stationary sequence of NA random variables with $EX_1 = 0, 0 < EX_1^2 < \infty$. Define $S_n = \sum_{j=1}^{n} X_j$,

$$\sigma_n^2 := \operatorname{Var} S_n, \qquad (2)$$

$$\sigma^2 := \operatorname{Var} X_1 + 2\sum_{j=2}^{\infty}\operatorname{Cov}\left(X_1, X_j\right). \qquad (3)$$

(1) Newman [4] and Matuła [5] showed that NA stationary sequences satisfy the central limit theorem (CLT) under $\sigma^2 > 0$, that is,

$$\sup_{x\in R}\left|P\left(\frac{S_n}{\sigma_n} < x\right) - \Phi(x)\right| = o(1). \qquad (4)$$

(2) Applying Matuła [6] and Wu's [7] methods, we can easily show that NA sequences satisfy the almost sure central limit theorem (ASCLT), that is,

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{S_k \le x\sigma k^{1/2}\right\} = \Phi(x) \quad \text{a.s. } \forall x \in \mathbb{R}, \qquad (5)$$

where $\Phi(x)$ is the standard normal distribution function and $I\{A\}$ denotes the indicator of the event A.

The ASCLT was stimulated by Brosamler [8] and Schatte [9]. Both were concerned with the partial sum of independent and identically distributed (i.i.d.) random variables with more than the second moment. The ASCLT was extensively studied in the past two decades and an interesting direction of the study is to prove it for dependent variables. There are some results for weakly dependent variables such as α, ρ, ϕ-mixing and associated random variables. Among those results, we refer to Peligrad and Shao [10], Matuła [11], and Wu [7].

More general version of ASCLT was proved by Csáki et al. [12]. The following theorem is due to them.

Theorem A. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $E|X_1|^3 < \infty$, let $EX_1 = 0$. a_k, b_k satisfy*

$$-\infty \leq a_k \leq 0 \leq b_k \leq \infty, \quad k = 1, 2, \ldots, \tag{6}$$

and assume that

$$\sum_{k=1}^{n} \frac{\log k}{k^{3/2} P\left(a_k \leq S_k < b_k\right)} = O\left(\log n\right), \quad \text{as } n \longrightarrow \infty, \tag{7}$$

and then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{I\left\{a_k \leq S_k < b_k\right\}}{kP\left(a_k \leq S_k < b_k\right)} = 1 \quad \text{a.s.} \tag{8}$$

This result may be called almost sure local central limit theorem, while (5) may be called almost sure global central limit theorem. Hurelbaatar [13] extended (8) to the case of ρ-mixing sequences and Weng et al. [14] derived an almost sure local central limit theorem for the product of partial sums of a sequence of i.i.d. positive random variables under some regular conditions. For more details, we refer to Berkes and Csáki [15] and Földes [16].

Our concern in this paper is to give a common generalization of (8) to the case of NA sequences.

In the next section we present the exact results, postponing some technical lemmas and the proofs to Section 3.

2. Main Results

Assume in the following that $\{X_n, n \geq 1\}$ is a strictly stationary sequence of NA random variables with $EX_1 = 0, 0 < EX_1^2 < \infty$. We consider the limit behavior of the logarithmic average

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{I\left\{a_k \leq S_k < b_k\right\}}{kP\left(a_k \leq S_k < b_k\right)} \tag{9}$$

with $-\infty \leq a_k \leq 0 \leq b_k \leq \infty$, where the terms in the sum above are defined to be 1 if their denominator happens to be 0.

More precisely let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences of real numbers and put

$$p_k := P\left(a_k \leq S_k < b_k\right),$$
$$\alpha_k := \begin{cases} \dfrac{I\left\{a_k \leq S_k < b_k\right\}}{p_k}, & \text{if } p_k \neq 0, \\ 1, & \text{if } p_k = 0. \end{cases} \tag{10}$$

So we need to investigate the limit behavior of

$$\mu_n := \sum_{k=1}^{n} \frac{\alpha_k}{k} \tag{11}$$

under certain conditions.

In our considerations, we will need the following Cox-Grimmett coefficient which describes the covariance structure of the sequence

$$u(n) := \sup_{k\in N} \sum_{j:|j-k|\geq n} \left|\text{Cov}\left(X_j, X_k\right)\right|, \quad n \in N \cup \{0\}. \tag{12}$$

We remark that for a stationary sequence of NA random variables

$$u(n) = -2 \sum_{k=n+1}^{\infty} \text{Cov}\left(X_1, X_k\right), \quad n \in N. \tag{13}$$

By Lemma 8 of Newman [4], we have $u(0) < \infty$ and $\lim_{n\to\infty} u(n) = 0$.

In the following, $\xi_n \sim \eta_n$ denotes $\xi_n/\eta_n \to 1, n \to \infty$. $\xi_n = O(\eta_n)$ denotes that there exists a constant $c > 0$ such that $\xi_n \leq c\eta_n$ for sufficiently large n. The symbols $c, c_1, c_2, \ldots$, stand for generic positive constants which may differ from one place to another.

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of NA random variables with $EX_1 = 0, E|X_1|^3 < \infty$ and let $\sigma^2 > 0$. a_k, b_k satisfy* (6). *Assume that*

$$\sum_{n=1}^{\infty} u(n) < \infty, \tag{14}$$

and for some $\beta > 1$,

$$\sum_{\substack{1\leq k\leq n \\ p_k \neq 0}} \frac{(\log k)^{1/3}}{kp_k} = O\left((\log n)^2 (\log \log n)^{-\beta}\right). \tag{15}$$

Then we have

$$\lim_{n\to\infty} \frac{\mu_n}{\log n} = 1, \quad \text{a.s.}, \tag{16}$$

where μ_n is defined by (11).

Remark 3. Let $a_k = -\infty$ and $b_k = x\sigma k^{1/2}$ in (6). By the central limit theorem (4), we have $p_k = P(S_k/\sigma k^{1/2} < x) \to \Phi(x)$, obviously (15) satisfies; then (16) becomes (5), which is the almost sure global central limit theorem. Thus the almost sure local central limit theorem is a general result which contains the almost sure global central limit theorem.

Remark 4. The condition (15) is satisfied with a wide range of p_k; for example, if

$$p_k = 0 \quad \text{or} \quad p_k \geq c \frac{(\log \log k)^{\beta}}{(\log k)^{2/3}} \tag{17}$$

holds, then the condition (15) is satisfied. In fact, letting $0 < \delta < 1$, we have

$$\begin{aligned}
\sum_{\substack{1\le k\le n\\ p_k\neq 0}} \frac{(\log k)^{1/3}}{kp_k}
&\le c\sum_{1\le k\le n}(\log\log k)^{-\beta}(\log k)^{\delta}\frac{(\log k)^{1-\delta}}{k}\\
&\le c(\log\log n)^{-\beta}(\log n)^{\delta}\sum_{1\le k\le n}\frac{(\log k)^{1-\delta}}{k}\\
&\le c(\log\log n)^{-\beta}(\log n)^{2}\\
&= O\left((\log n)^{2}(\log\log n)^{-\beta}\right).
\end{aligned} \tag{18}$$

In the given theorem below, we strengthen the condition (6) on a_k and b_k. Meanwhile, as a compensation, we do not need to impose restricting condition (15) on p_k.

Theorem 5. *Let $\{X_n, n\ge 1\}$ be a strictly stationary sequence of NA random variables with $EX_1 = 0, E|X_1|^3 < \infty$, and $\sigma^2 > 0$, and let a_k and b_k satisfy*

$$\begin{gathered}
-c_1k^{1/2-\alpha} \le a_k \le -c_2k^{1/2-\alpha},\\
c_3k^{1/2-\alpha} \le b_k \le c_4k^{1/2-\alpha},
\end{gathered} \tag{19}$$

where $0 < \alpha < 1/7$. Assume that (14) hold, and then we have (16).

3. Proofs

The following lemmas play important roles in the proof of our theorems. The proofs are given in the Appendix.

Lemma 6. *Assume that $\{\xi_n, n\ge 1\}$ are random variables such that*

$$\xi_k \ge 0, \quad E\xi_k = 1, \quad k = 1,2,\ldots \tag{20}$$

and furthermore there exists $d_k \ge 0$ such that $D_n = \sum_{k=1}^n d_k \uparrow \infty, D_n/D_{n-1} \to 1$, and

$$\operatorname{Var}\left(\sum_{k=1}^n d_k\xi_k\right) \le cD_n^2(\log D_n)^{-\beta}, \tag{21}$$

with some $\beta > 1$ and positive constant c, and then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k\xi_k = 1 \quad \text{a.s.} \tag{22}$$

Remark 7. Let $d_k = 1/k$ in (21), and then $D_n = \sum_{k=1}^n 1/k \sim \log n$. Thus, if

$$\operatorname{Var}\left(\sum_{k=1}^n \frac{1}{k}\xi_k\right) \le c(\log n)^2(\log\log n)^{-\beta}, \tag{23}$$

with some $\beta > 1$ and positive constant c, then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^n\frac{1}{k}\xi_k = 1 \quad \text{a.s.} \tag{24}$$

The following Lemma 8 is obvious.

Lemma 8. *Assume that the nonnegative random sequence $\{\xi_n, n\ge 1\}$ satisfies (24) and the sequence $\{\eta_n, n\ge 1\}$ is such that, for any $\varepsilon > 0$, there exists a $k_0 = k_0(\varepsilon,\omega)$ for which*

$$(1-\varepsilon)\,\xi_k \le \eta_k \le (1+\varepsilon)\,\xi_k, \quad k > k_0. \tag{25}$$

Then we have also

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^n\frac{1}{k}\eta_k = 1 \quad \text{a.s.} \tag{26}$$

The following Lemma 9 is an easy corollary to the Corollary 2.2 in Matuła [5] under strictly stationary condition, which studies the rate of convergence in the CLT under negative dependence. It was also studied in Pan [17]. Of course this is the Berry-Esseen inequality for the NA sequence.

Lemma 9. *Let $\{X_j, j\in N\}$ be a strictly stationary sequence of NA random variables with $EX_1 = 0, E|X_1|^3 < \infty, \sigma^2 > 0$ satisfying (14). Then one has*

$$\sup_{x\in R}\left|P\left(\frac{S_n}{\sigma_n} < x\right) - \Phi(x)\right| = O\left(n^{-1/5}\right). \tag{27}$$

Lemma 10. *If the conditions of Lemma 9 hold, a_k and b_k satisfy (19). Then one has*

$$c_1'k^{-\alpha} \le p_k \le c_2'k^{-\alpha} \quad (k\ge k_0), \tag{28}$$

where α is as (19).

Lemma 11. *If the conditions of Lemma 9 hold, a_k and b_k satisfy (19), and α is as (19). Assume that $\varepsilon_l = l^{3\alpha/2}$, and then the following asymptotic relations hold:*

$$\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}}\frac{1}{klp_l}P\left(|S_{l^{\alpha}}| \ge \varepsilon_l\right) = O(\log n), \tag{29}$$

$$\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}}\frac{1}{klp_l}\frac{1}{(l-k-l^{\alpha})^{1/5}} = O(\log n), \tag{30}$$

$$\begin{aligned}
&\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}}\frac{1}{klp_l}\left|\Phi\left(\frac{a_l-b_k-\varepsilon_l}{(l-k-l^{\alpha})^{1/2}}\right) - \Phi\left(\frac{a_l}{l^{1/2}}\right)\right|\\
&\quad = O(\log n),
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}}\frac{1}{klp_l}\left|\left(\Phi\left(\frac{b_l-a_k+\varepsilon_l}{(l-k-l^{\alpha})^{1/2}}\right) - \Phi\left(\frac{b_l}{l^{1/2}}\right)\right)\right|\\
&\quad = O(\log n).
\end{aligned} \tag{32}$$

The main point in our proof is to verify the condition (23). We use global central limit theorem with remainders and the following elementary inequalities:

$$\left|\Phi(x) - \Phi(y)\right| \le c\left|x - y\right|, \quad \text{for every } x, y \in \mathbb{R} \tag{33}$$

with some constant c. Moreover, for each $k > 0$, there exists $c_1 = c_1(k)$, such that

$$\begin{aligned} &\left|\Phi(x) - \Phi(y)\right| \\ &\quad \ge c_1\left|x - y\right|, \quad \text{for every } x, y \in \mathbb{R}, |x| + |y| \le k. \end{aligned} \tag{34}$$

Let $\{X_n, n \ge 1\}$ be a strictly stationary sequence of NA random variables with $\sigma^2 > 0$; we can immediately get $\sigma_n^2 \sim n\sigma^2$, that is,

$$c_1 n \le \operatorname{Var}(S_n) = \sigma_n^2 \le c_2 n \tag{35}$$

for some constant $c_1, c_2 > 0$ and sufficiently large n.

Proof of Theorem 2. First assume that

$$b_k - a_k \le c k^{1/2}, \quad k = 1, 2, \ldots \tag{36}$$

with some constant c. Let $1 \le k < l$ and $\varepsilon_k = k^{1/2}(\log k)^{1/3}$.

If either $p_k = 0$ or $p_l = 0$, then obviously $\operatorname{Cov}(\alpha_k, \alpha_l) = 0$, and so we may assume that $p_k p_l \ne 0$; then, we have

$$\begin{aligned} &\operatorname{Cov}(\alpha_k, \alpha_l) \\ &\quad = \frac{1}{p_l p_k}\left(P\left(a_k \le S_k < b_k, a_l \le S_l < b_l\right) - p_l p_k\right) \\ &\quad \le \frac{1}{p_l p_k}\left(P\left(a_l - b_k \le S_l - S_k < b_l - a_k,\right.\right. \\ &\qquad\qquad \left.\left. a_k \le S_k < b_k\right) - p_l p_k\right) \\ &\quad \le \frac{1}{p_l p_k}\left(P\left(a_l - b_k \le S_l - S_k < b_l - a_k\right)\right. \\ &\qquad\qquad \left. \times P\left(a_k \le S_k < b_k\right) - p_l p_k\right) \\ &\quad = \frac{1}{p_l}\left(P\left(a_l - b_k \le S_l - S_k < b_l - a_k\right) - p_l\right) \\ &\quad \le \frac{1}{p_l}\left(P\left(a_l - b_k - \varepsilon_k < S_l < b_l - a_k + \varepsilon_k\right)\right. \\ &\qquad\qquad \left. - p_l + P\left(\left|S_k\right| \ge \varepsilon_k\right)\right) \\ &\quad \le \frac{1}{p_l}\left(P\left(a_l - b_k - \varepsilon_k \le S_l < a_l\right)\right. \\ &\qquad\qquad \left. + P\left(b_l \le S_l < b_l - a_k + \varepsilon_k\right) + P\left(\left|S_k\right| \ge \varepsilon_k\right)\right). \end{aligned} \tag{37}$$

Applying Lemma 9, (33), (35), and (36) and noting that $\varepsilon_k = k^{1/2}(\log k)^{1/3}$, we obtain

$$\begin{aligned} &P\left(a_l - b_k - \varepsilon_k \le S_l < a_l\right) + P\left(b_l \le S_l < b_l - a_k + \varepsilon_k\right) \\ &\quad \le \left(\Phi\left(\frac{a_l}{\sigma_l}\right) - \Phi\left(\frac{a_l - b_k - \varepsilon_k}{\sigma_l}\right)\right) \\ &\qquad + \left(\Phi\left(\frac{b_l - a_k + \varepsilon_k}{\sigma_l}\right) - \Phi\left(\frac{b_l}{\sigma_l}\right)\right) + c\frac{1}{l^{1/5}} \\ &\quad \le \frac{b_k + \varepsilon_k}{\sigma_l} + \frac{-a_k + \varepsilon_k}{\sigma_l} + c\frac{1}{l^{1/5}} = \frac{b_k - a_k}{\sigma_l} + \frac{2\varepsilon_k}{\sigma_l} + c\frac{1}{l^{1/5}} \\ &\quad \le c\left(\frac{k^{1/2}}{l^{1/2}} + \frac{\varepsilon_k}{l^{1/2}} + \frac{1}{l^{1/5}}\right) \le c\left(\frac{k^{1/2}(\log k)^{1/3}}{l^{1/2}} + \frac{1}{l^{1/5}}\right). \end{aligned} \tag{38}$$

By the condition of (15), we have

$$\begin{aligned} &\sum_{l=1}^{n}\sum_{k=1}^{l}\frac{1}{klp_l}\frac{k^{1/2}(\log k)^{1/3}}{l^{1/2}} \\ &\quad \le \sum_{l=1}^{n}\frac{1}{l^{3/2}p_l}\sum_{k=1}^{l}\frac{(\log k)^{1/3}}{k^{1/2}} \\ &\quad \le c\sum_{l=1}^{n}\frac{(\log l)^{1/3}}{l^{3/2}p_l}\sum_{k=1}^{l}\frac{1}{k^{1/2}} \le c\sum_{l=1}^{n}\frac{(\log l)^{1/3}}{l^{3/2}p_l}l^{1/2} \\ &\quad \le c\sum_{l=1}^{n}\frac{(\log l)^{1/3}}{lp_l} = O\left((\log n)^2(\log\log n)^{-\beta}\right), \end{aligned} \tag{39}$$

$$\begin{aligned} &\sum_{l=1}^{n}\sum_{k=1}^{l-1}\frac{1}{klp_l}\frac{1}{l^{1/5}} \\ &\quad = \sum_{l=1}^{n}\frac{1}{l^{6/5}p_l}\sum_{k=1}^{l-1}\frac{1}{k} \le c\sum_{l=1}^{n}\frac{1}{l^{6/5}p_l}\log l \\ &\quad \le c\sum_{l=1}^{n}\frac{(\log l)^{1/3}}{lp_l} = O\left((\log n)^2(\log\log n)^{-\beta}\right). \end{aligned}$$

By Chebyshev's inequality and the condition of (15) and (35), we obtain

$$\begin{aligned} &\sum_{l=1}^{n}\sum_{k=1}^{l}\frac{1}{klp_l}P\left(\left|S_k\right| \ge \varepsilon_k\right) \\ &\quad \le \sum_{l=1}^{n}\frac{1}{lp_l}\sum_{k=1}^{l}\frac{\operatorname{Var}(S_k)}{k\varepsilon_k^2} \le c\sum_{l=1}^{n}\frac{1}{lp_l}\sum_{k=1}^{l}\frac{1}{k(\log k)^{2/3}} \\ &\quad \le c\sum_{l=1}^{n}\frac{(\log l)^{1/3}}{lp_l} = O\left((\log n)^2(\log\log n)^{-\beta}\right). \end{aligned} \tag{40}$$

Hence (37)–(40) imply that

$$\sum_{l=1}^{n}\sum_{k=1}^{l-1}\frac{\operatorname{Cov}(\alpha_k, \alpha_l)}{kl} = O\left((\log n)^2(\log\log n)^{-\beta}\right). \tag{41}$$

But $\operatorname{Var}(\alpha_k) = 0$ if $p_k = 0$ and

$$\operatorname{Var}(\alpha_k) = \frac{1-p_k}{p_k} \leq \frac{1}{p_k} \quad \text{if } p_k \neq 0. \tag{42}$$

Thus

$$\begin{aligned}\sum_{k=1}^{n} \frac{\operatorname{Var}(\alpha_k)}{k^2} &\leq \sum_{\substack{1\leq k\leq n\\ p_k\neq 0}} \frac{1}{k^2 p_k} \leq \sum_{\substack{1\leq k\leq n\\ p_k\neq 0}} \frac{(\log k)^{1/3}}{k p_k} \\ &= O\left((\log n)^2 (\log\log n)^{-\beta}\right).\end{aligned} \tag{43}$$

Equations (41) and (43) together imply that

$$\operatorname{Var}\left(\sum_{k=1}^{n} \frac{\alpha_k}{k}\right) = O\left((\log n)^2 (\log\log n)^{-\beta}\right), \quad \text{as } n \to \infty. \tag{44}$$

Hence applying Remark 7, our theorem is proved under the restricting condition (36).

Now we drop the restricting condition (36). Fix $x > 0$ and define

$$\begin{aligned}\tilde{a}_k &= \max\left(a_k, -x\sigma k^{1/2}\right), \\ \tilde{b}_k &= \min\left(b_k, x\sigma k^{1/2}\right), \\ \tilde{p}_k &= P\left(\tilde{a}_k \leq S_k < \tilde{b}_k\right),\end{aligned} \tag{45}$$

where σ is defined by (3).

Clearly $\tilde{p}_k \leq p_k$, and so assuming $\tilde{p}_k \neq 0$; then, also we have $p_k \neq 0$, and thus

$$\begin{aligned}\alpha_k &= \frac{1}{p_k} I\left\{a_k \leq S_k < b_k\right\} \\ &= \frac{1}{p_k}\Big(I\left\{\tilde{a}_k \leq S_k < \tilde{b}_k\right\} \\ &\qquad + I\left\{a_k \leq S_k < \tilde{a}_k\right\} + I\left\{\tilde{b}_k \leq S_k < b_k\right\}\Big) \\ &\leq \frac{1}{\tilde{p}_k} I\left\{\tilde{a}_k \leq S_k < \tilde{b}_k\right\} \\ &\quad + \frac{1}{p_k}\left(I\left\{a_k \leq S_k < \tilde{a}_k\right\} + I\left\{\tilde{b}_k \leq S_k < b_k\right\}\right) \\ &\leq \frac{1}{\tilde{p}_k} I\left\{\tilde{a}_k \leq S_k < \tilde{b}_k\right\} \\ &\quad + \frac{I\left\{S_k < -x\sigma k^{1/2}\right\}}{P\left(-x\sigma k^{1/2} \leq S_k < 0\right)} + \frac{I\left\{S_k \geq x\sigma k^{1/2}\right\}}{P\left(0 \leq S_k < x\sigma k^{1/2}\right)}.\end{aligned} \tag{46}$$

By (35) and the central limit theorem for NA random variables (4), that is,

$$\sup_{x\in R}\left|P\left(\frac{S_n}{\sigma_n} < x\right) - \Phi(x)\right| = o(1), \tag{47}$$

we obtain

$$\begin{aligned}&\lim_{k\to\infty} P\left(-x\sigma k^{1/2} \leq S_k < 0\right) \\ &\qquad = \lim_{k\to\infty} P\left(-x\sigma_k \leq S_k < 0\right) = \Phi(0) - \Phi(-x), \\ &\lim_{k\to\infty} P\left(0 \leq S_k < x\sigma k^{1/2}\right) \\ &\qquad = \lim_{k\to\infty} P\left(0 \leq S_k < x\sigma_k\right) = \Phi(x) - \Phi(0).\end{aligned} \tag{48}$$

Applying the almost sure central limit theorem for NA random variables (5), that is,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{S_k \leq x\sigma k^{1/2}\right\} = \Phi(x) \quad \text{a.s. } \forall x \in \mathbb{R}, \tag{49}$$

Lemma 8, and (48), we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{I\left\{S_k < -x\sigma k^{1/2}\right\}}{kP\left(-x\sigma k^{1/2} \leq S_k < 0\right)} \\ &\qquad = \frac{\Phi(-x)}{\Phi(0) - \Phi(-x)} \quad \text{a.s.}, \\ &\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{I\left\{S_k > x\sigma k^{1/2}\right\}}{kP\left(0 \leq S_k < x\sigma k^{1/2}\right)} \\ &\qquad = \frac{1 - \Phi(x)}{\Phi(x) - \Phi(0)} \quad \text{a.s.}\end{aligned} \tag{50}$$

Since $\tilde{a}_k$ and $\tilde{b}_k$ satisfy (36), we get

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\tilde{\alpha}_k}{k} = 1 \quad \text{a.s.}, \tag{51}$$

where

$$\tilde{\alpha}_k = \begin{cases} \dfrac{I\left\{\tilde{a}_k \leq S_k < \tilde{b}_k\right\}}{\tilde{p}_k}, & \text{if } \tilde{p}_k \neq 0, \\ 1, & \text{if } \tilde{p}_k = 0. \end{cases} \tag{52}$$

Equations (46) and (50)–(51) together imply that

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\alpha_k}{k} \leq 1 + 2\frac{1-\Phi(x)}{\Phi(x) - \Phi(0)} \quad \text{a.s.} \tag{53}$$

On the other hand, if $\tilde{p}_k \neq 0$, then we have

$$\begin{aligned}&\frac{1}{p_k} I\left\{a_k \leq S_k < b_k\right\} \\ &\geq \frac{1}{\tilde{p}_k} I\left\{\tilde{a}_k \leq S_k < \tilde{b}_k\right\}\left(1 - \frac{p_k - \tilde{p}_k}{p_k}\right) \\ &\geq \tilde{\alpha}_k\left(1 - \frac{P\left(S_k < -\sigma x k^{1/2}\right) + P\left(S_k > \sigma x k^{1/2}\right)}{\min\left\{P\left(0 \leq S_k < \sigma x k^{1/2}\right), P\left(-\sigma x k^{1/2} \leq S_k < 0\right)\right\}}\right),\end{aligned} \tag{54}$$

and by the central limit theorem,

$$\lim_{k\to\infty}\frac{P\left(S_k<-\sigma x k^{1/2}\right)+P\left(S_k>\sigma x k^{1/2}\right)}{\min\left\{P\left(0\le S_k<\sigma x k^{1/2}\right),P\left(-\sigma x k^{1/2}\le S_k<0\right)\right\}}=1-2\frac{1-\Phi(x)}{\Phi(x)-\Phi(0)}. \tag{55}$$

Applying Lemma 8, (51) and (54) imply that

$$\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\alpha_k}{k}\ge 1-2\frac{1-\Phi(x)}{\Phi(x)-\Phi(0)}\quad\text{a.s.,} \tag{56}$$

and hence

$$\begin{aligned}1+2\frac{1-\Phi(x)}{\Phi(x)-\Phi(0)}&\ge\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\alpha_k}{k}\\&\ge\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\alpha_k}{k}\\&\ge 1-2\frac{1-\Phi(x)}{\Phi(x)-\Phi(0)}\quad\text{a.s.}\end{aligned} \tag{57}$$

By the arbitrariness of x, let $x\to\infty$ in (57); we have

$$\begin{aligned}1&\ge\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\alpha_k}{k}\\&\ge\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\alpha_k}{k}\ge 1\quad\text{a.s.}\end{aligned} \tag{58}$$

Thus

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\alpha_k}{k}=1\quad\text{a.s.} \tag{59}$$

This completes the proof of Theorem 2. □

Proof of Theorem 5. Let $k<l-l^{\alpha}$, $1\le k<l$, and $\varepsilon_l=l^{3\alpha/2}$; we have

$$\begin{aligned}\operatorname{Cov}(\alpha_k,\alpha_l)&\le\frac{1}{p_lp_k}\left(P\left(a_l-b_k\le S_l-S_{k+l^{\alpha}}+S_{k+l^{\alpha}}-S_k\right.\right.\\&\qquad\left.\left.<b_l-a_k,a_k\le S_k<b_k\right)-p_lp_k\right)\\&\le\frac{1}{p_lp_k}\left(P\left(a_l-b_k\le S_l-S_{k+l^{\alpha}}+S_{k+l^{\alpha}}-S_k\right.\right.\\&\qquad\left.\left.<b_l-a_k\right)P\left(a_k\le S_k<b_k\right)-p_lp_k\right)\\&\le\frac{1}{p_l}\left(P\left(a_l-b_k-\varepsilon_l<S_l-S_{k+l^{\alpha}}<b_l-a_k+\varepsilon_l\right)\right.\\&\qquad\left.-p_l+P\left(\left|S_{k+l^{\alpha}}-S_k\right|\ge\varepsilon_l\right)\right)\\&\le\frac{1}{p_l}\left(P\left(a_l-b_k-\varepsilon_l<S_{l-k-l^{\alpha}}<b_l-a_k+\varepsilon_l\right)\right.\\&\qquad\left.-p_l+P\left(\left|S_{l^{\alpha}}\right|\ge\varepsilon_l\right)\right).\end{aligned} \tag{60}$$

Applying Lemma 9, (33), and (35), we obtain

$$\begin{aligned}P&\left(a_l-b_k-\varepsilon_l<S_{l-k-l^{\alpha}}<b_l-a_k+\varepsilon_l\right)-p_l\\&\le\left(\Phi\left(\frac{b_l-a_k+\varepsilon_l}{(l-k-l^{\alpha})^{1/2}}\right)-\Phi\left(\frac{b_l}{l^{1/2}}\right)\right)\\&\quad+\left(\Phi\left(\frac{a_l}{l^{1/2}}\right)-\Phi\left(\frac{a_l-b_k-\varepsilon_l}{(l-k-l^{\alpha})^{1/2}}\right)\right)\\&\quad+c\left(\frac{1}{l^{1/5}}+\frac{1}{(l-k-l^{\alpha})^{1/5}}\right).\end{aligned} \tag{61}$$

Hence Lemma 11, (60), and (61) imply that

$$\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}}\frac{\operatorname{Cov}(\alpha_k,\alpha_l)}{kl}=O(\log n). \tag{62}$$

On the other hand,

$$\sum_{\substack{1\le k<l\le n\\ k>l-l^{\alpha}}}\frac{\operatorname{Cov}(\alpha_k,\alpha_l)}{kl}=O(\log n), \tag{63}$$

because $l-l^{\alpha}<k<l$, $l^{\gamma}-(l-l^{\alpha})^{\gamma}\to 0$ as $l\to\infty$ for $\gamma<1,\alpha<1$.

But $\operatorname{Var}(\alpha_k)=0$ if $p_k=0$ and

$$\operatorname{Var}(\alpha_k)=\frac{1-p_k}{p_k}\le\frac{1}{p_k}\quad\text{if } p_k\ne 0. \tag{64}$$

Then

$$\sum_{k=1}^{n}\frac{\operatorname{Var}(\alpha_k)}{k^2}\le\sum_{\substack{1\le k\le n\\ p_k\ne 0}}\frac{1}{k^2p_k}\le\sum_{k=1}^{n}\frac{1}{k^{5/2-\alpha}}=O(\log n). \tag{65}$$

Noting that

$$\begin{aligned}\operatorname{Var}&\left(\sum_{k=1}^{n}\frac{\alpha_k}{k}\right)\\&=\sum_{k=1}^{n}\frac{\operatorname{Var}(\alpha_k)}{k^2}\\&\quad+2\sum_{\substack{1\le k<l\le n\\ k>l-l^{\alpha}}}\frac{\operatorname{Cov}(\alpha_k,\alpha_l)}{kl}+2\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}}\frac{\operatorname{Cov}(\alpha_k,\alpha_l)}{kl},\end{aligned} \tag{66}$$

thus (62)–(66) imply that

$$\operatorname{Var}\left(\sum_{k=1}^{n}\frac{\alpha_k}{k}\right)=O(\log n),\quad\text{as } n\longrightarrow\infty. \tag{67}$$

Hence applying Remark 7, we have

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\alpha_k}{k}=1\quad\text{a.s.} \tag{68}$$

This completes the proof of Theorem 5. □

Appendix

Proof of Lemma 6. Let $\mu_n = \sum_{k=1}^{n} d_k \xi_k$, and then $\forall \varepsilon > 0$

$$P\left(\left|\frac{\mu_n}{D_n} - \frac{E\mu_n}{D_n}\right| \geq \varepsilon\right) \leq \frac{\operatorname{Var}(\mu_n/D_n)}{\varepsilon^2} \leq c(\log D_n)^{-\beta}. \quad (A.1)$$

Let $r < 1$, $r\beta > 1$, and $n_k = \inf\{n_k, D_{n_k} \geq \exp(k^r)\}$, and then $D_{n_k} \geq \exp(k^r)$, $D_{n_k-1} < \exp(k^r)$, for $D_n \sim D_{n-1}$; we get

$$1 \leq \frac{D_{n_k}}{\exp(k^r)} \sim \frac{D_{n_k-1}}{\exp(k^r)} < 1 \longrightarrow 1, \quad (A.2)$$

that is,

$$D_{n_k} \sim \exp(k^r), \quad (A.3)$$

and thus

$$\begin{aligned} \frac{D_{n_k}}{D_{n_{k-1}}} &\sim \frac{\exp(k^r)}{\exp((k-1)^r)} \\ &= \exp\left(k^r\left[1-\left(1-\frac{1}{k}\right)^r\right]\right) \sim \exp\left(k^r \cdot r \cdot \frac{1}{k}\right) \\ &= \exp\left(r \cdot k^{r-1}\right). \end{aligned} \quad (A.4)$$

On account of $r < 1$, then

$$\begin{aligned} &\frac{D_{n_k}}{D_{n_{k-1}}} \sim \exp\left(r \cdot k^{r-1}\right) \longrightarrow 1, \quad \text{as } k \longrightarrow \infty, \\ &\sum_{k=1}^{\infty} P\left(\left|\frac{\mu_{n_k}}{D_{n_k}} - \frac{E\mu_{n_k}}{D_{n_k}}\right| \geq \varepsilon\right) \\ &\quad \leq c\sum_{k=1}^{\infty} \frac{1}{(\log D_n)^{-\beta}} \leq c\sum_{k=1}^{\infty} \frac{1}{k^{r\beta}} < \infty. \end{aligned} \quad (A.5)$$

By the Borel-Cantelli lemma,

$$\frac{\mu_{n_k}}{D_{n_k}} - \frac{E\mu_{n_k}}{D_{n_k}} \longrightarrow 0 \quad \text{a.s.} \quad (A.6)$$

Since

$$\frac{E\mu_{n_k}}{D_{n_k}} = \frac{\sum_{k=1}^{n_k} d_k}{D_{n_k}} = 1, \quad (A.7)$$

thus

$$\frac{\mu_{n_k}}{D_{n_k}} \longrightarrow 1, \quad \text{a.s.} \quad (A.8)$$

Now for $n_{k-1} \leq n < n_k$, for $D_n \uparrow \infty$, $D_{n_k}/D_{n_{k-1}} \rightarrow 1$, and by $\xi_k \geq 0$, then $\mu_n \uparrow$, and we have

$$1 \longleftarrow \frac{D_{n_k-1}}{D_{n_k}} \frac{\mu_{n_{k-1}}}{D_{n_{k-1}}} \leq \frac{\mu_n}{D_n} \leq \frac{\mu_{n_k}}{D_{n_k}} \frac{D_{n_k}}{D_{n_{k-1}}} \longrightarrow 1 \quad \text{a.s.} \quad (A.9)$$

hence

$$\frac{\mu_n}{D_n} \longrightarrow 1 \quad \text{a.s.} \quad (A.10)$$

This completes the proof of Lemma 6. □

Proof of Lemma 10. Applying Lemma 9, (33), and noting the conditions of (19) and $0 < \alpha < 1/7$, we get

$$\begin{aligned} p_k &= P(a_k \leq S_k \leq b_k) \\ &\leq \left(\Phi\left(\frac{b_k}{k^{1/2}}\right) - \Phi\left(\frac{a_k}{k^{1/2}}\right)\right) + c\frac{1}{k^{1/5}} \\ &\leq c\left(\frac{k^{1/2-\alpha}}{k^{1/2}} + \frac{1}{k^{1/5}}\right) \leq c_2' k^{-\alpha}. \end{aligned} \quad (A.11)$$

Applying Lemma 9, (34), and noting the conditions of (19) and $0 < \alpha < 1/7$, we have

$$\begin{aligned} p_k &= P(a_k \leq S_k \leq b_k) \\ &\geq \left(\Phi\left(\frac{b_k}{k^{1/2}}\right) - \Phi\left(\frac{a_k}{k^{1/2}}\right)\right) - c\frac{1}{k^{1/5}} \\ &\geq c\left(\frac{k^{1/2-\alpha}}{k^{1/2}} + \frac{1}{k^{1/5}}\right) \geq c_1' k^{-\alpha}. \end{aligned} \quad (A.12)$$

Thus Lemma 10 immediately follows from (A.11) and (A.12). □

Proof of Lemma 11. By Lemma 10, Chebyshev's inequality, and noting the condition of $0 < \alpha < 1/7$, we have

$$\begin{aligned} &\sum_{\substack{1 \leq k < l \leq n \\ k < l - l^{\alpha}}} \frac{1}{klp_l} P\left(\left|S_{l^{\alpha}}\right| \geq \varepsilon_l\right) \\ &\quad \leq \sum_{\substack{1 \leq k < l \leq n \\ k < l - l^{\alpha}}} \frac{1}{klp_l} \frac{\operatorname{Var}(S_{l^{\alpha}})}{l^{3\alpha}} \leq c\sum_{l=1}^{n} \frac{1}{l^{1+\alpha}} \sum_{k=1}^{l-l^{\alpha}} \frac{1}{k} \\ &\quad \leq c\sum_{l=1}^{n} \frac{\log(l-l^{\alpha})}{l^{1+\alpha}} = O(\log n). \end{aligned} \quad (A.13)$$

It proves (29). By Lemma 10 and $0 < \alpha < 1/7$, we get

$$\begin{aligned} &\sum_{\substack{1 \leq k < l \leq n \\ k < l - l^{\alpha}}} \frac{1}{klp_l} \frac{1}{(l-k-l^{\alpha})^{1/5}} \\ &\leq c\sum_{l=1}^{n} \frac{1}{l^{1-\alpha}} \left(\sum_{1 \leq k < (l-l^{\alpha})/2} \frac{1}{k(l-l^{\alpha}-k)^{1/5}}\right. \\ &\qquad \left. + \sum_{(l-l^{\alpha})/2 < k < l-l^{\alpha}} \frac{1}{k(l-l^{\alpha}-k)^{1/5}}\right) \\ &\leq c\sum_{l=1}^{n} \frac{1}{l^{1-\alpha}} \left(\frac{1}{(l-l^{\alpha})^{1/5}} \sum_{1 \leq k < (l-l^{\alpha})/2} \frac{1}{k}\right. \\ &\qquad \left. + \frac{1}{l-l^{\alpha}} \sum_{1 \leq k < (l-l^{\alpha})/2} \frac{1}{k^{1/5}}\right) \\ &\leq c\sum_{l=1}^{n} \frac{\log l}{l^{1-\alpha+1/5}} \leq c\sum_{l=1}^{n} \frac{1}{l} = O(\log n). \end{aligned} \quad (A.14)$$

It proves (30). Applying Lemma 9, (33), (35), $\varepsilon_l = l^{3\alpha/2}$, and noting the condition of $0 < \alpha < 1/7$, we obtain

$$\begin{aligned}
&\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}} \frac{1}{klp_l}\left|\Phi\left(\frac{a_l-b_k-\varepsilon_l}{(l-k-l^{\alpha})^{1/2}}\right)-\Phi\left(\frac{a_l}{l^{1/2}}\right)\right| \\
&\quad\le c\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}} \frac{-a_l}{klp_l}\left(\frac{1}{(l-k-l^{\alpha})^{1/2}}-\frac{1}{\sqrt{l}}\right) \\
&\qquad + c\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}} \frac{1}{klp_l}\frac{b_k}{(l-k-l^{\alpha})^{1/2}} \\
&\qquad + c\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}} \frac{1}{klp_l}\frac{\varepsilon_l}{(l-k-l^{\alpha})^{1/2}} := \Sigma_1+\Sigma_2+\Sigma_3.
\end{aligned} \tag{A.15}$$

Now applying the same procedure as before, we have

$$\begin{aligned}
\Sigma_1 &\le c\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}} \frac{-a_l}{klp_l}\left(\frac{1}{l^{1-\alpha}(l-k-l^{\alpha})^{1/2}}+\frac{k}{l(l-k-l^{\alpha})^{1/2}}\right) \\
&\le c\sum_{1\le l\le n}\frac{1}{l^{3/2-\alpha}} \\
&\quad\times\left(\frac{1}{(l-l^{\alpha})^{1/2}}\sum_{1\le k<(l-l^{\alpha})/2}\frac{1}{k}+\frac{1}{l-l^{\alpha}}\sum_{1\le k<(l-l^{\alpha})/2}\frac{1}{k^{1/2}}\right) \\
&\quad + c\sum_{1\le l\le n}\frac{1}{l^{3/2}}\sum_{1\le k<l-l^{\alpha}}\frac{1}{(l-k-l^{\alpha})^{1/2}} \\
&\le c\sum_{1\le l\le n}\frac{\log l}{l^{2-\alpha}}\le c\sum_{1\le l\le n}\frac{1}{l}=O(\log n), \\
\Sigma_2 &\le c\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}} \frac{1}{l^{1-\alpha}k^{1/2+\alpha}(l-l^{\alpha}-k)^{1/2}} \\
&\le c\sum_{1\le l\le n}\frac{1}{l^{1-\alpha}}\left(\frac{1}{(l-l^{\alpha})^{1/2}}\sum_{1\le k<(l-l^{\alpha})/2}\frac{1}{k^{1/2+\alpha}}\right. \\
&\qquad\qquad\left.+\frac{1}{(l-l^{\alpha})^{1/2+\alpha}}\sum_{1\le k<(l-l^{\alpha})/2}\frac{1}{k^{1/2}}\right) \\
&\le c\sum_{1\le l\le n}\frac{1}{l^{1-\alpha}}\left(\frac{1}{(l-l^{\alpha})^{1/2}}l^{1/2-\alpha}+\frac{1}{(l-l^{\alpha})^{1/2+\alpha}}(l-l^{\alpha})^{1/2}\right) \\
&\le c\sum_{1\le l\le n}\frac{1}{l}=O(\log n).
\end{aligned} \tag{A.16}$$

Noting that $0 < \alpha < 1/7$, we deduce

$$\begin{aligned}
\Sigma_3 &\le c\sum_{\substack{1\le k<l\le n\\ k<l-l^{\alpha}}} \frac{1}{l^{1-(7/2)\alpha}k^{1/2}(l-l^{\alpha}-k)^{1/2}} \\
&\le c\sum_{1\le l\le n}\frac{1}{l^{1-(7/2)\alpha}} \\
&\quad\times\left(\frac{1}{(l-l^{\alpha})^{1/2}}\sum_{1\le k<(l-l^{\alpha})/2}\frac{1}{k}+\frac{1}{(l-l^{\alpha})}\sum_{1\le k<(l-l^{\alpha})/2}\frac{1}{k^{1/2}}\right) \\
&\le c\sum_{1\le l\le n}\frac{\log l}{l^{3/2-(7/2)\alpha}}\le c\sum_{1\le l\le n}\frac{1}{l}=O(\log n).
\end{aligned} \tag{A.17}$$

It proves (31). The proof of (32) is similar to the proof of (31). This completes the proof of Lemma 11. □

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