Research Article

Viscosity Method for Hierarchical Fixed Point Problems with an Infinite Family of Nonexpansive Nonself-Mappings

Yaqin Wang[1,2]

[1] *Department of Mathematics, Shaoxing University, Shaoxing 312000, China*
[2] *Mathematical College, Sichuan University, Chengdu, Sichuan 610064, China*

Correspondence should be addressed to Yaqin Wang; wangyaqin0579@126.com

A viscosity method for hierarchical fixed point problems is presented to solve variational inequalities, where the involved mappings are nonexpansive nonself-mappings. Solutions are sought in the set of the common fixed points of an infinite family of nonexpansive nonself-mappings. The results generalize and improve the recent results announced by many other authors.

1. Introduction and Preliminaries

Let X a real Banach space and J be the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x\| = \|x^*\|\} \quad (1)$$

for all $x \in X$, where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between X and X^*. If $X = H$ is a Hilbert space, then J becomes the identity mapping on H. A point $x \in C$ is a fixed point of $T : C \subset X \to X$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

Let X be a normed linear space with $\dim X \geq 2$. The modulus of smoothness of X is the function $\rho_X : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_X(\tau) := \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}. \quad (2)$$

The space X is said to be smooth if $\rho_X(\tau) > 0$, for all $\tau > 0$. It is well known that if X is smooth then J is single valued. A Banach space X is said to be strictly convex if $\|x\| = \|y\| = 1$, $x \neq y$, implies $\|x+y\|/2 < 1$.

Let C be a nonempty closed convex subset of a real Banach space X. Recall the following concepts.

Definition 1. (i) A mapping $f : C \to C$ is a ρ-contraction if $\rho \in [0, 1)$ and if the following property is satisfied

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \quad (3)$$

(ii) A mapping $T : C \to E$ is nonexpansive provided

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (4)$$

(iii) A mapping $S : C \to X$ is

(a) accretive if for any $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y) \rangle \geq 0; \quad (5)$$

(b) β-strongly accretive if for any $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y) \rangle \geq \beta \|x-y\|^2, \quad (6)$$

for some real constant $\beta > 0$.

Noting that if $S : C \to X$ is nonexpansive, then $I - S$ is accretive; if $f : C \to C$ is a ρ-contraction, then $I - f$ is $(1 - \rho)$-strongly accretive. particulary, if $X = H$ is a Hilbert space, then (strongly) accretive mappings become (strongly) monotone mappings.

Definition 2. Let C and D be nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$.

  (i) A mapping $Q : C \to D$ is called sunny, if $Q(Qx + t(x - Qx)) = Qx$ for each $x \in C$ and $t \geq 0$ with $Q(Qx + t(x - Qx)) \in C$.

  (ii) A mapping $Q : C \to D$ is called a retraction from C to D if Q is continuous and $F(Q) = D$.

  (iii) A subset D of $C \subset E$ is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction Q of C onto D. For details, see [1–3].

Note that if $X = H$ is a Hilbert space, Q becomes the projection on C, denoted by P_C.

Let $P : C \to C$ a nonexpansive self-mapping on C and $\{T_n\}$ be a countable family of nonexpansive nonself-mappings of C into X such that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then we consider the following problem: find hierarchically a common fixed point of the infinite family $\{T_n\}$ with respect to a nonexpansive mapping P; namely, find $x^* \in \mathcal{F}$, such that

$$\langle x^* - Px^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{7}$$

Particularly, if $\{T_n\}$ is a finite family of nonexpansive nonself-mappings, problem (7) has been studied by Ceng and Petruşel [4]. If $X = H$ and $\{T_n\}$ is an infinite family of nonexpansive self-mappings, Problem (7) reduces to the following problem: find hierarchically a common fixed point of $\{T_n\}$ with respect to a nonexpansive mapping P, namely, find $x^* \in \mathcal{F}$, such that

$$\langle x^* - Px^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}, \tag{8}$$

which was studied by Zhang et al. [5]. If $X = H$ is a Hilbert space and $T_n = T$, for all $n \geq 1$, where T is a nonexpansive mapping on C, then problem (7) reduces to the following problem: finding hierarchically a fixed point of T with respect to another nonexpansive mapping P; namely, find $x^* \in F(T)$ such that

$$\langle x^* - Px^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{9}$$

Problem (7) includes many problems as special cases, so it is very important in the area of optimization and related fields, such as signal processing and image reconstruction (see [6–9]).

In 2007, Moudafi [10] introduced the following Krasnoselski-Mann's algorithm in Hilbert spaces:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n (\sigma_n Px_n + (1 - \sigma_n) Tx_n),$$
$$\forall n \geq 0, \tag{10}$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are two real sequences in (0,1) and T and P are two nonexpansive mappings of C into itself. Furthermore, he established a weak convergence result for Algorithm (10) for solving problem (9).

Subsequently, Yao and Liou [11] derived a weak convergence result of algorithm (10) under the restrictions on parameters weaker than those in [10, Theorem 2.1].

Recently, Marino and Xu [12] introduced the following explicit hierarchical fixed point algorithm in Hilbert spaces:

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)(\alpha_n Vx_n + (1 - \alpha_n) Tx_n),$$
$$\forall n \geq 0, \tag{11}$$

where f is a contraction on C and V, T are two nonexpansive mappings of C into itself and proved that the sequence $\{x_n\}$ generated by (11) converges strongly to a solution of problem (9).

Very recently, Zhang et al. [5] introduced the following iterative algorithm in order to find hierarchically a fixed point of Problem (8):

$$x_0 \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \tag{12}$$
$$y_n = \beta_n P(x_n) + (1 - \beta_n) Tx_n,$$

where $f : C \to C$ is a contraction, $P : C \to C$ is a nonexpansive mapping, $\{T_n\} : C \to C$ is a countable family of nonexpansive mappings, and $T : C \to C$ is a mapping defined by

$$T = \sum_{n=1}^{\infty} \lambda_n T_n, \quad \lambda_n \geq 0 \ (n = 1, 2, \ldots) \text{ with } \sum_{n=1}^{\infty} \lambda_n = 1. \tag{13}$$

Under suitable conditions on parameters $\{\alpha_n\}$ and $\{\beta_n\}$, they established some strong and weak convergence theorems. Note that, in [5], $\{T_n\}$ is an infinite family of self-mappings and P is also a self-mapping. And they obtained the results in the setting of Hilbert spaces.

Motivated and inspired by the above researches, in a reflexive Banach space which admits a weakly sequentially continuous duality mapping J, we propose and analyze an iteration process for a countable family of nonexpansive nonself-mappings $\{T_n\} : C \to X$ and $S : C \to X$ is a nonexpansive nonself-mapping as follows:

$$x_0 \in C,$$
$$x_{n+1} = Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n), \tag{14}$$
$$y_n = \beta_n Sx_n + (1 - \beta_n) Tx_n, \quad n \geq 0,$$

where Q is a sunny nonexpansive retraction of X onto C and establishes a convergence theorem. particularly, if $X = H$ is a Hilbert space, we obtain some convergence results.

To prove the main results, we need the following lemmas.

Lemma 3 (see [1]). *Let C be a nonempty and convex subset of a smooth Banach space X, $D \subset C$, $J : X \to X^*$ the normalized duality mapping of X, and $Q : C \to D$ a retraction. Then the following conditions are equivalent:*

  (i) $\langle x - Qx, J(y - Qx) \rangle \leq 0$, *for all $x \in C$ and $y \in D$;*

  (ii) Q *is both sunny and nonexpansive.*

Lemma 4 (see [13, Lemma 3.1, 3.3]). *Let X be a real smooth and strictly convex Banach space and C a nonempty closed and*

convex subset of X which is also a sunny nonexpansive retract of X. Assuming that $T : C \rightarrow X$ is a nonexpansive mapping and Q is a sunny nonexpansive retraction of X onto C, then $F(T) = F(QT)$.

Lemma 5 (see [1]). *Let X be a real Banach space and $J : X \rightarrow 2^{X^*}$ the normalized duality mapping. Then for any $x, y \in X$, the following hold:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$, *for all* $j(x + y) \in J(x + y)$;

(ii) $\|x\|^2 + 2\langle y, j(x)\rangle \leq \|x + y\|^2$, *for all* $j(x) \in J(x)$.

Lemma 6 (see [14]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying*

$$\sum_{n=0}^{\infty} b_n < \infty, \tag{15}$$

$$a_{n+1} \leq a_n + b_n, \quad n = 0, 1, 2, \ldots .$$

Then $\lim_{n \to \infty} a_n$ exists.

Lemma 7 (see [15]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \lambda_n) a_n + \lambda_n b_n + c_n, \quad \forall n \geq 0, \tag{16}$$

where $\{\lambda_n\}, \{b_n\}$ and $\{c_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\Pi_{n=0}^{\infty}(1 - \lambda_n) = 0$;

(ii) $\limsup_{n \to \infty} b_n \leq 0$;

(iii) $c_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

If Banach space X admits sequentially continuous duality mapping J from weak topology to weak $*$ topology, then by [16, Lemma 1] we get that duality mapping J is single-valued. In this case, duality mapping J is also said to be weakly sequentially continuous, that is, for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, then $J(x_n) \rightharpoonup Jx$ [16, 17].

Recall that a Banach space X is said to be satisfying Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \rightarrow \infty)$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad \forall y \in E, \text{ with } y \neq x. \tag{17}$$

By [16, Lemma 1], we know that if X admits a weakly sequentially continuous duality mapping, then X satisfies Opial's condition.

In the sequel, we also need the following lemmas.

Lemma 8 (see [17]). *Let C be a nonempty, closed and convex subset of a reflexive Banach space X which satisfies Opial's*

condition and $T : C \rightarrow X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed at zero, that is,

$$x_n \rightharpoonup x$$

$$x_n - Tx_n \longrightarrow 0 \tag{18}$$

$$imply \ x = Tx.$$

Let C be a nonempty and convex subset of a Banach space X. Then for $x \in C$, one defines the inward set $I_C(x)$ as follows [2, 3]:

$$I_C(x) = \{y \in X : y = x + \lambda (z - x), z \in C, \lambda \geq 0\}. \tag{19}$$

A mapping $T : C \rightarrow X$ is said to satisfy the inward condition if $Tx \in I_C(x)$ for all $x \in C$. T is also said to satisfy the weakly inward condition if for each $x \in C$, $Tx \in \overline{I_C(x)}$ ($\overline{I_C(x)}$ is the closure of $I_C(x)$). Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set if C does.

Lemma 9 (see [18, Theorem 2.4]). *Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonempty closed convex subset of X which is also a sunny nonexpansive retract of X, and $T : C \rightarrow X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \neq \emptyset$. Let $\{u_n\}$ be defined by*

$$u_0 \in C,$$

$$u_{n+1} = Q(\alpha_n f(u_n) + (1 - \alpha_n) Tu_n), \tag{20}$$

where Q is a sunny nonexpansive retract of X onto C and $\alpha_n \in (0, 1)$ satisfy the following conditions:

(i) $\alpha_n \rightarrow 0$, *as* $n \rightarrow \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n \to \infty} (\alpha_n/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ converges strongly to a fixed point p of T such that p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f) p, j(p - u)\rangle \leq 0, \quad \forall u \in F(T). \tag{21}$$

Remark 10. If a Banach space X admits a sequentially continuous duality mapping J from weak topology to weak star topology, from Lemma 1 of [16] it follows that X is smooth. So for Lemma 9, if X is a reflexive and strictly convex Banach space which admits a weakly sequentially continuous duality mapping J, by Lemma 4, the weakly inward condition of T can be removed.

2. Main Results

Theorem 11. *Let X be a reflexive and strictly convex Banach space which admits a weakly sequentially continuous duality mapping $J : X \rightarrow X^*$ and C a nonempty, closed and convex subset of X which is also a sunny nonexpansive retract of X. Let $S : C \rightarrow X$ be a nonexpansive nonself-mapping, $f : C \rightarrow C$*

a contractive mapping with a contractive constant $\rho \in (0, 1)$ *and* $T_i : C \to X$ $(i = \{1, 2, \ldots\})$ *an infinite family of nonexpansive nonself-mappings such that* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* $T : C \to X$ *be defined by* (13) *and* Q *a sunny nonexpansive retraction of* X *onto* C. *Let* $\{x_n\}$ *be the sequence generated by* (14), *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *the sequences in (0,1) satisfying the following conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n \to \infty} (\beta_n / \alpha_n) = 0$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converges strongly to some point* $x^* \in F(T) = \bigcap_{i=1}^{\infty} F(T_i)$, *which is the unique solution to the following variational inequality:*

$$\langle (I - f) x^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in F(T). \tag{22}$$

Proof. From condition (ii), without loss of generality, we can assume that $\beta_n \leq \alpha_n$, for all $n \geq 0$.

First we prove that the sequence $\{x_n\}$ is bounded.

In fact, for any $u \in F(T)$, we have

$$\|x_{n+1} - u\|$$
$$= \|Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n) - Qu\|$$
$$\leq \alpha_n \|f(x_n) - u\| + (1 - \alpha_n) \|\beta_n S x_n + (1 - \beta_n) T x_n - u\|$$
$$\leq \alpha_n (\rho \|x_n - u\| + \|f(u) - u\|)$$
$$\quad + (1 - \alpha_n) (\beta_n \|S x_n - u\| + (1 - \beta_n) \|T x_n - u\|)$$
$$\leq (1 - \alpha_n (1 - \rho)) \|x_n - u\| + \alpha_n \|f(u) - u\|$$
$$\quad + (1 - \alpha_n) \beta_n \|S u - u\|$$
$$\leq (1 - \alpha_n (1 - \rho)) \|x_n - u\|$$
$$\quad + \alpha_n (\|f(u) - u\| + \|S u - u\|)$$
$$\leq \max \left\{ \|x_n - u\|, \frac{\|f(u) - u\| + \|S u - u\|}{1 - \rho} \right\}. \tag{23}$$

By induction,

$$\|x_{n+1} - u\| \leq \max \left\{ \|x_0 - u\|, \frac{\|f(u) - u\| + \|S u - u\|}{1 - \rho} \right\}. \tag{24}$$

Thus $\{x_n\}$ is bounded, so $\{S x_n\}$ and $\{T x_n\}$ are also bounded.

Next we prove that $\|x_n - u_n\| \to 0$, as $n \to \infty$, where the sequence $\{u_n\}$ is defined by

$$u_0 = x_0 \in C,$$
$$u_{n+1} = Q(\alpha_n f(u_n) + (1 - \alpha_n) T u_n). \tag{25}$$

By Lemma 9 and Remark 10, $\{u_n\}$ converges strongly to some point $x^* \in F(T)$, which is the unique solution to the following variational inequality:

$$\langle (I - f) x^*, j(x^* - x) \rangle \leq 0, \quad \forall x \in F(T). \tag{26}$$

Furthermore, we obtain

$$\|x_{n+1} - u_{n+1}\|$$
$$\leq \|Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n)$$
$$\quad - Q(\alpha_n f(u_n) + (1 - \alpha_n) T u_n)\|$$
$$\leq \|\alpha_n (f(x_n) - f(u_n)) + (1 - \alpha_n)(y_n - T u_n)\|$$
$$\leq \alpha_n \rho \|x_n - u_n\| + (1 - \alpha_n)$$
$$\quad \times (\beta_n \|S x_n - T u_n\| + (1 - \beta_n) \|T x_n - T u_n\|)$$
$$\leq (1 - \alpha_n (1 - \rho)) \|x_n - u_n\| + (1 - \alpha_n) \beta_n M$$
$$\leq (1 - \alpha_n (1 - \rho)) \|x_n - u_n\| + \beta_n M, \tag{27}$$

where $M = \sup_{n \geq 0} \|S x_n - T u_n\|$. It follows from conditions (i)-(ii) and Lemma 7 we have $\lim_{n \to \infty} \|x_n - u_n\| = 0$. Since as $n \to \infty$, $u_n \to x^* \in F(T)$, we get $x_n \to x^* (n \to \infty)$, which is the unique solution to the variational inequality (22). $\square$

Remark 12. Theorem 11 extends Theorem 2.1 in [5] from the following aspects: (i) from Hilbert spaces to reflexive and strictly convex Banach spaces which admits a weakly sequentially continuous duality mapping; (ii) for the infinite family of mappings $\{T_i\}$ from self-mappings to nonself-mappings. In addition, the existence of the sunny nonexpansive retraction has been proved in [19, Theorem 3.10].

Remark 13. If we take

$$\alpha_n = \frac{1}{(1 + n)^\alpha},$$
$$\beta_n = \frac{1}{(1 + n)^\beta}, \tag{28}$$
$$0 < \alpha < \beta < 1,$$

then since $|\alpha_{n+1} - \alpha_n| \approx 1/n^{\alpha+1}$ and $|\beta_{n+1} - \beta_n| \approx 1/n^{\beta+1}$ (as $n \to \infty$), it is not hard to find that the conditions (i)–(iii) are satisfied. For details, see [12, Remark 3.2].

In the sequel, we consider the result in the setting of Hilbert spaces.

Theorem 14. *Let H be a Hilbert space and C a nonempty, closed and convex subset of H. Let $S : C \to H$ be a nonexpansive nonself-mapping, $f : C \to C$ a contractive mapping with a contractive constant $\rho \in (0, 1)$, and $T_i : C \to H$ $(i = \{1, 2, \ldots\})$ an infinite family of nonexpansive nonself-mappings such that $F(T) = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by* (14) *and $\{\alpha_n\}$ and $\{\beta_n\}$ the sequences in (0, 1) satisfying the following conditions:*

(i) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n \to \infty} (\beta_n / \alpha_n) = \tau \in (0, +\infty)$;

(iii) $\lim_{n \to \infty} ((|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|) / \alpha_n \beta_n) = 0$;

(iv) *there exists a constant $K > 0$ such that $1/\alpha_n |(1/\beta_n) - (1/\beta_{n-1})| \leq K$ for all $n > 0$.*

Then $\{x_n\}$ converges strongly to some point $x^* \in F(T)$, which is the unique solution to the following variational inequality:

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(T).$$
(29)

Proof. By condition (ii), without loss of generality, we can assume that $\beta_n \leq (\tau + 1)\alpha_n$, for all $n \geq 0$. Similar to the proof of (24), for any $u \in F(T)$, we have

$$\|x_{n+1} - u\|$$
$$\leq \max \left\{ \|x_0 - u\|, \frac{(\tau + 1)(\|f(u) - u\| + \|Su - u\|)}{1 - \rho} \right\}.$$
(30)

Thus $\{x_n\}$ is bounded. Furthermore, $\{f(x_n)\}, \{Tx_n\}, \{y_n\}, \{Sx_n\}$ are all bounded. Put $u_n = \alpha_n f(x_n) + (1 - \alpha_n)y_n$ and $M = \sup_{n \geq 0}\{\|f(x_n)\| + \|y_n\|, \|Tx_n\| + \|Sx_n\|\}$. So $\{u_n\}$ and $\{P_C(u_n)\}$ are also bounded.

Step 1. We prove that $\|x_{n+1} - x_n\| \to 0 \ (n \to \infty)$.
From (14), we obtain

$$\|x_{n+1} - x_n\|$$
$$= \|P_C(u_n) - P_C(u_{n-1})\| \leq \|u_n - u_{n-1}\|$$
$$\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + (1 - \alpha_n)\|y_n - y_{n-1}\|$$
$$+ |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|y_{n-1}\|)$$
$$\leq \alpha_n \rho \|x_n - x_{n-1}\| + (1 - \alpha_n)$$
$$\times \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|M,$$
(31)

$$\|y_n - y_{n-1}\|$$
$$\leq \beta_n\|Sx_n - Sx_{n-1}\| + (1 - \beta_n)\|Tx_n - Tx_{n-1}\|$$
$$+ |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|Tx_{n-1}\|)$$
$$\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M.$$
(32)

Substituting (32) into (31), we have

$$\|x_{n+1} - x_n\| \leq (1 - \alpha_n(1 - \rho))\|x_n - x_{n-1}\|$$
$$+ \alpha_n \frac{(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M}{\alpha_n}.$$
(33)

By conditions (i), (iii), and Lemma 7, we have $\|x_{n+1} - x_n\| \to 0 \ (n \to \infty)$.

Step 2. We prove that $\omega_w(x_n) \subset F(T)$, where $\omega_w(x_n)$ is the ω-limit point set of $\{x_n\}$ in the weak topology:

$$\|x_{n+1} - QTx_n\|$$
$$\leq \alpha_n\|f(x_n)\| + \beta_n\|Sx_n\| + (\alpha_n + \beta_n + \alpha_n\beta_n)\|Tx_n\|.$$
(34)

Noting that $\alpha_n \to 0$ and $\beta_n \to 0$, we have $\|x_{n+1} - QTx_n\| \to 0 \ (n \to \infty)$. Then from Step 1 we have $\|x_n - QTx_n\| \to$

$0 \ (n \to \infty)$. Furthermore, it follows from Lemmas 4 and 8 that $\omega_w(x_n) \subset F(QT) = F(T)$, where $Q = P_C$.

Step 3. We show that $\|x_{n+1} - x_n\|/\beta_n \to 0 \ (n \to \infty)$.
It follows from (31) and (33) that

$$\frac{\|x_{n+1} - x_n\|}{\beta_n}$$
$$\leq \frac{\|u_n - u_{n-1}\|}{\beta_n} \leq (1 - \alpha_n(1 - \rho))\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}}$$
$$+ (1 - \alpha_n(1 - \rho))\|x_n - x_{n-1}\|\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right|$$
$$+ \frac{(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M}{\beta_n}$$
$$\leq (1 - \alpha_n(1 - \rho))\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}}$$
$$+ \alpha_n\|x_n - x_{n-1}\|K$$
$$+ \frac{(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M}{\alpha_n\beta_n}\alpha_n.$$
(35)

By conditions (i) and (iii), $\|x_n - x_{n-1}\| \to 0 \ (n \to \infty)$, and Lemma 7, we have

$$\frac{\|x_{n+1} - x_n\|}{\beta_n} \to 0 \quad (n \to \infty).$$
(36)

Thus from (35), we get

$$\frac{\|u_n - u_{n-1}\|}{\beta_n} \to 0 \quad (n \to \infty).$$
(37)

Step 4. We show that $\{x_n\}$ converges strongly to some point $x' \in F(T)$, which is the unique solution of (29).
Setting $W_n = \beta_n S + (1 - \beta_n)T$, we have

$$x_{n+1} = P_C(u_n) - u_n + \alpha_n f(x_n) + (1 - \alpha_n)W_n x_n.$$
(38)

Then

$$x_n - x_{n+1}$$
$$= u_n - P_C(u_n) + \alpha_n(I - f)x_n + (1 - \alpha_n)(I - W_n)x_n.$$
(39)

Letting $v_n = (x_n - x_{n+1})/(1 - \alpha_n)\beta_n$, from condition (i) and (36), we have $v_n \to 0 \ (n \to \infty)$. Noting that $I - W_n$ is

monotone and $I - f$ is $(1 - \rho)$-strongly monotone, for any $x^* \in F(T)$, from Lemma 3 we obtain

$$
\begin{aligned}
&\langle v_n, x_n - x^* \rangle \\
&= \frac{1}{(1 - \alpha_n)\beta_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
&\quad + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \langle (I - f) x_n, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I - W_n) x_n, x_n - x^* \rangle \\
&= \frac{1}{(1 - \alpha_n)\beta_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
&\quad + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \langle (I - f) x_n, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I - W_n) x_n - (I - W_n) x^*, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I - W_n) x^*, x_n - x^* \rangle \\
&= \frac{1}{(1 - \alpha_n)\beta_n} \langle u_n - P_C(u_n), P_C(u_n) - x^* \rangle \\
&\quad + \frac{1}{(1 - \alpha_n)\beta_n} \\
&\quad \times \langle u_n - P_C(u_n), -(P_C(u_n) - x^*) + (P_C(u_{n-1}) - x^*) \rangle \\
&\quad + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \langle (I - f) x_n - (I - f) x^*, x_n - x^* \rangle \\
&\quad + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \langle (I - f) x^*, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I - W_n) x_n - (I - W_n) x^*, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I - W_n) x^*, x_n - x^* \rangle \\
&\geq \frac{1}{(1 - \alpha_n)\beta_n} \langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) \rangle \\
&\quad + \frac{\alpha_n (1 - \rho)}{(1 - \alpha_n)\beta_n} \| x_n - x^* \|^2 \\
&\quad + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \langle (I - f) x^*, x_n - x^* \rangle \\
&\quad + \langle (I - S) x^*, x_n - x^* \rangle .
\end{aligned}
\tag{40}
$$

Thus we have

$$
\begin{aligned}
\| x_n - x^* \|^2 \\
\leq \frac{(1 - \alpha_n)\beta_n}{\alpha_n (1 - \rho)} \langle v_n, x_n - x^* \rangle
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{(1 - \alpha_n)\beta_n}{\alpha_n (1 - \rho)} \langle (I - S) x^*, x_n - x^* \rangle \\
&\quad - \frac{1}{\alpha_n (1 - \rho)} \langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) \rangle \\
&\quad - \frac{1}{(1 - \rho)} \langle (I - f) x^*, x_n - x^* \rangle \\
&\leq \frac{(1 - \alpha_n)\beta_n}{\alpha_n (1 - \rho)} \| v_n \| \| x_n - x^* \| \\
&\quad - \frac{(1 - \alpha_n)\beta_n}{\alpha_n (1 - \rho)} \langle (I - S) x^*, x_n - x^* \rangle \\
&\quad + \frac{1}{(1 - \rho)} \| u_n - P_C(u_n) \| \left\| \frac{u_{n-1} - u_n}{\alpha_n} \right\| \\
&\quad - \frac{1}{(1 - \rho)} \langle (I - f) x^*, x_n - x^* \rangle .
\end{aligned}
\tag{41}
$$

Since $\beta_n \leq (\tau + 1)\alpha_n$, by (37) we have

$$
\frac{\| u_n - u_{n-1} \|}{\alpha_n} \longrightarrow 0 \quad (n \longrightarrow \infty) .
\tag{42}
$$

Combining condition (ii), $v_n \to 0 \ (n \to \infty)$, (41), and (42), every weak cluster point of $\{x_n\}$ is also a strong cluster point. From (40), we obtain

$$
\begin{aligned}
&\langle (I - f) x_n, x_n - x^* \rangle \\
&= \frac{(1 - \alpha_n)\beta_n}{\alpha_n} \langle v_n, x_n - x^* \rangle \\
&\quad - \frac{1}{\alpha_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
&\quad - \frac{(1 - \alpha_n)}{\alpha_n} \langle (I - W_n) x_n, x_n - x^* \rangle \\
&= \frac{(1 - \alpha_n)\beta_n}{\alpha_n} \langle v_n, x_n - x^* \rangle \\
&\quad - \frac{1}{\alpha_n} \langle u_n - P_C(u_n), P_C(u_n) - x^* \rangle \\
&\quad - \frac{1}{\alpha_n} \langle u_n - P_C(u_n), P_C(u_{n-1}) \\
&\qquad\qquad - P_C(u_n) \rangle - \frac{(1 - \alpha_n)}{\alpha_n} \\
&\quad \times \langle (I - W_n) x_n - (I - W_n) x^*, x_n - x^* \rangle \\
&\quad - \frac{(1 - \alpha_n)}{\alpha_n} \langle (I - W_n) x^*, x_n - x^* \rangle
\end{aligned}
$$

$$\leq \frac{(1 - \alpha_n)\beta_n}{\alpha_n} \|v_n\| \|x_n - x^*\|$$

$$+ \frac{1}{\alpha_n} \|u_n - P_C(u_n)\| \|P_C(u_{n-1}) - P_C(u_n)\|$$

$$- \frac{(1 - \alpha_n)\beta_n}{\alpha_n} \langle (I - S) x^*, x_n - x^* \rangle$$

$$\leq \frac{(1 - \alpha_n)\beta_n}{\alpha_n} \|v_n\| \|x_n - x^*\|$$

$$+ \frac{\|u_{n-1} - u_n\|}{\alpha_n} \|u_n - P_C(u_n)\|$$

$$- \frac{(1 - \alpha_n)\beta_n}{\alpha_n} \langle (I - S) x^*, x_n - x^* \rangle. \tag{43}$$

Note that the sequence $\{x_n\}$ is bounded; thus there exists a subsequence $\{x_{n_j}\}$ converging to a point $x' \in H$. From Step 2, we have $x' \in F(T)$. Then it follows from the above inequality, (42), and $v_n \to 0$ $(n \to \infty)$ that

$$\langle (I - f) x', x' - x^* \rangle$$
$$\leq -\tau \langle (I - S) x^*, x' - x^* \rangle, \quad \forall x^* \in F(T). \tag{44}$$

Replacing x^* with $x' + \mu(x^* - x')$, where $\mu \in (0, 1)$ and $x^* \in F(T)$, we have

$$\langle (I - f) x', x' - x^* \rangle$$
$$\leq -\tau \langle (I - S)(x' + \mu(x^* - x')), x' - x^* \rangle, \tag{45}$$
$$\forall x^* \in F(T).$$

Letting $\mu \to 0$, we have

$$\langle (I - f) x', x' - x^* \rangle$$
$$\leq -\tau \langle (I - S) x', x' - x^* \rangle, \quad \forall x^* \in F(T). \tag{46}$$

If there exists another subsequence $\{x'_{n_j}\}$ of $\{x_n\}$ converging to a point $x'' \in H$. From Step 2, we also have $x'' \in F(T)$. Then from (46) we obtain

$$\langle (I - f) x', x' - x'' \rangle \leq -\tau \langle (I - S) x', x' - x'' \rangle \tag{47}$$

and, via interchanging x' and x'',

$$\langle (I - f) x'', x'' - x' \rangle \leq -\tau \langle (I - S) x'', x'' - x' \rangle. \tag{48}$$

Adding up these two inequalities yields

$$(1 - \rho) \|x' - x''\|^2 \leq \langle (I - f) x' - (I - f) x'', x' - x'' \rangle \leq 0, \tag{49}$$

which implies $x' = x''$. Then $\{x_n\}$ converges strongly to $x' \in F(T)$, which is the solution to the following variational inequality:

$$\left\langle \frac{1}{\tau}(I - f) x' + (I - S) x', x - x' \right\rangle \geq 0, \quad \forall x \in F(T). \tag{50}$$

Since $I - f$ is $(1 - \rho)$-strongly monotone and $I - S$ is monotone, it is easy to see that the above variational inequality has a unique solution. □

Remark 15. Theorem 14 extends Theorem 3.2 in [12] from the following aspects: (i) from a nonexpansive mapping T to an infinite family of nonexpansive mappings $\{T_i\}$; (ii) from self-mappings to nonself-mappings.
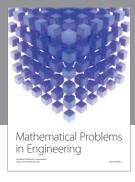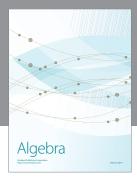
Acknowledgments

References

[1] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, Japan, 2000.

[2] H.-K. Xu, "Approximating curves of nonexpansive nonself-mappings in Banach spaces," *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, vol. 325, no. 2, pp. 151–156, 1997.

[3] W. Takahashi and G.-E. Kim, "Strong convergence of approximants to fixed points of nonexpansive nonself-mappings in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 32, no. 3, pp. 447–454, 1998.

[4] L. C. Ceng and A. Petruşel, "Krasnoselski-Mann iterations for hierarchical fixed point problems for a finite family of nonself mappings in Banach spaces," *Journal of Optimization Theory and Applications*, vol. 146, no. 3, pp. 617–639, 2010.

[5] S.-A. Zhang, X.-R. Wang, H. W. J. Lee, and C.-K. Chan, "Viscosity method for hierarchical fixed point and variational inequalities with applications," *Applied Mathematics and Mechanics. English Edition*, vol. 32, no. 2, pp. 241–250, 2011.

[6] D. C. Youla, "Mathematical theory of image restoration by the method of convex projections," in *Image Recovery: Theory and Applications*, H. Stark, Ed., pp. 29–78, Academic Press, New York, NY, USA, 1987.

[7] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[8] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000)*, vol. 8 of *Studies in Computational Mathematics*, pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[9] I. Yamada, N. Ogura, and N. Shirakawa, "A numerically robust hybrid steepest descent method for the convexly constrained generalized inverse problems," in *Inverse Problems, Image*
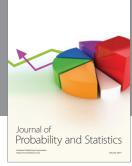
Analysis, and Medical Imaging (New Orleans, LA, 2001), vol. 313 of *Contemporary Mathematics*, pp. 269–305, American Mathematical Society, Providence, RI, USA, 2002.

[10] A. Moudafi, "Krasnoselski-Mann iteration for hierarchical fixed-point problems," *Inverse Problems*, vol. 23, no. 4, pp. 1635–1640, 2007.

[11] Y. Yao and Y.-C. Liou, "Weak and strong convergence of Krasnoselski-Mann iteration for hierarchical fixed point problems," *Inverse Problems*, vol. 24, no. 1, Article ID 015015, 8 pages, 2008.

[12] G. Marino and H.-K. Xu, "Explicit hierarchical fixed point approach to variational inequalities," *Journal of Optimization Theory and Applications*, vol. 149, no. 1, pp. 61–78, 2011.

[13] S.-Y. Matsushita and W. Takahashi, "Strong convergence theorems for nonexpansive nonself-mappings without boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 2, pp. 412–419, 2008.

[14] P. L. Combettes, "Quasi-Fejérian analysis of some optimization algorithms," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000)*, D. Butnariu, Y. Censor, and S. Reich, Eds., vol. 8 of *Studies in Computational Mathematics*, pp. 115–152, North-Holland, Amsterdam, The Netherlands, 2001.

[15] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society. Second Series*, vol. 66, no. 1, pp. 240–256, 2002.

[16] J.-P. Gossez and E. Lami Dozo, "Some geometric properties related to the fixed point theory for nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 40, no. 3, pp. 565–573, 1972.

[17] J. S. Jung, "Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 509–520, 2005.

[18] Y. Song and R. Chen, "Viscosity approximation methods for nonexpansive nonself-mappings," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 1, pp. 316–326, 2006.

[19] J. G. O'Hara, P. Pillay, and H.-K. Xu, "Iterative approaches to convex feasibility problems in Banach spaces," *Nonlinear Analysis. Theory, M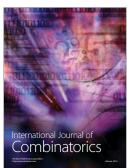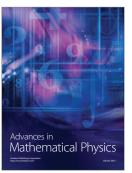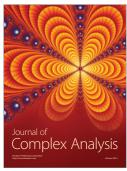ethods & Applications*, vol. 64, no. 9, pp. 2022–2042, 2006.