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Research Article

Coupled Fixed Point Theorems for a Pair of Weakly Compatible Maps along with *CLRg* Property in Fuzzy Metric Spaces

Manish Jain,[1] Kenan Tas,[2] Sanjay Kumar,[3] and Neetu Gupta[4]

[1] *Department of Mathematics, Ahir College, Rewari 123401, India*
[2] *Department of Mathematics and Computer Science, Cankaya University, Ankara, Turkey*
[3] *Department of Mathematics, DCRUST, Murthal, Sonepat 131039, India*
[4] *HAS Department, YMCAUST, Faridabad, India*

Correspondence should be addressed to Kenan Tas, kenan@cankaya.edu.tr

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The aim of this paper is to extend the notions of E.A. property and *CLRg* property for coupled mappings and use these notions to generalize the recent results of Xin-Qi Hu (2011). The main result is supported by a suitable example.

1. Introduction and Preliminaries

The concept of fuzzy set was introduced by Zadeh [1] and after his work there has been a great endeavor to obtain fuzzy analogues of classical theories. This problem has been searched by many authors from different points of view. In 1994, George and Veeramani [2] introduced and studied the notion of fuzzy metric space and defined a Hausdorff topology on this fuzzy metric space.

Bhaskar and Lakshmikantham [3] introduced the notion of coupled fixed points and proved some coupled fixed point results in partially ordered metric spaces. The work [3] was illustrated by proving the existence and uniqueness of the solution for a periodic boundary value problem. These results were further extended and generalized by Lakshmikantham and Cirić [4] to coupled coincidence and coupled common fixed point results for nonlinear contractions in partially ordered metric spaces.

Sedghi et al. [5] proved some coupled fixed point theorems under contractive conditions in fuzzy metric spaces. The results proved by Fang [6] for compatible and weakly compatible mappings under ϕ-contractive conditions in Menger spaces that provide a tool to Hu [7] for proving fixed points results for coupled mappings and these results are the genuine generalization of the result of [5].

Aamri and Moutawakil [8] introduced the concept of E.A. property in a metric space. Recently, Sintunavarat and Kuman [9] introduced a new concept of (*CLRg*). The importance of *CLRg* property ensures that one does not require the closeness of range subspaces.

In this paper, we give the concept of E.A. property and (*CLRg*) property for coupled mappings and prove a result which provides a generalization of the result of [7].

2. Preliminaries

Before we give our main result, we need the following preliminaries.

Definition 2.1 (see [1]). A fuzzy set A in X is a function with domain X and values in $[0,1]$.

Definition 2.2 (see [10]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm, if $([0,1], *)$ is a topological abelian monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Some examples are below:

(i) $*(a,b) = ab$,

(ii) $*(a,b) = \min(a,b)$.

Definition 2.3 (see [11]). Let $\sup_{t \in (0,1)} \Delta(t,t) = 1$. A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = t, \quad \Delta(\Delta^m) = \Delta^{m+1}(t) = t. \tag{2.1}$$

A t-norm Δ is an H-type t-norm if and only if for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that $\Delta^m(t) > (1-\lambda)$ for all $m \in \mathbb{N}$, when $t > (1-\delta)$.

The t-norm $\Delta_M = \min$ is an example of t-norm, of H-type.

Definition 2.4 (see [2]). The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:

(FM-1) $M(x,y,0) > 0$ for all $x, y \in X$,

(FM-2) $M(x,y,t) = 1$ if and only if $x = y$, for all $x, y \in X$ and $t > 0$,

(FM-3) $M(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t > 0$,

(FM-4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ for all $x, y, z \in X$ and $t, s > 0$,

(FM-5) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is continuous for all $x, y \in X$.

In present paper, we consider M to be fuzzy metric space with, the following condition:

(FM-6) $\lim_{t \to \infty} M(x,y,t) = 1$, for all $x, y \in X$ and $t > 0$.

Definition 2.5 (see [2]). Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\} \in X$ is said to be:

(i) convergent to a point $x \in X$, if for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \tag{2.2}$$

(ii) a Cauchy sequence, if for all $t > 0$ and $p > 0$,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1. \tag{2.3}$$

A fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

We note that $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$.

Lemma 2.6 (see [12]). *Let $x_n \to x$ and $y_n \to y$, then for all $t > 0$:*

(i) $\lim_{n\to\infty} M(x_n, y_n, t) \geq M(x, y, t)$,

(ii) $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ *if $M(x, y, t)$ is continuous.*

Definition 2.7 (see [7]). Define $\Phi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+\}$, and each $\phi \in \Phi$ satisfies the following conditions:

(ϕ-1) ϕ is nondecreasing;

(ϕ-2) ϕ is upper semicontinuous from the right;

(ϕ-3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t)), n \in \mathbb{N}$.

Clearly, if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

Definition 2.8 (see [4]). An element $(x, y) \in X \times X$ is called:

(i) a coupled fixed point of the mapping $f : X \times X \to X$ if $f(x, y) = x$, $f(y, x) = y$,

(ii) a coupled coincidence point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $f(x, y) = g(x)$, $f(y, x) = g(y)$,

(iii) a common coupled fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $x = f(x, y) = g(x)$, $y = f(y, x) = g(y)$.

Definition 2.9 (see [6]). An element $x \in X$ is called a common fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $x = f(x, x) = g(x)$.

Definition 2.10 (see [6]). The mappings $f : X \times X \to X$ and $g : X \to X$ are called:

(i) commutative if $gf(x, y) = f(gx, gy)$ for all $x, y \in X$,

(ii) compatible if

$$\begin{aligned} &\lim_{n\to\infty} M(gf(x_n, y_n), f(g(x_n), g(y_n)), t) = 1, \\ &\lim_{n\to\infty} M(gf(y_n, x_n), f(g(y_n), g(x_n)), t) = 1, \end{aligned} \tag{2.4}$$

for all $t > 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, and $\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$, for some $x, y \in X$.

Definition 2.11 (see [13]). The maps $f : X \times X \to X$ and $g : X \to X$ are called w-compatible if $gf(x,y) = f(gx,gy)$ whenever $f(x,y) = g(x), f(y,x) = g(y)$.

We note that the maps $f : X \times X \to X$ and $g : X \to X$ are called weakly compatible if

$$f(x,y) = g(x), \qquad f(y,x) = g(y), \tag{2.5}$$

implies $gf(x,y) = f(gx,gy)$, $gf(y,x) = f(gy,gx)$, for all $x,y \in X$.

There exist pair of mappings that are neither compatible nor weakly compatible, as shown in the following example.

Example 2.12. Let $(X,M,*)$ be a fuzzy metric space, $*$ being a continuous norm with $X = [0,1)$. Define $M(x,y,t) = t/(t+|x-y|)$ for all $t > 0$, $x,y \in X$. Also define the maps $f : X\times X \to X$ and $g : X \to X$ by $f(x,y) = (x^2/2) + (y^2/2)$ and $g(x) = x/2$, respectively. Note that $(0,0)$ is the coupled coincidence point of f and g in X. It is clear that the pair (f,g) is weakly compatible on X.

We next show that the pair (f,g) is not compatible.

Consider the sequences $\{x_n\} = \{(1/2)+(1/n)\}$ and $\{y_n\} = \{(1/2)-(1/n)\}$, $n \geq 3$, then

$$\begin{aligned} \lim_{n\to\infty} f(x_n,y_n) &= \frac{1}{4} = \lim_{n\to\infty} g(x_n), \\ \lim_{n\to\infty} f(y_n,x_n) &= \frac{1}{4} = \lim_{n\to\infty} g(y_n), \end{aligned} \tag{2.6}$$

but

$$M(f(gx_n,gy_n),gf(x_n,y_n),t) = \frac{t}{t+\left|f(gx_n,gy_n)-gf(x_n,y_n)\right|} = \frac{t}{t+(1/8)((1/2)+(2/n^2))}, \tag{2.7}$$

which is not convergent to 1 as $n \to \infty$.

Hence the pair (f,g) is not compatible.

We note that, if f and g are compatible then they are weakly compatible. But the converse need not be true, as shown in the following example.

Example 2.13. Let $(X,M,*)$ be a fuzzy metric space, $*$ being a continuous norm with $X = [2,20]$. Define $M(x,y,t) = t/(t+|x-y|)$ for all $t > 0$, $x,y \in X$. Define the maps $f : X\times X \to X$ and $g : X \to X$ by

$$\begin{gathered} f(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5,\ y \in X, \\ 6, & \text{if } 2 < x \leq 5,\ y \in X, \end{cases} \\ g(x) = \begin{cases} 2, & \text{if } x = 2, \\ 12, & \text{if } 2 < x \leq 5, \\ x-3, & x > 5. \end{cases} \end{gathered} \tag{2.8}$$

The only coupled coincidence point of the pair (f, g) is $(2, 2)$. The mappings f and g are noncompatible, since for the sequences $\{x_n\} = \{y_n\} = \{5+(1/n)\}$, $n \geq 1$ we have $f(x_n, y_n) = 2$, $g(x_n) \to 2$, $f(y_n, x_n) = 2$, $g(y_n) \to 2$, $M(f(gx_n, gy_n), g(f(x_n, y_n)), t) = t/(t+4) \nrightarrow 1$ as $n \to \infty$. But they are weakly compatible since they commute at their coupled coincidence point $(2, 2)$.

Now we introduce our notions.

Aamri and El Moutawakil [8] introduced the concept of E.A. property in a metric space as follows.

Let (X, d) be a metric space. Self mappings $f : X \to X$ and $g : X \to X$ are said to satisfy E.A. property if there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \tag{2.9}$$

for some $t \in X$.

Now we extend this notion for a pair of coupled maps as follows.

Definition 2.14. Let (X, d) be a metric space. Two mappings $f : X \times X \to X$ and $g : X \to X$ are said to satisfy E.A. property if there exists sequences $\{x_n\}, \{y_n\} \in X$ such that

$$\begin{aligned}
\lim_{n\to\infty} f(x_n, y_n) &= \lim_{n\to\infty} g(x_n) = x, \\
\lim_{n\to\infty} f(y_n, x_n) &= \lim_{n\to\infty} g(y_n) = y,
\end{aligned} \tag{2.10}$$

for some $x, y \in X$.

In a similar mode, we state E.A. property for coupled mappings in fuzzy metric spaces as follows.

Let $(X, M, *)$ be a FM space. Two maps $f : X \times X \to X$ and $g : X \to X$ satisfy E.A. property if there exists sequences $\{x_n\}$ and $\{y_n\} \in X$ such that $f(x_n, y_n)$, $g(x_n)$ converges to x and $f(y_n, x_n)$, $g(y_n)$ converges to y in the sense of Definition 2.5.

Example 2.15. Let $(-\infty, \infty)$ be a usual metric space. Define mappings $f : X \times X \to X$ and $g : X \to X$ by $f(x, y) = x^2 + y^2$ and $g(x) = 2x$ for all $x, y \in X$. Consider the sequences $\{x_n\} = \{1/n\}$ and $\{y_n\} = \{-1/n\}$. Since

$$\begin{aligned}
\lim_{n\to\infty} f(x_n, y_n) &= \lim_{n\to\infty} f\left(\frac{1}{n}, -\frac{1}{n}\right) = 0 = \lim_{n\to\infty} g\left(\frac{1}{n}\right) = \lim_{n\to\infty} g(x_n), \\
\lim_{n\to\infty} f(y_n, x_n) &= \lim_{n\to\infty} f\left(-\frac{1}{n}, \frac{1}{n}\right) = 0 = \lim_{n\to\infty} g\left(-\frac{1}{n}\right) = \lim_{n\to\infty} g(y_n),
\end{aligned} \tag{2.11}$$

therefore, f and g satisfy E.A. property, since $0 \in X$.

Remark 2.16. It is to be noted that property E.A. need not imply compatibility, since in Example 2.12, the maps f and g defined are not compatible, but satisfy property E.A., since for the sequences $\{x_n\} = \{(1/2) + (1/n)\}$ and $\{x_n\} = \{(1/2) - (1/n)\}$ we have

$$\lim_{n\to\infty} f(x_n, y_n) = \frac{1}{4} = \lim_{n\to\infty} g(x_n),$$
$$\lim_{n\to\infty} f(y_n, x_n) = \frac{1}{4} = \lim_{n\to\infty} g(y_n), \tag{2.12}$$

since $1/4 \in X$.

Recently, Sintunavarat and Kuman [9] introduced a new concept of *the common limit in the range of* g, $(CLRg)$ property, as follows.

Definition 2.17. Let (X, d) be a metric space. Two mappings $f : X \to X$ and $g : X \to X$ are said to satisfy $(CLRg)$ property if there exists a sequence $\{x_n\} \in X$ such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(p)$ for some $p \in X$.

Now we extend this notion for a pair of coupled mappings as follows.

Definition 2.18. Let (X, d) be a metric space. Two mappings $f : X \times X \to X$ and $g : X \to X$ are said to satisfy $(CLRg)$ property if there exists sequences $\{x_n\}, \{y_n\} \in X$ such that

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = g(p),$$
$$\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = g(q), \tag{2.13}$$

for some $p, q \in X$.

Similarly, we state $(CLRg)$ property for coupled mappings in fuzzy metric spaces.

Let $(X, M, *)$ be an FM space. Two maps $f : X \times X \to X$ and $g : X \to X$ satisfy $(CLRg)$ property if there exists sequences $\{x_n\}, \{y_n\} \in X$ such that $f(x_n, y_n), g(x_n)$ converge to $g(p)$ and $f(y_n, x_n)$, $g(y_n)$ converge to $g(q)$, in the sense of Definition 2.5.

Example 2.19. Let $X = [0, \infty)$ be a metric space under usual metric. Define mappings $f : X \times X \to X$ and $g : X \to X$ by $f(x, y) = x + y + 2$ and $g(x) = 2(1 + x)$ for all $x, y \in X$. We consider the sequences $\{x_n\} = \{1 + (1/n)\}$ and $\{x_n\} = \{1 - (1/n)\}$. Since

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} f\left(1 + \frac{1}{n}, 1 - \frac{1}{n}\right) = 4 = g(1) = \lim_{n\to\infty} g\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} g(x_n),$$
$$\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} f\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = 4 = g(1) = \lim_{n\to\infty} g\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} g(y_n), \tag{2.14}$$

therefore, the maps f and g satisfy $(CLRg)$ property.

In the next example, we show that the maps satisfying $(CLRg)$ property need not be continuous, that is, continuity is not the necessary condition for self maps to satisfy $(CLRg)$ property.

Example 2.20. Let $X = [0,\infty)$ be a metric space under usual metric. Define mappings $f : X\times X \to X$ and $g : X \to X$ by

$$f(x,y) = \begin{cases} x+y, & \text{if } x \in [0,1),\ y \in X, \\ \dfrac{x+y}{2}, & \text{if } x \in [1,\infty),\ y \in X, \end{cases}$$
$$g(x) = \begin{cases} 1+x, & \text{if } x \in [0,1), \\ \dfrac{x}{2}, & \text{if } x \in [1,\infty). \end{cases} \tag{2.15}$$

We consider the sequences $\{x_n\} = \{1/n\}$ and $\{y_n\} = \{1+(1/n)\}$. Since

$$\lim_{n\to\infty} f(x_n,y_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, 1+\frac{1}{n}\right) = 1 = g(0) = \lim_{n\to\infty} g\left(\frac{1}{n}\right) = \lim_{n\to\infty} g(x_n),$$
$$\lim_{n\to\infty} f(y_n,x_n) = \lim_{n\to\infty} f\left(1+\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} = g(1) = \lim_{n\to\infty} g\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} g(y_n), \tag{2.16}$$

therefore, the maps f and g satisfy $(CLRg)$ property but the maps are not continuous.

We next show that the pair of maps satisfying $(CLRg)$ property may not be compatible.

Example 2.21. Let $(X, M, *)$ be a fuzzy metric space, $*$ being a continuous norm, $X = [0, 1/2)$, and $M(x,y,t) = t/(t+|x-y|)$ for all $x,y \in X$ and $t > 0$.

Define the maps $f : X\times X \to X$ and $g : X \to X$ by $f(x,y) = (x^2+y^2)/2$ and $g(x) = x/3$, respectively.

Consider the sequences $\{x_n\} = \{(1/3)+(1/n)\}$ and $\{y_n\} = \{(1/3)-(1/n)\}, n > 7$. Then

$$\lim_{n\to\infty} f(x_n,y_n) = \frac{1}{9} = \lim_{n\to\infty} g(x_n),$$
$$\lim_{n\to\infty} f(y_n,x_n) = \frac{1}{9} = \lim_{n\to\infty} g(y_n). \tag{2.17}$$

Further there exists the point $1/3$ in X such that $g(1/3) = 1/9$, so that the pair (f,g) satisfies $(CLRg)$ property. But,

$$\begin{aligned} &M(f(gx_n, gy_n), gf(x_n,y_n), t) \\ &\quad = \frac{t}{t+|f(gx_n,gy_n) - gf(x_n,y_n)|} = \frac{t}{t+(1/18)((1/9)+(1/n^2))} \end{aligned} \tag{2.18}$$

does not converge to 1 as $n \to \infty$.

Hence, the pair (f,g) is not compatible.

3. Main Results

For convenience, we denote

(1)

$$\left[M(x,y,t)\right]^n = \frac{M(x,y,t) * M(x,y,t) * \cdots * M(x,y,t)}{n}, \tag{3.1}$$

for all $n \in \mathbb{N}$.

Hu [7] proved the following result.

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space where $*$ is a continuous t-norm of H-type. Let $f : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ such that*

(2)

$$M(f(x,y), f(u,v), \phi(t)) \geq M(gx, gu, t) * M(gy, gv, t), \tag{3.2}$$

for all $x, y, u, v \in X$ and $t > 0$. Suppose that $f(X \times X) \subseteq g(X)$, g is continuous, f and g are compatible maps. Then there exists a unique point $x \in X$ such that $x = g(x) = f(x,x)$, that is, f and g have a unique common fixed point in X.

We now give our main result which provides a generalization of Theorem 3.1.

Theorem 3.2. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type. Let $f : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ satisfying (2) with the following conditions:*

(3) *the pair (f, g) is weakly compatible,*

(4) *the pair (f, g) satisfy $(CLRg)$ property.*

Then f and g have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = f(x,x) = g(x)$.

Proof. Since f and g satisfy $(CLRg)$ property, there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = g(p), \qquad \lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = g(q), \tag{3.3}$$

for some $p, q \in X$.

Step 1. To show that f and g have a coupled coincidence point. From (2),

$$M(f(x_n, y_n), f(p,q), t) \geq M(f(x_n, y_n), f(p,q), \phi(t)) \geq M(gx_n, g(p), t) * M(gy_n, g(q), t). \tag{3.4}$$

Taking limit $n \to \infty$, we get $M(g(p), f(p,q), t) = 1$, that is, $f(p,q) = g(p) = x$.

Similarly, $f(q,p) = g(q) = y$.

Since f and g are weakly compatible, so that $f(p,q) = g(p) = x(\text{say})$ and $f(q,p) = g(q) = y(\text{say})$ implies $gf(p,q) = f(g(p),g(q))$ and $gf(q,p) = f(g(q),g(p))$, that is, $g(x) = f(x,y)$ and $g(y) = f(y,x)$. Hence f and g have a coupled coincidence point.

Step 2. To show that $g(x) = x$, and $g(y) = y$. Since $*$ is a t-norm of H-type, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * \cdots * (1-\delta)}_{p} \geq (1-\epsilon), \tag{3.5}$$

for all $p \in \mathbb{N}$.

Since $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$, there exists $t_0 > 0$ such that

$$M(gx,x,t_0) \geq (1-\delta), \qquad M(gy,y,t_0) \geq (1-\delta). \tag{3.6}$$

Also since $\phi \in \Phi$ using condition $(\phi - 3)$, we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$.

Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$. From (2), we have

$$\begin{aligned} M(gx,x,\phi(t_0)) &= M(f(x,y),f(p,q),\phi(t_0)) \geq M(gx,gp,t_0) * M(gy,gq,t_0) \\ &= M(gx,x,t_0) * M(gy,y,t_0), \\ M(gy,y,\phi(t_0)) &= M(f(y,x),f(q,p),\phi(t_0)) \geq M(gy,gq,t_0) * M(gx,gp,t_0) \\ &= M(gy,y,t_0) * M(gx,x,t_0). \end{aligned} \tag{3.7}$$

Similarly, we can also get

$$\begin{aligned} M\left(gx,x,\phi^2(t_0)\right) &= M\left(f(x,y),f(p,q),\phi^2(t_0)\right) \\ &\geq M(gx,gp,\phi(t_0)) * M(gy,gq,\phi(t_0)) \\ &= M(gx,x,\phi(t_0)) * M(gy,y,\phi(t_0)) \\ &\geq [M(gx,x,t_0)]^2 * [M(gy,y,t_0)]^2, \\ M\left(gy,y,\phi^2(t_0)\right) &= M\left(f(y,x),f(q,p),\phi^2(t_0)\right) \\ &\geq [M(gy,y,t_0)]^2 * [M(gx,x,t_0)]^2. \end{aligned} \tag{3.8}$$

Continuing in the same way, we can get for all $n \in \mathbb{N}$,

$$\begin{aligned} M(gx,x,\phi^n(t_0)) &= M\left(gx,x,\phi^{n-1}(t_0)\right) * M\left(gy,y,\phi^{n-1}(t_0)\right) \\ &\geq M(gx,x,t_0)^{2^{n-1}} * M(gy,y,t_0)^{2^{n-1}}, \\ M(gy,y,\phi^n(t_0)) &\geq [M(gy,y,t_0)]^{2^{n-1}} * [M(gx,x,t_0)]^{2^{n-1}}. \end{aligned} \tag{3.9}$$

Then, we have

$$\begin{aligned}
M(gx,x,t) &\geq M\left(gx,x,\sum_{k=n_0}^{\infty}\phi^k(t_0)\right)\\
&\geq M(gx,x,\phi^{n_0}t_0)\\
&\geq \left[M(gx,x,t_0)\right]^{2^{n_0-1}} * \left[M(gy,y,t_0)\right]^{2^{n_0-1}}\\
&\geq \underbrace{(1-\delta)*\cdots*(1-\delta)}_{2^{n_0}} \geq (1-\epsilon).
\end{aligned} \tag{3.10}$$

So, for any $\epsilon > 0$, we have $M(gx,x,t) \geq (1-\epsilon)$ for all $t > 0$.

This implies $g(x) = x$. Similarly, $g(y) = y$.

Step 3. Next we shall show that $x = y$. Since $*$ is a t-norm of H-type, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\underbrace{(1-\delta)*\cdots*(1-\delta)}_{p} \geq (1-\epsilon), \tag{3.11}$$

for all $p \in \mathbb{N}$.

Since $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$, there exists $t_0 > 0$ such that $M(x,y,t_0) \geq (1-\delta)$.

Also since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty}\phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$t > \sum_{k=n_0}^{\infty}\phi^k(t_0). \tag{3.12}$$

Using condition (2), we have

$$\begin{aligned}
M(x,y,\phi(t_0)) &= M(f(p,q),f(q,p),\phi(t_0)) \geq M(gp,gq,t_0) * M(gq,gp,t_0)\\
&= M(x,y,t_0) * M(y,x,t_0).
\end{aligned} \tag{3.13}$$

Continuing in the same way, we can get for all $n_0 \in \mathbb{N}$,

$$M(x,y,\phi^n(t_0)) \geq \left[M(x,y,t_0)\right]^{2^{n_0-1}} * \left[M(y,x,t_0)\right]^{2^{n_0-1}}. \tag{3.14}$$

Then we have

$$\begin{aligned}
M(x,y,t) &\geq M\left(x,y,\sum_{k=n_0}^{\infty}\phi^k(t_0)\right)\\
&\geq M(x,y,\phi^{n_0}t_0)\\
&\geq \left[M(x,y,t_0)\right]^{2^{n_0-1}} * \left[M(y,x,t_0)\right]^{2^{n_0-1}}\\
&\geq \underbrace{(1-\delta)*\cdots*(1-\delta)}_{2^{n_0}} \geq (1-\epsilon),
\end{aligned} \tag{3.15}$$

which implies that $x = y$. Thus, we have proved that f and g have a common fixed point $x \in X$.

Step 4. We now prove the uniqueness of x. Let z be any point in X such that $z \neq x$ with $g(z) = z = f(z,z)$. Since $*$ is a t-norm of H-type, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta)*\cdots*(1-\delta)}_{p} \geq (1-\epsilon), \tag{3.16}$$

for all $p \in \mathbb{N}$. Since $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$, there exists $t_0 > 0$ such that $M(x,z,t_0) \geq (1-\delta)$. Also since $\phi \in \Phi$ and using condition (ϕ-3), we have $\sum_{n=1}^{\infty}\phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$t > \sum_{k=n_0}^{\infty}\phi^k(t_0). \tag{3.17}$$

Using condition (2), we have

$$\begin{aligned}
M(x,z,\phi(t_0)) &= M(f(x,x),f(z,z),\phi(t_0))\\
&\geq M(g(x),g(z),t_0) * M(g(x),g(z),t_0)\\
&\geq M(x,z,t_0) * M(x,z,t_0) = \left[M(x,z,t_0)\right]^2.
\end{aligned} \tag{3.18}$$

Continuing in the same way, we can get for all $n \in \mathbb{N}$,

$$M(x,z,\phi^n(t_0)) \geq \left(\left[M(x,z,t_0)\right]^{2^{n_0-1}}\right)^2. \tag{3.19}$$

Then we have

$$
\begin{aligned}
M(x,z,t) &\geq M\left(x,z,\sum_{k=n_0}^{\infty}\phi^k(t_0)\right) \\
&\geq M\left(x,z,\phi^{n_0}(t_0)\right) \\
&\geq \left([M(x,z,t_0)]^{2^{n_0-1}}\right)^2 = [M(x,z,t_0)]^{2^{n_0}} \\
&\geq \underbrace{(1-\delta) * \cdots * (1-\delta)}_{2^{n_0}} \geq (1-\epsilon),
\end{aligned}
\tag{3.20}
$$

which implies that $x = z$.

Hence f and g have a unique common fixed point in X. □

Remark 3.3. We still get a unique common fixed point if weakly compatible notion is replaced by w-compatible notion.

Now we give another generalization of Theorem 3.1.

Corollary 3.4. *Let $(X, M, *)$ be a fuzzy metric space where $*$ is a continuous t-norm of H-type. Let $f : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ satisfying (2) and (3) with the following condition:*

(5) *the pair (f, g) satisfy E.A. property.*

If $g(X)$ is a closed subspace of X, then f and g have a unique common fixed point in X.

Proof. Since f and g satisfy E.A. property, there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$
\begin{aligned}
\lim_{n\to\infty} f(x_n, y_n) &= \lim_{n\to\infty} g(x_n) = x, \\
\lim_{n\to\infty} f(y_n, x_n) &= \lim_{n\to\infty} g(y_n) = y,
\end{aligned}
\tag{3.21}
$$

for some $x, y \in X$.

It follows from $g(X)$ being a closed subspace of X that $x = g(p)$, $y = g(q)$ for some $p, q \in X$ and then f and g satisfy the $(CLRg)$ property. By Theorem 3.2, we get that f and g have a unique common fixed point in X. □

Corollary 3.5. *Let $(X, M, *)$ be a fuzzy metric space where $*$ is a continuous t-norm of H-type. Let $f : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ satisfying (2), (3), and (5).*

Suppose that $f(X \times X) \subseteq g(X)$, if range of one of the maps f or g is a closed subspace of X, then f and g have a unique common fixed point in X.

Proof. It follows immediately from Corollary 3.5. □

Taking $g = I_X$ in Theorem 3.2, the Corollary 3.6 follows immediately the following.

Corollary 3.6. *Let $(X, M, *)$ be a fuzzy metric space where $*$ is a continuous t-norm of H-type. Let $f : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ satisfying the following conditions, for all $x, y, u, v \in X$ and $t > 0$:*

(6) $M(f(x,y), f(u,v), \phi(t)) \geq M(x,u,t) * M(y,v,t)$,

(7) *there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that*

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} x_n = x,$$
$$\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} y_n = y, \tag{3.22}$$

for some $x, y \in X$.

Then, there exists a unique $z \in X$ such that $z = f(z,z)$.

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