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*Research Article*

# Bernstein-Polynomials-Based Highly Accurate Methods for One-Dimensional Interface Problems

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A new numerical method based on Bernstein polynomials expansion is proposed for solving one-dimensional elliptic interface problems. Both Galerkin formulation and collocation formulation are constructed to determine the expansion coefficients. In Galerkin formulation, the flux jump condition can be imposed by the weak formulation naturally. In collocation formulation, the results obtained by B-polynomials expansion are compared with that obtained by Lagrange basis expansion. Numerical experiments show that B-polynomials expansion is superior to Lagrange expansion in both condition number and accuracy. Both methods can yield high accuracy even with small value of $N$.

## 1. Introduction

In this paper, we consider the following two-point boundary value problem:

$$(\beta u_x)_x + u = f + v\delta(x-\alpha) + \frac{1}{2}(\beta^- + \beta^+)w\delta'(x-\alpha), \quad x \in \Omega = (a,b), \tag{1.1}$$

with boundary conditions

$$u(a) = u_a, \qquad u(b) = u_b, \tag{1.2}$$

where $a < \alpha < b$, $\delta(x)$ is the Dirac delta function and $\delta'(x)$ is the dipole source term. The function $\beta(x)$ is allowed to be discontinuous at $x = \alpha$. For simplicity here we assume that $f(x)$

is smooth function. Due to the presence of singular source, the solution $u(x)$ possesses the following jump relations [1]:

$$[u] = w, \qquad [\beta u_x] = v, \tag{1.3}$$

where the jump notation $[\cdot]$ is defined as

$$[u] = \lim_{x \to \alpha^+} u(x) - \lim_{x \to \alpha^-} u(x) = u^+ - u^-. \tag{1.4}$$

This problem is referred to as the interface problem and is used in various applications of physics, engineering, and biological sciences, see [2–4] and the references therein.

For interface problems, since sharp interfaces or local jumps exist within the solution domain, any high-order method, such as the spectral method, suffers from the Gibbs phenomenon [5]. Here we evoke items we care most for solving interface problems. In 1993, the immersed interface method (IIM) was proposed for interface problems [1]. It is a two-order finite difference method based on Cartesian grids by incorporating the jump relations into difference schemes. The authors have constructed the IIM-based ADI finite difference scheme for 2D nonlinear convection diffusion interface problems [6]. In [7], a high-order method was developed for both discontinuous coefficients and singular source based on finite element method. To enhance the accuracy, the modified Hermite polynomials are used for the basic functions in each element. The matched interface and boundary method was proposed in [8] for elliptic interface problems with discontinuous coefficients and singular source. In [9], the coupling interface method was developed for elliptic interface problems. Recently, Shin and Jung [5] developed the spectral collocation method for one-dimensional interface problems (1.1) with $w = 0$, in which Lagrange basis functions are chosen as the trial functions.

In this work, we also consider this problem and take Bernstein polynomials basis as the trial functions. Bernstein polynomials are useful polynomials in computer-aided geometric design because of their excellent properties [10]. Recently there are some works that used Bernestein polynomials as basis for numerically solving differential equations [11, 12], integral equations [13, 14], and so on, but none of them is about interface problems. Our method is different from Shin and Jung's in three aspects.

(i) The trial functions are chosen as Bernstein polynomials due to its nice properties. These polynomials defined on an interval form a complete basis over the interval. Each of these polynomials is positive and their sum is unity.

(ii) The Galerkin formulation is constructed for this problem. Since the Bernstein polynomials are algebraic polynomials, the mass matrix and stiff matrices can be evaluated efficiently.

(iii) The B-polynomial-based collocation formulation is given collocated with both equidistant points and spectral points, respectively. Unlike Lagrange basis functions, the B-polynomial differential matrix can be computed easily and the condition number of resulting linear system is smaller than that of Lagrange basis, which is shown in the numerical examples.

The rest of this paper is organized as follows. In Section 2, the Bernstein polynomials on interval $[a, b]$ and its derivatives are given. The Galerkin formulation and B-polynomial-based collocation formulation are shown in Section 3. Two numerical examples are given and analyzed in Section 4 before the conclusions are made in Section 5.

## 2. Bernstein Polynomials Basis

The general form of Bernstein polynomials of $n$th degree on interval $[a, b]$ is defined as [12]

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \quad i = 0, 1, \ldots, n, \tag{2.1}$$

where the binomial coefficients are given by $\binom{n}{i} = n!/i!(n-i)!$, with $n! = 1 \times 2 \times \cdots \times n$ for $n \geq 1$ and $0! = 1$. These $(n+1)$ B-polynomials of degree $n$ form a complete basis over the interval $[a, b]$. It is easy to show that any given polynomial of degree $n$ can be expressed in terms of linear combination of the B-polynomials basis functions. The B-polynomials can be generated by a recursive definition:

$$B_{i,n}(x) = \frac{b-x}{b-a} B_{i,n-1}(x) + \frac{x}{b-a} B_{i-1,n-1}(x). \tag{2.2}$$

The derivatives of the $n$th degree B-polynomials are combinations of B-polynomials of degree $n-1$, which can be formulated as

$$\begin{aligned} B'_{i,n} &= \frac{n}{b-a}\left(B_{i-1,n-1} - B_{i,n-1}\right), \\ B''_{i,n} &= \frac{n(n-1)}{(b-a)^2}\left(B_{i-2,n-2} - 2B_{i-1,n-2} + B_{i,n-2}\right), \\ B'''_{i,n} &= \frac{n(n-1)(n-2)}{(b-a)^3}\left(B_{i-3,n-3} - 3B_{i-2,n-3} + 3B_{i-1,n-3} - B_{i,n-3}\right), \end{aligned} \tag{2.3}$$

where we set $B_{i,n} = 0$ if $i < 0$ or $i > n$. In favor of these recursive relations, the differentiation matrix of Bernstein basis can be evaluated conveniently, while the computation of differentiation matrix of high-order Lagrange basis function may suffer from certain difficulties [15].

## 3. Numerical Methods

In this section, we give the numerical method for solving interface problem (1.1)–(1.3). It includes Galerkin formulation and collocation formulation. Without loss of generality, we assume that only one interface $x = \alpha$ exists in $\Omega$ and $\beta$ is piecewise constant in each subdomain. The multiple interface can be handled analogously. Thus the entire domain is divided by $\alpha$ into two parts $\Omega_1 = (a, \alpha)$ and $\Omega_2 = (\alpha, b)$.

Suppose that the solutions in $\Omega_1$ and $\Omega_2$ are $u_1$ and $u_2$, respectively, then the problem (1.1)–(1.3) can be regarded as the following two smooth problems:

$$\begin{aligned} &\left(\beta_1 u_1'\right)' + u_1 = f, \quad x \in \Omega_1, \text{ with } u_1(a) = u_a, \\ &\left(\beta_2 u_2'\right)' + u_2 = f, \quad x \in \Omega_2, \text{ with } u_2(b) = u_b, \end{aligned} \tag{3.1}$$

together with jump conditions (1.3), which make (3.1) closed.

In each subdomain $\Omega_k$, $k = 1, 2$, the solution $u_k(x)$ can be approximated by

$$U_k(x) = \sum_{i=0}^{N_k} C_{k,i} B_{i,N_k}(x), \quad x \in \Omega_k,\ k = 1, 2, \tag{3.2}$$

where $N_k$ is the degree of B-polynomials on each subdomain. According to the interpolation property of B-polynomials at two endpoints, we can easily get $C_{1,0} = u_a$ and $C_{2,N_2} = u_b$.

### *3.1. Galerkin Formulations*

The variational formulation of (3.1) reads

$$\begin{aligned} &-\int_a^\alpha \beta_1 u_1' v_1' \mathrm{d}x + \int_a^\alpha u_1 v_1 \mathrm{d}x = \int_a^\alpha f v_1 \mathrm{d}x - \beta_1 u_1' v_1 \Big|_a^\alpha, \\ &-\int_\alpha^b \beta_2 u_2' v_2' \mathrm{d}x + \int_\alpha^b u_2 v_2 \mathrm{d}x = \int_\alpha^b f v_2 \mathrm{d}x - \beta_2 u_2' v_1 \Big|_\alpha^b, \end{aligned} \tag{3.3}$$

where $v_k \in H^1(\Omega_k)$ is arbitrary and satisfies certain boundary conditions.

Plugging (3.2) into (3.3) and replacing $v_k$ with B-polynomials basis produce

$$\begin{aligned} &-\sum_{i=0}^{N_1}\left(\int_a^\alpha \beta_1 B_{i,N_1}' B_{j,N_1}' \mathrm{d}x + \int_a^\alpha B_{i,N_1} B_{j,N_1} \mathrm{d}x\right) C_{1,i} = \int_a^\alpha f(x) B_{j,N_1} \mathrm{d}x - \beta_1 u_1' B_{j,N_1} \Big|_a^\alpha, \\ &-\sum_{i=0}^{N_2}\left(\int_\alpha^b \beta_2 B_{i,N_2}' B_{m,N_2}' \mathrm{d}x + \int_\alpha^b B_{i,N_2} B_{m,N_2} \mathrm{d}x\right) C_{2,i} \\ &\quad = \int_\alpha^b f(x) B_{m,N_2} dx - \beta_2 u_2' B_{m,N_2} \Big|_\alpha^b, \end{aligned} \tag{3.4}$$

with $j = 0, 1, \ldots, N_1$, $m = 0, 1, \ldots, N_2$.

Observing that $C_{1,0} = u_a$, $C_{2,N_2} = u_b$ and $B_{j,N_k}(x)$, $j = 1, 2, \ldots, N_k - 1$, have zeros at the endpoints, (3.4) can be reformulated as

$$\begin{aligned}
&\sum_{i=1}^{N_1} C_{1,i} \int_a^\alpha \left( -\beta_1 B'_{i,N_1} B'_{j,N_1} + B_{i,N_1} B_{j,N_1} \right) \mathrm{d}x \\
&\quad = \int_a^\alpha \left( f(x) B_{j,N_1} + \beta_1 u_a B'_{0,N_1} B'_{j,N_1} - u_a B_{0,N_1} B_{j,N_1} \right) \mathrm{d}x, \quad j = 1, \ldots, N_1 - 1,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\sum_{i=0}^{N_2-1} C_{2,i} \int_\alpha^b \left( -\beta_2 B'_{i,N_2} B'_{m,N_2} + B_{i,N_2} B_{m,N_2} \right) \mathrm{d}x \\
&\quad = \int_\alpha^b \left( f(x) B_{m,N_2} + \beta_2 u_b B'_{N_2,N_2} B'_{m,N_2} - u_b B_{N_2,N_2} B_{m,N_2} \right) \mathrm{d}x, \quad m = 1, \ldots, N_2 - 1.
\end{aligned} \tag{3.6}$$

Let $j = N_1$ in (3.5) and $m = 0$ in (3.6), we get

$$\begin{aligned}
&\sum_{i=1}^{N_1} C_{1,i} \int_a^\alpha \left( -\beta_1 B'_{i,N_1} B'_{N_1,N_1} + B_{i,N_1} B_{N_1,N_1} \right) \mathrm{d}x \\
&\quad = \int_a^\alpha \left( f(x) B_{N_1,N_1} + \beta_1 u_a B'_{0,N_1} B'_{N_1,N_1} - u_a B_{0,N_1} B_{N_1,N_1} \right) \mathrm{d}x - \beta_1 u_x^-,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\sum_{i=0}^{N_2-1} C_{2,i} \int_\alpha^b \left( -\beta_2 B'_{i,N_2} B'_{0,N_2} + B_{i,N_2} B_{0,N_2} \right) \mathrm{d}x \\
&\quad = \int_\alpha^b \left( f(x) B_{0,N_2} + \beta_2 u_b B'_{N_2,N_2} B'_{0,N_2} - u_b B_{N_2,N_2} B_{0,N_2} \right) \mathrm{d}x + \beta_2 u_x^+.
\end{aligned} \tag{3.8}$$

Adding (3.7) and (3.8) together, combined with jump condition $[\beta u_x] = \beta_2 u_x^+ - \beta_1 u_x^- = v$, yields

$$\begin{aligned}
&\sum_{i=1}^{N_1} C_{1,i} \int_a^\alpha \left( -\beta_1 B'_{i,N_1} B'_{N_1,N_1} + B_{i,N_1} B_{N_1,N_1} \right) \mathrm{d}x \\
&\qquad + \sum_{i=0}^{N_2-1} C_{2,i} \int_\alpha^b \left( -\beta_2 B'_{i,N_2} B'_{0,N_2} + B_{i,N_2} B_{0,N_2} \right) \mathrm{d}x \\
&\quad = \int_a^\alpha \left( f(x) B_{N_1,N_1} + \beta_1 u_a B'_{0,N_1} B'_{N_1,N_1} - u_a B_{0,N_1} B_{N_1,N_1} \right) \mathrm{d}x \\
&\qquad + \int_\alpha^b \left( f(x) B_{0,N_2} + \beta_2 u_b B'_{N_2,N_2} B'_{0,N_2} - u_b B_{N_2,N_2} B_{0,N_2} \right) \mathrm{d}x + v.
\end{aligned} \tag{3.9}$$

Another jump condition $[u] = w$ implies

$$-C_{1,N_1} + C_{2,0} = w. \tag{3.10}$$

Equations (3.5), (3.6), (3.9), and (3.10) form a linear system with $N_1 + N_2$ equations and $N_1 + N_2$ unknowns:

$$C_{1,1}, C_{1,2}, \dots, C_{1,N_1}, C_{2,0}, C_{2,1}, \dots, C_{2,N_2-1}. \tag{3.11}$$

### 3.2. Bernstein Collocation Methods

Substitution of (3.2) into (3.1) produces residuals

$$R_k = \sum_{i=0}^{N_k} C_{k,i}\beta_k B''_{i,N_k} + \sum_{i=0}^{N_k} C_{k,i} B_{i,N_k} - f, \quad k = 1, 2. \tag{3.12}$$

In each interval $[a, \alpha]$ and $[\alpha, b]$, define the collocation points:

$$\begin{aligned} X_1 &= \{x_{1,i} \mid a = x_{1,0} < x_{1,1} < \cdots < x_{1,N_1-1} < x_{1,N_1} = \alpha\}, \\ X_2 &= \{x_{2,i} \mid \alpha = x_{2,0} < x_{2,1} < \cdots < x_{2,N_2-1} < x_{2,N_2} = b\}. \end{aligned} \tag{3.13}$$

Here the points in $X_k$ can be equidistant points, Legendre-Gauss-Lobatto (L-G-L) points, or Chebyshev-Gauss-Lobatto (C-G-L) points.

Note that $C_{1,0}$ and $C_{2,N_2}$ in (3.2) are known. Collocation of (3.12) at points $X_k$ yields

$$\begin{aligned} &\sum_{i=1}^{N_1} C_{1,i}\left(\beta_1 B''_{i,N_1}(x_{1,j}) + B_{i,N_1}(x_{1,j})\right) \\ &\quad = f(x_{1,j}) - u_a\left(\beta_1 B''_{0,N_1}(x_{1,j}) + B_{0,N_1}(x_{1,j})\right), \quad j = 1, \dots, N_1 - 1, \\ &\sum_{i=0}^{N_2-1} C_{2,i}\left(\beta_2 B''_{i,N_2}(x_{2,m}) + B_{i,N_2}(x_{2,m})\right) \\ &\quad = f(x_{2,m}) - u_b\left(\beta_2 B''_{N_2,N_2}(x_{2,m}) + B_{N_2,N_2}(x_{2,m})\right), \quad m = 1, \dots, N_2 - 1. \end{aligned} \tag{3.14}$$

From (3.2) we can easily get

$$U'_k(x) = \sum_{i=0}^{N_k} C_{k,i} B'_{i,N_k}(x), \quad x \in \Omega_k, \ k = 1, 2. \tag{3.15}$$

Jump condition $[u] = w$ and $[\beta u_x] = v$ imply

$$\begin{aligned} -C_{1,N_1} + C_{2,0} &= w, \\ -\sum_{i=1}^{N_1} C_{1,i}\beta_1 B'_{i,N_1}(\alpha) + \sum_{i=0}^{N_2-1} C_{2,i}\beta_2 B'_{i,N_2}(\alpha) &= v + \beta_1 u_a B'_{0,N_1}(\alpha) - \beta_2 u_b B'_{N_2,N_2}(\alpha). \end{aligned} \tag{3.16}$$

**Table 1:** Convergence analysis of Example 4.1 by Galerkin formulation.

| $N$ | $\beta_1 = 100, \beta_2 = 10$ | | | $\beta_1 = 10, \beta_2 = 100$ | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $2.8902e+002$ | $8.0336e-006$ | $3.1656e-005$ | $1.2005e+002$ | $6.0268e-007$ | $4.4909e-006$ |
| 6 | $1.9948e+003$ | $1.2627e-008$ | $7.1402e-008$ | $5.0076e+002$ | $2.3531e-010$ | $2.6075e-009$ |
| 8 | $2.6058e+004$ | $1.1896e-011$ | $8.7994e-011$ | $6.5562e+003$ | $5.5614e-014$ | $8.1463e-013$ |
| 10 | $3.6110e+005$ | $8.9538e-014$ | $7.2848e-013$ | $9.0944e+004$ | $1.2798e-014$ | $2.0911e-013$ |
| 12 | $5.1689e+006$ | $5.7748e-014$ | $5.2493e-013$ | $1.3025e+006$ | $5.5197e-015$ | $8.2017e-014$ |

Equations (3.14) and (3.16) produce the linear systems of $N_1 + N_2$ equations with $N_1 + N_2$ unknowns

$$C_{1,1}, C_{1,2}, \ldots, C_{1,N_1}, C_{2,0}, C_{2,1}, \ldots, C_{2,N_2-1}. \tag{3.17}$$

## 4. Numerical Experiments

In this section, we give two examples to verify the accuracy of proposed numerical method. The first example is the one in which $v \neq 0$ and $w = 0$, while in the second example both $w$ and $v$ are not zeros. We compare our results (B-polynomials-based) with that of Shin and Jung's (Lagrange-polynomials-based) [5]. In all cases, we take $N_1 = N_2 = N$ and the resulting linear systems are almost block diagonal and solved by BiCGStab algorithm.

*Example 4.1.* Consider the interface problem given in [5]

$$\begin{gathered} \left(\beta u_x\right)_x + u = 1 + v\delta(x-\alpha), \quad x \in (a,b), \\ u(a) = u(b) = 0, \end{gathered} \tag{4.1}$$

with the following exact solution:

$$u(x) = \begin{cases} C_1 \cos\left(\dfrac{x}{\sqrt{\beta_1}}\right) + C_2 \sin\left(\dfrac{x}{\sqrt{\beta_1}}\right) + 1, & x \in (a,\alpha), \\ C_3 \cos\left(\dfrac{x}{\sqrt{\beta_2}}\right) + C_4 \sin\left(\dfrac{x}{\sqrt{\beta_2}}\right) + 1, & x \in (\alpha,b), \end{cases} \tag{4.2}$$

where $a = 0$, $b = 5$, $\alpha = 5/3$, and $v = 10$. The jump conditions read $[u] = 0$, $[\beta u_x] = v$. $C_i$s can be determined by the boundary and jump conditions.

Both Galerkin formulation and collocation method are used to solve this problem numerically. The condition number (Cond) of the resulting linear system is given in each computation. Table 1 gives the convergence analysis of Galerkin formulation with different coefficients $\beta_1$ and $\beta_2$, in which the $L_2$ and $H_1$ norms are shown. It can be seen that the error reduces rapidly as the order of the B-polynomials increases.

The convergence analysis of collocation formulation is shown in Tables 2–5, in which the B-polynomials basis and Lagrange basis are compared. In each table, we use two types of collocation points to compare the results. It shows that the results of spectral collocation

**Table 2:** Convergence analysis of Example 4.1 by Bernstein collocation method ($\beta_1 = 100$, $\beta_2 = 10$).

| $N$ | Equidistant collocation points | | | L-G-L collocation points | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $8.0697e+002$ | $3.2739e-005$ | $7.2959e-005$ | $9.7943e+002$ | $9.3669e-006$ | $3.5591e-005$ |
| 6 | $1.7192e+003$ | $1.0254e-007$ | $2.8205e-007$ | $2.6631e+003$ | $1.8687e-008$ | $7.6800e-008$ |
| 8 | $3.5459e+003$ | $2.7004e-010$ | $6.8954e-010$ | $5.9340e+003$ | $2.7678e-011$ | $9.5494e-011$ |
| 10 | $2.0224e+004$ | $6.0013e-013$ | $1.1492e-012$ | $1.1258e+004$ | $2.5097e-014$ | $7.5211e-014$ |
| 12 | $1.2993e+005$ | $8.5704e-013$ | $3.2062e-012$ | $2.0852e+004$ | $1.4182e-015$ | $3.3308e-015$ |

**Table 3:** Convergence analysis of Example 4.1 by Lagrange collocation method ($\beta_1 = 100$, $\beta_2 = 10$).

| $N$ | Equidistant collocation points | | | L-G-L collocation points | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $3.3057e+003$ | $5.2000e-005$ | $7.2298e-005$ | $3.9362e+003$ | $1.4511e-005$ | $2.8433e-005$ |
| 6 | $1.5322e+004$ | $1.9556e-007$ | $2.7515e-007$ | $1.5295e+004$ | $3.1861e-008$ | $5.9502e-008$ |
| 8 | $8.3734e+004$ | $4.5539e-010$ | $6.4269e-010$ | $4.6202e+004$ | $3.9798e-011$ | $7.1989e-011$ |
| 10 | $4.6854e+005$ | $6.4179e-013$ | $9.4096e-013$ | $1.1468e+005$ | $1.8912e-014$ | $4.8567e-014$ |
| 12 | $4.0396e+006$ | $8.1907e-014$ | $1.8001e-013$ | $2.4580e+005$ | $2.2058e-014$ | $3.3532e-014$ |

points (L-G-L or C-G-L points) are more accurate than the equidistant collocation points. Comparing Table 2 with Table 3, we can conclude that the condition number of linear systems derived from B-polynomials is much smaller than that derived from Lagrange polynomials. And the error from the former is smaller than the latter. Similar analysis can be got by comparing Table 4 with Table 5.

*Example 4.2.* In this example, the solution $u$ has nonzero jump across $\alpha$. The following interface problem is considered

$$\begin{gathered}(\beta u_x)_x + u = 1 + v\delta(x-\alpha) + \frac{1}{2}(\beta_1+\beta_2)w\delta'(x-\alpha), \quad x \in (a,b), \\ u(a) = u(b) = 0.\end{gathered} \tag{4.3}$$

The jump conditions are $[u] = w$, $[\beta u_x] = v$. The exact solution is

$$u(x) = \begin{cases} C_1 \cos\left(\dfrac{x}{\sqrt{\beta_1}}\right) + C_2 \sin\left(\dfrac{x}{\sqrt{\beta_1}}\right) + 1, & x \in (a,\alpha), \\ C_3 \cos\left(\dfrac{x}{\sqrt{\beta_2}}\right) + C_4 \sin\left(\dfrac{x}{\sqrt{\beta_2}}\right) + 1, & x \in (\alpha,b), \end{cases} \tag{4.4}$$

where $a = 0$, $b = 5$, $\alpha = 5/3$, $w = 10$, $v = 10$, and $C_i$s can be determined by boundary and jump conditions.

The similar convergence analysis results can be obtained compared with Example 4.1. Since the nonzero jump $w$ just affects the right-hand side of the resulting linear systems, the condition numbers in Tables 6, 7, 8, 9, and 10 are unchanged compared with that in

**Table 4:** Convergence analysis of Example 4.1 by Bernstein collocation method ($\beta_1 = 10$, $\beta_2 = 100$).

| $N$ | Equidistant collocation points | | | C-G-L collocation points | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $2.4186e+002$ | $1.3416e-005$ | $1.8145e-005$ | $2.7609e+002$ | $3.7065e-006$ | $6.8234e-006$ |
| 6 | $4.6720e+002$ | $1.4350e-008$ | $1.9575e-008$ | $6.8378e+002$ | $1.5692e-009$ | $3.5133e-009$ |
| 8 | $8.0785e+002$ | $9.1093e-012$ | $1.2390e-011$ | $1.4516e+003$ | $4.1601e-013$ | $1.0452e-012$ |
| 10 | $1.8803e+003$ | $1.1076e-014$ | $1.3953e-014$ | $2.6872e+003$ | $1.3912e-015$ | $1.9211e-015$ |
| 12 | $9.7253e+003$ | $2.9255e-014$ | $3.5622e-014$ | $4.4882e+003$ | $1.5746e-015$ | $2.4994e-015$ |

**Table 5:** Convergence analysis of Example 4.1 by Lagrange collocation method ($\beta_1 = 10$, $\beta_2 = 100$).

| $N$ | Equidistant collocation points | | | C-G-L collocation points | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $9.8076e+002$ | $1.3416e-005$ | $1.8145e-005$ | $1.1152e+003$ | $3.7065e-006$ | $6.8234e-006$ |
| 6 | $4.1388e+003$ | $1.4350e-008$ | $1.9575e-008$ | $4.6618e+003$ | $1.5692e-009$ | $3.5133e-009$ |
| 8 | $2.0589e+004$ | $9.1456e-012$ | $1.2443e-011$ | $1.4939e+004$ | $3.9966e-013$ | $1.0327e-012$ |
| 10 | $1.2877e+005$ | $1.4536e-014$ | $2.5636e-014$ | $3.8643e+004$ | $1.8444e-014$ | $2.2488e-014$ |
| 12 | $1.4925e+006$ | $2.5445e-013$ | $3.2439e-013$ | $8.5365e+004$ | $1.5287e-014$ | $2.1577e-014$ |

**Table 6:** Convergence analysis of Example 4.2 by Galerkin formulation.

| $N$ | $\beta_1 = 100$, $\beta_2 = 10$ | | | $\beta_1 = 10$, $\beta_2 = 100$ | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $2.8902e+002$ | $3.5634e-004$ | $1.3781e-003$ | $1.2005e+002$ | $1.4132e-005$ | $1.0699e-004$ |
| 6 | $1.9948e+003$ | $5.6809e-007$ | $3.1820e-006$ | $5.0076e+002$ | $5.6321e-009$ | $6.2488e-008$ |
| 8 | $2.6058e+004$ | $5.3895e-010$ | $3.9608e-009$ | $6.5562e+003$ | $1.3353e-012$ | $1.9540e-011$ |
| 10 | $3.6110e+005$ | $3.4189e-013$ | $3.0718e-012$ | $9.0944e+004$ | $4.3556e-014$ | $3.5483e-013$ |
| 12 | $5.1689e+006$ | $7.9344e-014$ | $7.1592e-013$ | $1.3025e+006$ | $2.6693e-014$ | $4.7276e-013$ |

**Table 7:** Convergence analysis of Example 4.2 by Bernstein collocation method ($\beta_1 = 100$, $\beta_2 = 10$).

| N | Equidistant collocation points | | | L-G-L collocation points | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $8.0697e+002$ | $1.4582e-003$ | $2.8374e-003$ | $9.7943e+002$ | $5.7278e-004$ | $1.4832e-003$ |
| 6 | $1.7192e+003$ | $7.2538e-006$ | $1.3324e-005$ | $2.6631e+003$ | $1.4192e-006$ | $3.5539e-006$ |
| 8 | $3.5459e+003$ | $1.9942e-008$ | $3.4812e-008$ | $5.9340e+003$ | $1.9544e-009$ | $4.5723e-009$ |
| 10 | $2.0224e+004$ | $3.4252e-011$ | $5.7468e-011$ | $1.1258e+004$ | $1.6657e-012$ | $3.6562e-012$ |
| 12 | $1.2993e+005$ | $7.4279e-012$ | $1.0651e-011$ | $2.0852e+004$ | $1.1997e-014$ | $2.0057e-014$ |

**Table 8:** Convergence analysis of Example 4.2 by Lagrange collocation method ($\beta_1 = 100$, $\beta_2 = 10$).

| $N$ | Equidistant collocation points | | | L-G-L collocation points | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $3.3057e+003$ | $1.4582e-003$ | $2.8374e-003$ | $3.9362e+003$ | $5.7278e-004$ | $1.4832e-003$ |
| 6 | $1.5322e+004$ | $7.2538e-006$ | $1.3324e-005$ | $1.5295e+004$ | $1.4192e-006$ | $3.5539e-006$ |
| 8 | $8.3734e+004$ | $1.9942e-008$ | $3.4812e-008$ | $4.6202e+004$ | $1.9544e-009$ | $4.5723e-009$ |
| 10 | $4.6854e+005$ | $3.4418e-011$ | $5.7661e-011$ | $1.1468e+005$ | $2.1240e-012$ | $3.9355e-012$ |
| 12 | $4.0396e+006$ | $1.5353e-013$ | $5.0360e-013$ | $2.4580e+005$ | $2.7272e-013$ | $4.6544e-013$ |

**Table 9:** Convergence analysis of Example 4.2 by Bernstein collocation method ($\beta_1 = 10$, $\beta_2 = 100$).

| $N$ | Equidistant collocation points | | | C-G-L collocation points | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $2.4186e+002$ | $1.7618e-004$ | $2.8143e-004$ | $2.7609e+002$ | $4.9906e-005$ | $1.3489e-004$ |
| 6 | $4.6720e+002$ | $2.6676e-007$ | $3.8509e-007$ | $6.8378e+002$ | $2.9181e-008$ | $7.8955e-008$ |
| 8 | $8.0785e+002$ | $2.0042e-010$ | $2.7687e-010$ | $1.4516e+003$ | $9.1270e-012$ | $2.4596e-011$ |
| 10 | $1.8803e+003$ | $1.4215e-013$ | $1.8609e-013$ | $2.6872e+003$ | $8.3967e-015$ | $1.7718e-014$ |
| 12 | $9.7253e+003$ | $6.1320e-014$ | $8.4108e-014$ | $4.4882e+003$ | $6.8128e-015$ | $1.5746e-014$ |

**Table 10:** Convergence analysis of Example 4.2 by Lagrange collocation method ($\beta_1 = 10$, $\beta_2 = 100$).

| $N$ | Equidistant collocation points | | | C-G-L collocation points | | |
|---|---|---|---|---|---|---|
| | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ | Cond | $\|u-U\|_{L_2}$ | $\|u-U\|_{H_1}$ |
| 4 | $9.8076e+002$ | $1.7618e-004$ | $2.8143e-004$ | $1.1152e+003$ | $4.9906e-005$ | $1.3489e-004$ |
| 6 | $4.1388e+003$ | $2.6676e-007$ | $3.8509e-007$ | $4.6618e+003$ | $2.9181e-008$ | $7.8955e-008$ |
| 8 | $2.0589e+004$ | $2.0054e-010$ | $2.7699e-010$ | $1.4939e+004$ | $9.1229e-012$ | $2.4579e-011$ |
| 10 | $1.2877e+005$ | $2.4447e-013$ | $4.5634e-013$ | $3.8643e+004$ | $7.1025e-014$ | $9.6688e-014$ |
| 12 | $1.4925e+006$ | $3.0489e-012$ | $4.3164e-012$ | $8.5365e+004$ | $8.5589e-013$ | $1.2211e-012$ |

Tables 1–5, while the error in Tables 6–10 is much larger than that in Tables 1–5. The regularity of the solution can affect the accuracy of the numerical algorithm enormously.

## 5. Conclusions

In this paper, a new numerical method based on B-polynomials expansion is proposed for solving one-dimensional interface problems. We give two methods to evaluate the expansion coefficients, the Galerkin formulation, and the collocation formulation. Both methods can yield highly accurate results with small number of B-polynomials. In collocation method, the Lagrange polynomials are used to compare with B-polynomials. It is shown by numerical examples that B-polynomials are superior to Lagrange polynomials in both condition number and accuracy, especially when collocated with equidistant points. In theoretical aspect, since the B-polynomials basis is equivalent to power basis or Lagrange basis under certain invertible transformations, theoretical analysis of the proposed method may be done similarly, which is a part of our future research plan. The method can be extended to problems with multiple interfaces easily.

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