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Research Article

Adaptive Control of a Two-Item Inventory Model with Unknown Demand Rate Coefficients

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This paper considers a multiitem inventory model with unknown demand rate coefficients. An adaptive control approach with a nonlinear feedback is applied to track the output of the system toward the inventory goal level. The Lyapunov technique is used to prove the asymptotic stability of the adaptive controlled system. Also, the updating rules of the unknown demand rate coefficients are derived from the conditions of the asymptotic stability of the perturbed system. The linear stability analysis of the model is discussed. The adaptive controlled system is modeled by a system of nonlinear differential equations, and its solution is discussed numerically.

1. Introduction

The area of adaptive control has grown rapidly to be one of the richest fields in the control theory. Many books and research monographs already exist on the topics of parameter estimation and adaptive control. Adaptive control theory is found to be very useful in solving many problems in different fields, such as management science, dynamic systems, and inventory systems [1–3].

(i) El-Gohary and Yassen [4] used an adaptive control approach and synchronization procedures to the coupled dynamo system with unknown parameters. Based on the Lyapunov stability technique, an adaptive control laws were derived such that the coupled dynamo system is asymptotically stable and the two identical dynamo systems are asymptotically synchronized. Also the updating rules of the unknown parameters were derived;

(ii) El-Gohary and Al-Ruzaiza [5] studied the adaptive control of a continuous-time three-species prey-predator populations. They have derived the nonlinear feedback control inputs which asymptotically stabilized the system about its steady states;

(iii) Tadj et al. [6] discussed the optimal control of an inventory system with ameliorating and deteriorating items. They considered different cases for the difference between the ameliorating and deteriorating items;

(iv) Foul et al. [7] studied the problem of adaptive control of a production and inventory system in which a manufacturing firm produces a single product, then it sells some of its production and stocks the remaining. They applied a model reference adaptive control system with a feedback to track the output of the system toward the inventory goal level;

(v) Alshamrani and El-Gohary [8] studied the problem of optimal control of the two-item inventory system with different types of deterioration. They derived the optimal inventory levels and continuous rates of supply from the optimality conditions;

(vi) Many other studies which are concerned with the production and inventory systems, multiitem inventory control, and inventory analysis can be found in references [9–14]. Such studies have discussed the optimal control of a multi-item inventory model, the stability conditions of a multi-item inventory model with different demand rates, and the optimal control of multi-item inventory systems with budgetary constraints.

This paper is concerned with a two-item inventory system with different types of deteriorating items subjected to unknown demand rate coefficients. We derive the controlled inventory levels and continuous rates of supply. Further, the updating rules of the unknown demand rate coefficients are derived from the conditions of asymptotic stability of the reference model. The resulting controlled system is modeled by a system of nonlinear differential equations, and its solution is discussed numerically for different sets of parameters and initial states.

The motivation of this study is to extend and generalize the two-item inventory system with different types of deterioration and applying an adaptive control approach to this system in order to get an asymptotic controlled system. This paper generalizes some of the models available in the literature, see for example, [6, 9].

The rest of this paper is organized as follows. In Section 2, we present the mathematical model of the two-item system. Also, stability analysis of the model is discussed in Section 2. Section 3 discusses the adaptive control problem of the system. Numerical solution and examples are presented in Section 4. Finally, Section 5 concludes the paper.

2. The Two-Item Inventory System

This section uses the mathematical methods to formulate the two-item inventory system with two different type of deteriorations. In this model, we consider a factory producing two items and having a finished goods warehouse.

2.1. The Model Assumptions and Formulation

This subsection is devoted to introduce the model assumptions and its formulation. It is assumed that the inventory supply rates are equal to the production rates, while the demand rates may adopt two different types. Throughout this paper we use $i, j = 1, 2$ for the two different types of items. Moreover, the following variables and parameters are used:

$x_i(t)$: the ith item inventory level at time t;

$u_i(t)$: rate of continuous supply to x_i at time t;

x_{io}: the ith item initial state inventory level;

$d_i x_i$: linear demand rate at instantaneous level of inventory x_i, where d_i is a constant;

θ_{ii}: the deterioration coefficient due to self-contact of the ith item inventory level x_i;

a_{ij}: the demand coefficient of x_i due to presence of units of x_j, where $(i \neq j)$;

θ_i: the natural deterioration rate of the ith item inventory level x_i;

$\overline{x}_i$: the value of the ith item inventory level at the steady state;

$\overline{u}_i$: the value of the ith item continuous rate of supply at the steady state;

$\widehat{a}_{ij}(t)$: the dynamic estimator of demand coefficient of x_i due to presence of units of x_j, where $(i \neq j)$.

The main problem of this paper is to present the adaptive control problem for the two-item inventory system as a control problem with two state variables and two control variables which are the inventory levels $x_i(t)$, $i = 1, 2$ and the two continuous supply rates $u_i(t)$, $i = 1, 2$, respectively.

Also, since an analytical solution of the resulting control system is nonlinear and its analytical solution is not available, we solve it numerically and display the solution graphically. We show that the solution of the adaptive controlled system covers different modes of demand rates.

2.2. The Mathematical Model and Stability Analysis

In this subsection, we present a suitable mathematical form for a two-item inventory system with two types of deteriorations. This mathematical form must be simple to deal with any response of the two-item inventory model with deterioration to any given input. The differential equations system that governs the time evolution of the two-item inventory system is found to be as follows [8]:

$$\begin{aligned} \dot{x}_1(t) &= u_1(t) - x_1(t)\left(d_1 + \theta_1 + a_{12}x_2(t) + \theta_{11}x_1(t)\right), \\ \dot{x}_2(t) &= u_2(t) - x_2(t)\left(d_2 + \theta_2 + a_{21}x_1(t) + \theta_{22}x_2(t)\right), \end{aligned} \tag{2.1}$$

with the following nonnegatively conditions:

$$x_i(t) \geq 0, \quad u_i(t) \geq 0, \quad d_i(t) > 0, \quad \theta_i(t) > 0, \quad \theta_{ii} > 0, \quad i = 1, 2, \tag{2.2}$$

and with the following boundary conditions:

$$x_i(0) = x_{i0}, \quad i = 1, 2. \tag{2.3}$$

In this paper, we consider the inventory goal levels $\overline{x}_i$ and the goal rates of the continuous rate of supply $\overline{u}_i$ to be their values at the steady state of the system. The advantage of this study is to prove the asymptotic stability of the two-item inventory system using the Liapunov technique about the steady state of the system.

Next, we will derive the steady state solution of (2.1). The steady state of the system (2.1) can be derived by putting both of $\dot{x}_1(t)$ and $\dot{x}_2(t)$ equal zero, that is,

$$\begin{aligned} &\overline{x}_1(d_1 + \theta_1 + a_{12}\overline{x}_2 + \theta_{11}\overline{x}_1) - \overline{u}_1 = 0, \\ &\overline{x}_2(d_2 + \theta_2 + a_{21}\overline{x}_1 + \theta_{22}\overline{x}_2) - \overline{u}_2 = 0. \end{aligned} \tag{2.4}$$

Solving (2.4), we get $\overline{x}_1$ as a function of $\overline{x}_2$ as follows:

$$\overline{x}_1 = \left[-(d_1 + \theta_1 + a_{12}\overline{x}_2) \pm \sqrt{(d_1 + \theta_1 + a_{12}\overline{x}_2)^2 + 4\theta_{11}\overline{u}_1}\right](2\theta_{11})^{-1}, \tag{2.5}$$

where the values of $\overline{x}_2$ are the roots of the equation:

$$\left(\theta_{22} - \frac{a_{12}a_{21}}{2\theta_{11}}\right)\overline{x}_2^2 + \frac{a_{21}}{2\theta_{11}}\left[\sqrt{(d_1 + \theta_1 + a_{12}\overline{x}_2)^2 + 4\theta_{11}\overline{u}_1} - d_1 - \theta_1\right]\overline{x}_2 + d_2 + \theta_2 - \overline{u}_2 = 0. \tag{2.6}$$

In what follows, we discuss the numerical solution for the (2.4) for fixed values of the parameters $d_i, \theta_i, \theta_{ii}, (i = 1, 2)$ and a_{12}, a_{21}:

(1) in this example, we discuss the numerical solution of (2.4) for constant rates of supply $\overline{u}_1 = 2.25$ and $\overline{u}_2 = 3.25$, the steady states are given in Table 1;

(2) in this example, we discuss the numerical solution of (2.4) for the supply rates $\overline{u}_1 = 5\overline{x}_1 + 6\overline{x}_2$ and $\overline{u}_2 = 45\overline{x}_1 + 25\overline{x}_2$ of the inventory levels, the steady states are given in Table 2;

(3) in this example, we discuss the numerical solution of (2.4) for supply rates $\overline{u}_1 = 2\overline{x}_1^2 + 3\overline{x}_2^2 + 5\overline{x}_1\overline{x}_2$ and $\overline{u}_2 = 5\overline{x}_1^2 + 15\overline{x}_2^2 + 45\overline{x}_1\overline{x}_2$ of the inventory levels, the steady states are given in Table 3.

Figure 1 displays the numerical solution for the two-item inventory system with the quadratic continuous rates of supply $\overline{u}_1 = \alpha\overline{x}_1\overline{x}_2$ and $\overline{u}_2 = \beta\overline{x}_1\overline{x}_2$, with the initial inventory levels $x_1(0) = 3$ and $x_2(0) = 15$, and with the following set of parameters in Table 4.

Figure 2 displays the numerical solution for the two-item inventory with constant continuous rates of supply $\overline{u}_1 = \alpha$ and $\overline{u}_2 = \beta$, with the initial inventory levels $x_1(0) = 3$ and $x_2(0) = 5$, and the following set of parameters in Table 5.

2.3. Linear Stability Analysis

The concept of stability is concerned with the investigation and characterization of the behavior of dynamic systems. Stability analysis plays a crucial role in system theory and

Table 1

Model parameters	Steady states $(\overline{x}_1, \overline{x}_2)$
$d_1 = 10,\ d_2 = 20,\ a_{12} = 2.5, a_{21} = 3.5,$ $\theta_1 = 0.1,\ \theta_2 = 0.15,\ \theta_{11} =$ $0.01,$ and $\theta_{22} = 0.01$	$(-1010, -0.00996)$, negative $(-6.4, -4.624)$, negative $(0.225, 0.156)$, positive

Table 2

Model parameters	Steady states $(\overline{x}_1, \overline{x}_2)$
$d_1 = 10,\ d_2 = 20,\ a_{12} = 25,\ a_{21} =$ $45,\ \ \theta_1 = 0.05,\ \theta_2 = 0.25,\ \theta_{11} =$ $0.2,$ and $\theta_{22} = 0.2$	$(0, 0)$, null state $(-149.81, 0.995)$, negative $(0.22, 1.93)$, positive

control engineering and has been investigated extensively in the past century. Some of the most fundamental concepts of stability were introduced by the Russian mathematician and engineer Alexandr Lyapunov in [15].

In this section, we discuss the linear stability analysis of the system (2.1) about its steady states (2.4). We classify the roots of the characteristic equation of the Jacobian matrix of the system (2.1) about its steady states (2.4).

The characteristic equation is given by:

$$\lambda^2 - b\lambda + c = 0, \tag{2.7}$$

where the coefficients b and c are:

$$b = \left\{ \sum_{i=1}^{2} \left(\frac{\partial u_i}{\partial x_i} - \frac{u_i}{x_i} - \theta_{ii} x_i \right) \right\}_{(x_1,x_2)=(\overline{x}_1,\overline{x}_2)}$$

$$c = \left\{ \left(\frac{\partial u_1}{\partial x_1} - \frac{u_1}{x_1} - \theta_{11} x_1 \right) \left(\frac{\partial u_2}{\partial x_2} - \frac{u_2}{x_2} - \theta_{22} x_2 \right) - \left(\frac{\partial u_1}{\partial x_2} - a_{12} x_1 \right) \left(\frac{\partial u_2}{\partial x_1} - a_{21} x_2 \right) \right\}_{(x_1,x_2)=(\overline{x}_1,\overline{x}_2)}. \tag{2.8}$$

The roots of the characteristic equation are:

$$\lambda = b \pm \sqrt{b^2 - 4c}. \tag{2.9}$$

The roots of the characteristic equation will be complex numbers with negative real parts if the following conditions can be satisfied:

$$\sum_{i=1}^{2} \left[\frac{\partial u_i}{\partial x_i} - \frac{u_i}{x_i} - \theta_{ii} x_i \right]_{(x_1,x_2)=(\overline{x}_1,\overline{x}_2)} < 0,$$

$$\left\{ \left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{u_1}{x_1} + \frac{u_2}{x_2} - \theta_{11} x_1 + \theta_2 x_2 \right)^2 - 4 \left(\frac{\partial u_1}{\partial x_2} - a_{12} x_1 \right) \left(\frac{\partial u_2}{\partial x_1} - a_{21} x_2 \right) \right\}_{(x_1,x_2)=(\overline{x}_1,\overline{x}_2)} < 0. \tag{2.10}$$

Table 3

Model parameters	Steady states $(\overline{x}_1, \overline{x}_2)$
$d_1 = 50,\ d_2 = 40,\ a_{12} = 75,\ a_{21} = 55,\ \theta_1 = 0.15,\ \theta_2 = 0.25,\ \theta_{11} = 0.2$, and $\theta_{22} = 0.2$	$(0, 0)$, null state $(-2.4 + 1.72I, -0.786 + 0.04I)$, imaginary $(0.099, 2.9)$, positive

Table 4

Parameter	θ_{11}	θ_{22}	a_{12}	a_{21}	d_1	d_2	θ_1	θ_2	$\overline{u}_1$	$\overline{u}_2$	α	β
Value	0.12	0.15	6	5	6	8	0.02	0.04	100	50	8	5

Therefore the system (2.1) is stable in the linear sense if the conditions (2.10) are satisfied, otherwise this system is absolutely unstable. The absolutely stability of system (2.1) needs further complicated mathematical analysis.

The roots of the characteristic equation will be negative real numbers if the following conditions can be satisfied:

$$\sum_{i=1}^{2}\left[\frac{\partial u_i}{\partial x_i} - \frac{u_i}{x_i} - \theta_{ii}x_i\right]_{(x_1,x_2)=(\overline{x}_1,\overline{x}_2)} < 0,$$
$$\left\{\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{u_1}{x_1} + \frac{u_2}{x_2} - \theta_{11}x_1 + \theta_2 x_2\right)^2 + 4\left(\frac{\partial u_1}{\partial x_2} - a_{12}x_1\right)\left(\frac{\partial u_2}{\partial x_1} - a_{21}x_2\right)\right\}_{(x_1,x_2)=(\overline{x}_1,\overline{x}_2)} > 0$$
$$\left\{\left(\frac{\partial u_1}{\partial x_1} - \frac{u_1}{x_1} - \theta_{11}x_1\right)\left(\frac{\partial u_2}{\partial x_2} - \frac{u_2}{x_2} - \theta_{22}x_2\right) - \left(\frac{\partial u_1}{\partial x_2} - a_{12}x_1\right)\left(\frac{\partial u_2}{\partial x_1} - a_{21}x_2\right)\right\}_{(x_1,x_2)=(\overline{x}_1,\overline{x}_2)} > 0. \tag{2.11}$$

If the conditions (2.11) are satisfied, then the system (2.1) is stable in the linear sense, otherwise this system is absolutely unstable. The absolutely stability of system (2.1) needs further complicated mathematical analysis.

Next, we discuss some special cases in which the rates of supply take different functions of the inventory levels:

(1) when the supply rates do not depend on the inventory levels, the linear stability conditions are reduced to

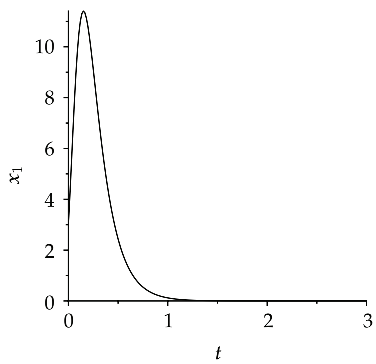
$$\frac{\overline{x}_1}{\overline{x}_2} < \frac{\overline{u}_1 + \theta_{11}\overline{x}_1^2}{\overline{u}_2 + \theta_{22}\overline{x}_2^2} \tag{2.12}$$

or

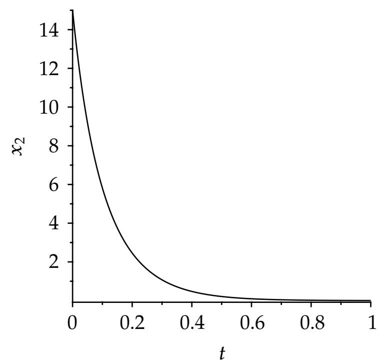
$$\frac{\overline{u}_1 + \theta_{11}\overline{x}_1^2}{a_{12}\overline{x}_1^2} < \frac{\overline{u}_2 + \theta_{22}\overline{x}_2^2}{a_{21}\overline{x}_2^2}; \tag{2.13}$$

Table 5

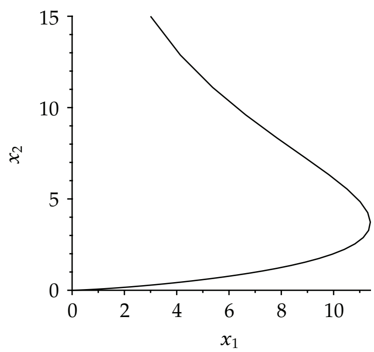
Parameter	θ_{11}	θ_{22}	a_{12}	a_{21}	d_1	d_2	θ_1	θ_2	α	β
Value	0.2	0.5	0.6	0.5	0.6	0.8	0.02	0.04	8	5



(a) First inventory level



(b) Second inventory level



(c) The two items inventory limit cycle

Figure 1: (a) and (b) are the first and the second inventory levels, respectively, of the uncontrolled system, with quadratic rates of supply. (c) is the trajectory of the inventory system in x_1x_2-plane.

(2) when the supply rates are linear function of the inventory levels, $u_i = \alpha_i x_i$, $i = 1, 2$, the linear stability conditions are reduced to

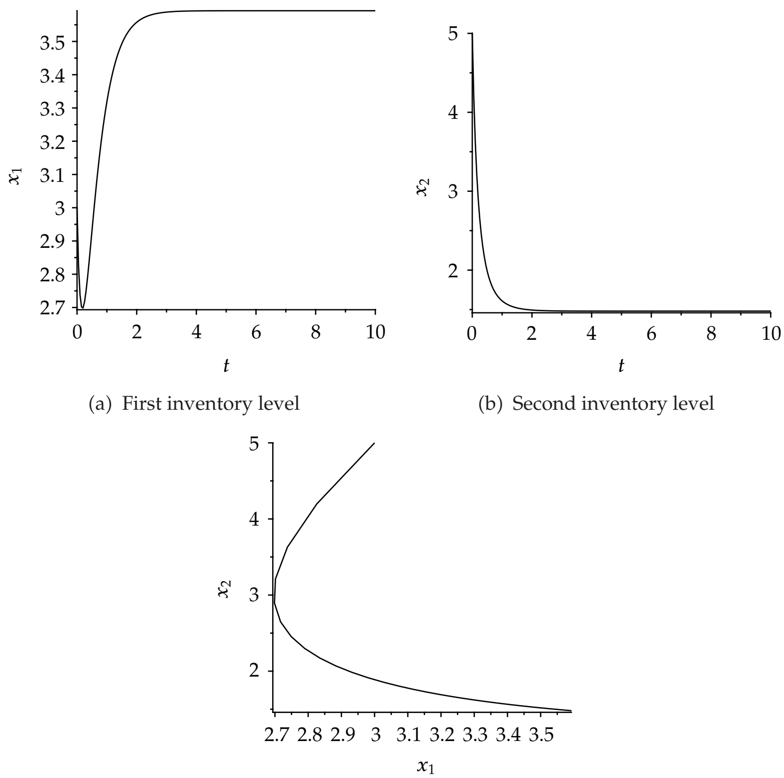
$$(\theta_{22}\overline{x}_2 - \theta_1\overline{x}_1)^2 < 4a_{12}a_{21}\overline{x}_1\overline{x}_2 \tag{2.14}$$

or

$$(\alpha_1 + \theta_{11}\overline{x}_1)(\alpha_2 + \theta_{22}\overline{x}_2) > a_{12}a_{21}\overline{x}_1\overline{x}_2; \tag{2.15}$$

(3) when the supply rates are quadratic functions of the inventory levels, $u_i = \alpha_i x_1 x_2, i = 1, 2$, the linear stability conditions are reduced to

$$(\theta_{22}x_2 - \theta_{11}\overline{x}_1)^2 < 4(\alpha_1 - a_{12})(\alpha_2 - a_{21})\overline{x}_1\overline{x}_2 \tag{2.16}$$



(a) First inventory level

(b) Second inventory level

(c) The two items inventory limit cycle

Figure 2: (a) and (b) are the first and the second inventory levels of the uncontrolled system, with constant rates of supply. (c) is the trajectory of the inventory system in x_1x_2-plane.

or

$$\begin{gathered}(\theta_{22}x_2 - \theta_{11}\overline{x}_1)^2 + 4(\alpha_1 - a_{12})(\alpha_2 - a_{21})\overline{x}_1\overline{x}_2 > 0, \\ \theta_{11}\theta_{22} > (\alpha_1 - a_{12})(\alpha_2 - a_{21}).\end{gathered} \tag{2.17}$$

In what follows, we study the problem of adaptive control. In order to study this problem, we start by obtaining the perturbed system of the two-item inventory model about its steady states $(\overline{x}_1, \overline{x}_2)$. To obtain this perturbed system, we introduce the following new variables:

$$\xi_i(t) = x_i(t) - \overline{x}_i, \quad v_i(t) = u_i(t) - \overline{u}_i, \quad (i = 1, 2), \tag{2.18}$$

Substituting from (2.18) into (2.1) and using (2.4), we get

$$\begin{aligned}\dot{\xi}_1(t) &= -\xi_1[d_1 + \theta_1 + a_{12}\xi_2 + a_{12}\overline{x}_2 + \theta_{11}(\xi_1 + 2\overline{x}_1)] - a_{12}\overline{x}_1\xi_2 + v_1, \\ \dot{\xi}_2(t) &= -\xi_2[d_2 + \theta_2 + a_{21}\xi_1 + a_{21}\overline{x}_1 + \theta_{22}(\xi_2 + 2\overline{x}_2)] - a_{21}\overline{x}_2\xi_1 + v_2.\end{aligned} \tag{2.19}$$

The system (2.19) will be used to study the problem of adaptive control of the two-item inventory model with deteriorating item and unknown demand coefficients.

In adaptive control systems, we are concerned with changing the properties of dynamic systems so that they can exhibit acceptable behavior when perturbed from their operating point using a feedback approach.

3. The Adaptive Control Problem

The problem that we address in this section is the adaptive control of the two-item inventory system with different types of deterioration which are subjected to unknown demand rate coefficients. In such study, we assume that the demand coefficients a_{12} and a_{21} are unknown parameters. So we assume that the functions $\widehat{a}_{12}(t)$ and $\widehat{a}_{21}(t)$ represent their dynamic estimators. Using this assumption, we can rewrite the system (2.19) in the following form:

$$\begin{aligned}\dot{\xi}_1(t) &= -\xi_1\left[d_1+\theta_1+\widehat{a}_{12}\xi_2+\widehat{a}_{12}\overline{x}_2+\theta_{11}(\xi_1+2\overline{x}_1)\right]-\widehat{a}_{12}\overline{x}_1\xi_2+v_1,\\ \dot{\xi}_2(t) &= -\xi_2\left[d_2+\theta_2+\widehat{a}_{21}\xi_1+\widehat{a}_{21}\overline{x}_1+\theta_{22}(\xi_2+2\overline{x}_2)\right]-\widehat{a}_{21}\overline{x}_2\xi_1+v_2.\end{aligned} \tag{3.1}$$

The adaptive law is usually a differential equation whose state is designed using stability considerations or simple optimization techniques to minimize the difference between the state and its estimator with respect to the state at each time t.

In what follows, we discuss the asymptotic stability of the special solution of the system (3.1) which is given by

$$\xi_i(t)=0,\quad v_i=0,\quad (i=1,2),\qquad \widehat{a}_{12}(t)=a_{12},\qquad \widehat{a}_{21}(t)=a_{21}. \tag{3.2}$$

This solution corresponds to the steady states solution of the system (2.1). So the asymptotic stability of this solution leads to the asymptotic stability of the (2.1) about its steady state.

The following theorem determines both of the perturbations of the continuous rates of supply v_i and the updating rules of $\widehat{a}_{12}(t)$ and $\widehat{a}_{21}(t)$ of demand rate coefficients from the conditions of the asymptotic stability of the solution (3.2).

Theorem 3.1. *If the perturbations of the continuous supply rates and the updating rules of the unknown parameters $\widehat{a}_{12}(t)$ and $\widehat{a}_{21}(t)$ are given by*

$$\begin{aligned}v_1(t) &= a_{12}\overline{x}_1\xi_2+a_{12}\xi_1\xi_2+\theta_{11}\xi_1^3-k_1\xi_1,\\ v_2(t) &= a_{21}\overline{x}_2\xi_1+a_{21}\xi_1\xi_2+\theta_{22}\xi_2^3-k_2\xi_2,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\dot{\widehat{a}}_{12}(t) &= \overline{x}_2\xi_1\xi_2+\overline{x}_2\xi_1^2+\xi_1^2\xi_2-m_1(\widehat{a}_{12}-a_{12}),\\ \dot{\widehat{a}}_{21}(t) &= \overline{x}_1\xi_1\xi_2+\overline{x}_1\xi_2^2+\xi_1\xi_2^2-m_2(\widehat{a}_{21}-a_{21}),\end{aligned} \tag{3.4}$$

where k_i, m_i, and $(i=1,2)$ are positive real constant, then the solution (3.2) *is asymptotically stable in the Liapunov sense.*

Proof. The proof of this theorem can be reached by using the Liapunov technique. Assume that the Liapunov function of the system of equations (3.2) and (3.4) is

$$2V(\xi_i, \hat{a}_{12}, \hat{a}_{21}) = \sum_{i=1}^{2} \xi_i^2 + (\hat{a}_{12} - a_{12})^2 + (\hat{a}_{21} - a_{21})^2. \tag{3.5}$$

Differentiating the function V in (3.5):

$$\dot{V} = \xi_1 \ \dot{\xi}_1 + \xi_2 \ \dot{\xi}_2 + (\hat{a}_{12} - a_{12})\dot{a}_{12} + (\hat{a}_{21} - a_{21})\dot{a}_{21}. \tag{3.6}$$

Substituting from (3.1) into (3.6), we get

$$\begin{aligned} \dot{V} = {} & \xi_1 \left[-(d_1 + \theta_1)\xi_1 - \overline{x}_1\hat{a}_{12}\xi_2 - \overline{x}_2\hat{a}_{12}\xi_1 - \hat{a}_{12}\xi_1\xi_2 - \theta_{11}\xi_1^2 - 2\overline{x}_1\xi_1 + v_1 \right] \\ & + \xi_2 \left[-(d_2 + \theta_2)\xi_2 - \overline{x}_2\hat{a}_{21}\xi_1 - \overline{x}_1\hat{a}_{21}\xi_2 - \hat{a}_{21}\xi_1\xi_2 - \theta_{22}\xi_2^2 - 2\overline{x}_2\xi_2 + v_2 \right] \\ & + (\hat{a}_{12} - a_{12})\dot{a}_{12} + (\hat{a}_{21} - a_{21})\dot{a}_{21}. \end{aligned} \tag{3.7}$$

Substituting from (3.1), (3.3), and (3.4) into (3.7), and after some simple calculations, we get

$$\begin{aligned} \dot{V} = {} & -\Big[m_1(\hat{a}_{12} - a_{12})^2 + m_2(\hat{a}_{21} - a_{21})^2 + (d_1 + \theta_1 + a_{12}\overline{x}_2 + 2\theta_{11}\overline{x}_1)\xi_1^2, \\ & +(d_2 + \theta_2 + a_{21}\overline{x}_1 + 2\theta_{22}\overline{x}_2)\xi_2^2 \Big], \end{aligned} \tag{3.8}$$

since the coefficients $d_1 + \theta_1 + a_{12}\overline{x}_2 + 2\theta_{11}\overline{x}_1$ and $d_2 + \theta_2 + a_{21}\overline{x}_1 + 2\theta_{22}\overline{x}_2$ are positive, then $\dot{V}$ is a negative definite function of ξ_i, $\hat{a}_{12}$, and $\hat{a}_{21}$, so the solution (3.3) is asymptotically stable in the Liapunov sense, which completes the proof. □

In Section 4, we will discuss the numerical solution of the controlled two-item inventory system with unknown demand rate coefficients for different values of the parameters and different initial states.

4. Numerical Solution and Examples

The objective of this section is to study the numerical solution of the problem of determining an adaptive control strategy for the two-item inventory system subjected to different types of deterioration and unknown demand rate coefficients. To illustrate the solution procedure, let us consider simple examples in which the system parameters and initial states take different values. In these examples, the numerical solutions of the controlled two-item inventory system with unknown demand rate coefficients are presented. The numerical solution algorithm is based on the numerical integration of the system using the Runge-Kutta method.

Table 6

Parameter	θ_{11}	θ_{22}	a_{12}	a_{21}	$\overline{x}_1$	$\overline{x}_2$	k_1	k_2	m_1	m_2	θ_1	θ_2	d_1	d_2
Value	0.18	0.26	6.0	8.0	0.4	0.2	0.25	0.5	1.0	0.5	0.1	0.25	5.0	6.0

Table 7

Parameter	θ_{11}	θ_{22}	a_{12}	a_{21}	$\overline{x}_1$	$\overline{x}_2$	k_1	k_2	m_1	m_2	θ_1	θ_2	d_1	d_2
Value	0.15	0.3	15.0	10.0	5.0	0.6	5	3	9.0	7.0	0.3	0.2	10.0	15.0

Substituting from (3.3) into (3.1) and adding the system (3.4), we can get the adaptive control system as follows:

$$\begin{aligned}
\dot{\xi}_1(t) = &-(d_1+\theta_1)\xi_1(t) - \widehat{a}_{12}\xi_1(t)\xi_2(t) - a_{12}\overline{x}_2\xi_1(t) - 2\theta_{11}\overline{x}_1\xi_1(t) \\
&- \widehat{a}_{12}\overline{x}_1\xi_2(t) + a_{12}\xi_1(t)\xi_2(t) + a_{12}\overline{x}_1\xi_2(t) + \theta_{11}\xi_1(t)^2 - k_1\xi_1(t), \\
\dot{\xi}_2(t) = &-(d_2+\theta_2)\xi_2(t) - \widehat{a}_{21}\xi_1(t)\xi_2(t) - a_{21}\overline{x}_1\xi_2(t) - 2\theta_{22}\overline{x}_2\xi_2(t) \\
&- \widehat{a}_{21}\overline{x}_2\xi_1(t) + a_{21}\xi_1(t)\xi_2(t) + a_{21}\overline{x}_2\xi_1(t) + \theta_{22}\xi_2(t)^2 - k_2\xi_2(t), \\
\dot{\widehat{a}}_{12}(t) = &\ \xi_1(t)^2\xi_2(t) + \overline{x}_2\xi_1(t)^2 + \overline{x}_1\xi_1(t)\xi_2(t) - m_1(\widehat{a}_{12}(t) - a_{12}), \\
\dot{\widehat{a}}_{21}(t) = &\ \xi_2(t)^2\xi_1(t) + \overline{x}_1\xi_2(t)^2 + \overline{x}_2\xi_1(t)\xi_2(t) - m_2(\widehat{a}_{21}(t) - a_{21}).
\end{aligned} \tag{4.1}$$

Clearly, this system is non-linear and its general solution is not available, so we will discuss its solution numerically. Next, we solve this system numerically for some particular values of the parameters and initial states.

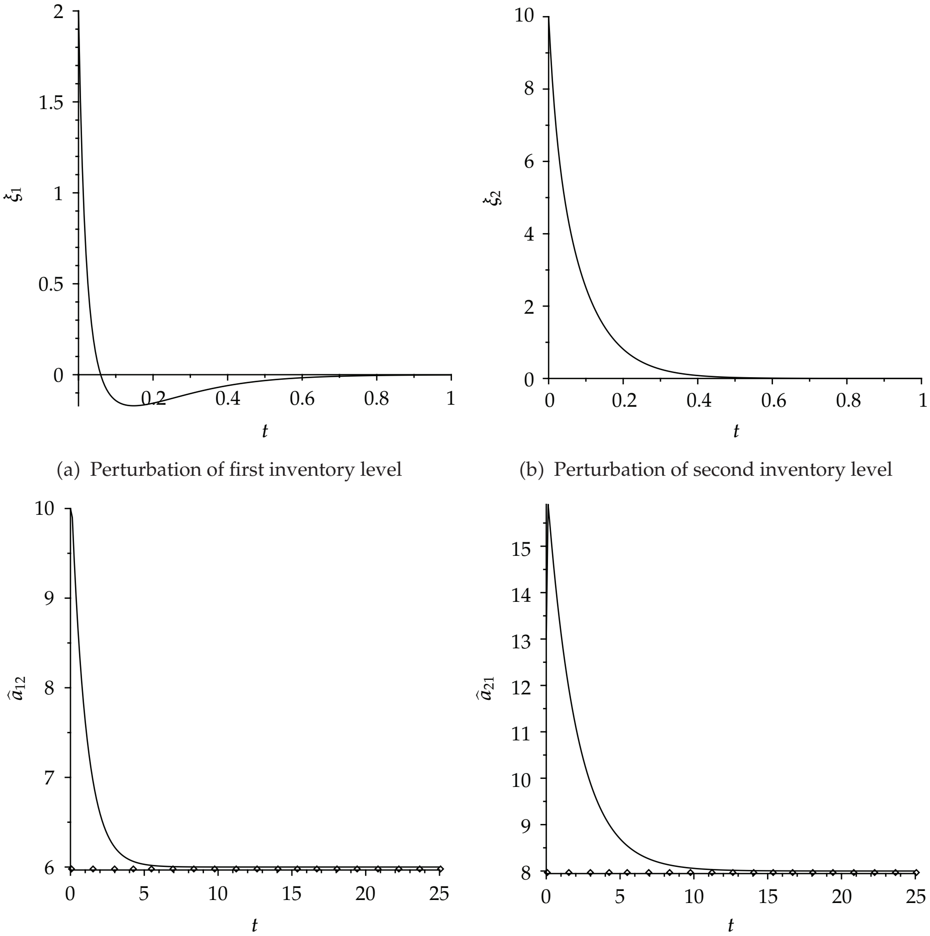
4.1. Example 1

In this example, a numerical solution of the adaptive controlled system (4.1) is displayed graphically assuming constant demand rates. The following set of parameter values is used in Table 6 with the following initial values of perturbations of inventory levels and estimators of demand rate coefficients: $\xi_1(0) = 2$; $\xi_2(0) = 10$; $\widehat{a}_{12}(0) = 10$; $\widehat{a}_{21}(0) = 13$.

The numerical results are illustrated in Figure 3. We conclude that both of the perturbations of inventory levels and the estimators of demand rate coefficients tend to zero and their real values, respectively. This means that both of the inventory levels and demand rate coefficients asymptotically tend to their values at the steady state.

4.2. Example 2

In this example, a numerical solution of the adaptive controlled system (4.1) is displayed graphically when the demand rate is a linear function of the inventory level. The following set of parameter values is used in Table 7 with the following initial values of perturbations of inventory levels and estimators of demand rate coefficients: $\xi_1(0) = 5$; $\xi_2(0) = 15$; $\widehat{a}_{12}(0) = 5$; $\widehat{a}_{21}(0) = 8$.



(a) Perturbation of first inventory level

(b) Perturbation of second inventory level

(c) Estimator of the first demand coefficient

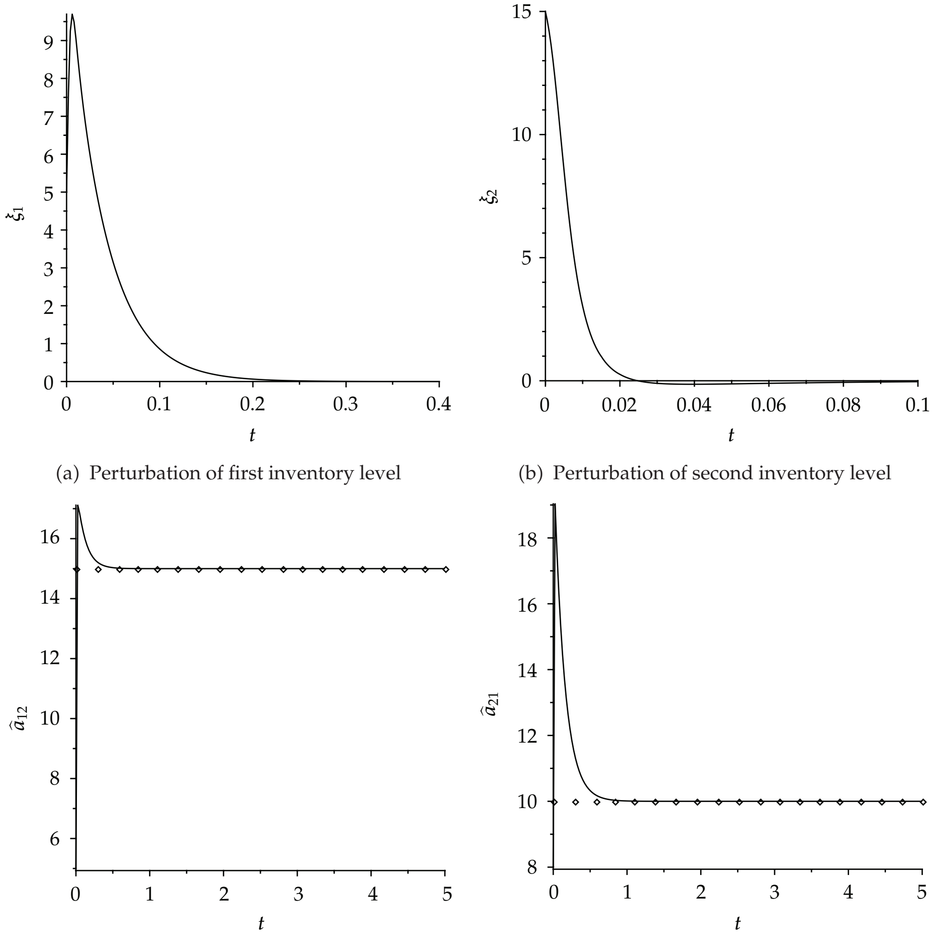
(d) Estimator of the second demand coefficient

Figure 3: (a) and (b) are the perturbation of the first and second inventory levels about their inventory goal levels as the demand rate is a constant. (c) and (d) are the difference between dynamic estimators of the first and second demand rates and their real values.

The numerical results are illustrated in Figure 4. We conclude that both of the perturbations of inventory levels and the estimators of demand rate coefficients tend to zero and their real values, respectively. This means that both of the inventory levels and demand rate coefficients asymptotically tend to their values at the steady state. Also, we can easily notice that the estimators of the unknown demand rate coefficients are exponentially tend to the exact values.

4.3. Example 3

In this example, a numerical solution of the adaptive controlled system (4.1) is displayed graphically when the demand rate is an exponential function of time. The following set of parameter values is used in Table 8 with the following initial values of perturbations of



(a) Perturbation of first inventory level

(b) Perturbation of second inventory level

(c) Estimator of the first demand coefficient

(d) Estimator of the second demand coefficient

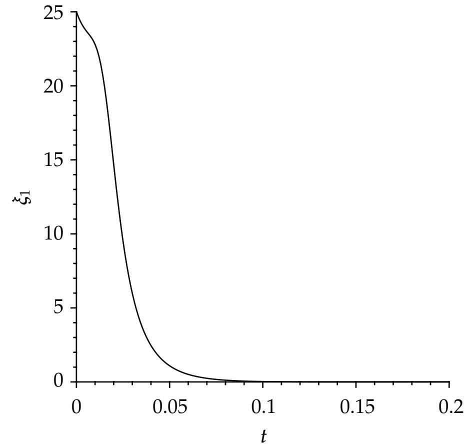
Figure 4: (a) and (b) are the perturbation of the first and second inventory levels about their inventory goal levels as the demand rate is a linear function of the inventory level. (c) and (d) are the difference between dynamic estimators of the first and second demand rates and their real values.

Table 8

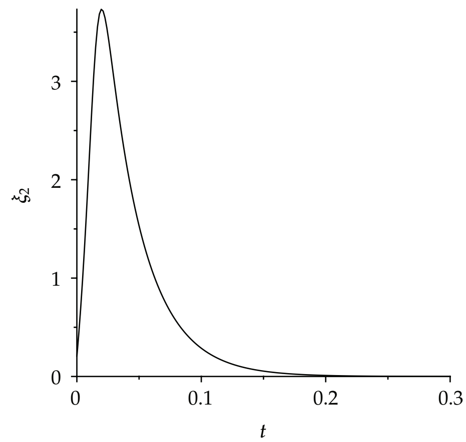
Parameter	θ_{11}	θ_{22}	a_{12}	a_{21}	$\overline{x}_1$	$\overline{x}_2$	k_1	k_2	m_1	m_2	θ_1	θ_2	d_1	d_2
Value	0.08	0.06	20.0	15.0	0.2	1.0	0.02	0.015	0.1	0.2	0.15	0.1	20.0	30.0

inventory levels and estimators of demand rate coefficients: $\xi_1(0) = 25; \xi_2(0) = 0.2; \hat{a}_{12}(0) = 0.2; \hat{a}_{21}(0) = 10$.

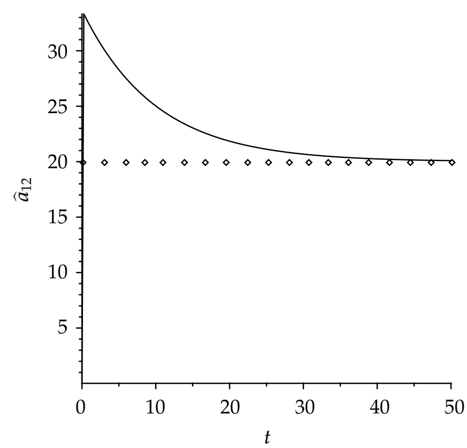
The numerical results are illustrated in Figure 5. We conclude that both of the perturbations of inventory levels and the estimators of demand rate coefficients tend to zero and their real values, respectively. This means that both of the inventory levels and demand rate coefficients asymptotically tend to their values at the steady state.



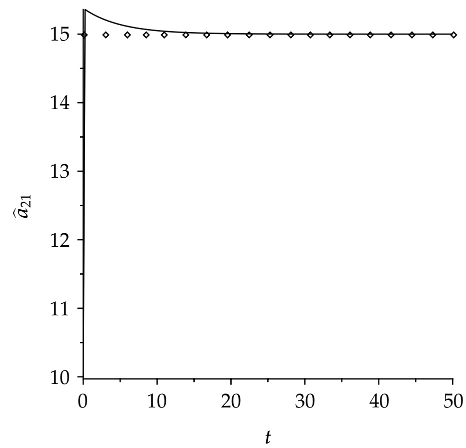
(a) Perturbation of first inventory level



(b) Perturbation of second inventory level



(c) Estimator of the first demand coefficient



(d) Estimator of the second demand coefficient

Figure 5: (a) and (b) are the perturbation of the first and second inventory levels about their inventory goal levels as the demand rate is an exponential function of the time. (c) and (d) are the difference between dynamic estimators of the first and second demand rates and their real values.

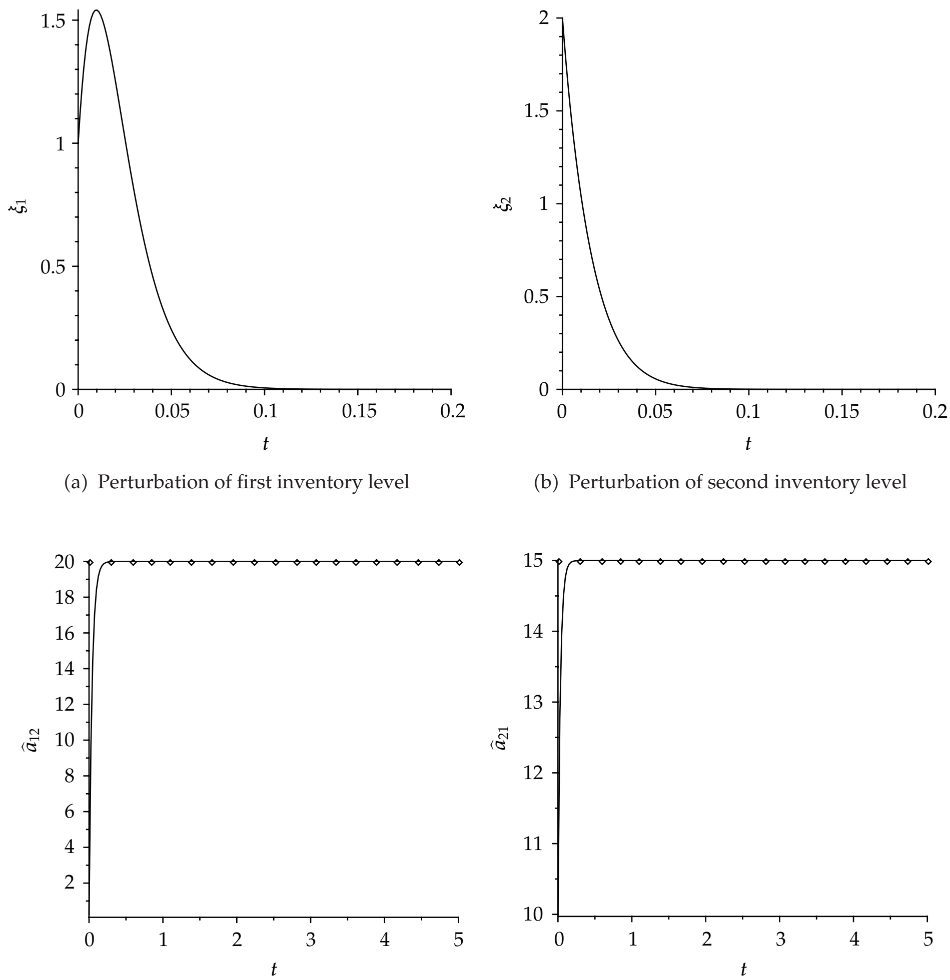
Table 9

Parameter	θ_{11}	θ_{22}	a_{12}	a_{21}	$\overline{x}_1$	$\overline{x}_2$	k_1	k_2	m_1	m_2	θ_1	θ_2	d_1	d_2
Value	0.08	0.06	20	15	2	1	20	15	25	30	0.15	0.1	20	30

4.4. Example 4

In this example, a numerical solution of the adaptive controlled system (4.1) is displayed graphically when the demand rate is an exponential function of time. The following set of parameter values is used in Table 9 with the following initial values of perturbations of inventory levels and estimators of demand rate coefficients: $\xi_1(0) = 1; \xi_2(0) = 2; \hat{a}_{12}(0) = 0.2; \hat{a}_{21}(0) = 10$.

The numerical results are illustrated in Figure 6. We conclude that both of the perturbations of inventory levels and the estimators of demand rate coefficients tend to zero



(a) Perturbation of first inventory level

(b) Perturbation of second inventory level

(c) Estimator of the first demand coefficient

(d) Estimator of the second demand coefficient

Figure 6: (a) and (b) are the perturbation of the first and second inventory levels about their inventory goal levels as the demand rate is an exponential function of the time. (c) and (d) are the difference between dynamic estimators of the first and second demand rates and their real values.

and their real values, respectively. This means that both of the inventory levels and demand rate coefficients asymptotically tend to their values at the steady state.

5. Conclusion

We have shown in this paper how to use an adaptive control approach to study the asymptotic stabilization of a two-item inventory model with unknown demand rate coefficients. A non-linear feedback approach is used to derive the continuous rate of supply. The Liapunov technique is used to prove the asymptotic stability of the adaptive controlled system. Also, the updating rules of the unknown demand rate coefficients have been derived by using the conditions of the asymptotic stability of the perturbed system. Some numerical examples are presented to:

(1) investigate the asymptotic behavior of both inventory levels and demand rate coefficient at the steady state;

(2) estimate the unknown demand rate coefficients.

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