Research Article

Random Attractors for Stochastic Three-Component Reversible Gray-Scott System with Multiplicative White Noise

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The paper is devoted to proving the existence of a compact random attractor for the random dynamical system generated by stochastic three-component reversible Gray-Scott system with multiplicative white noise.

1. Introduction

Let $\mathcal{O}$ be an open bounded set of $\mathbf{R}^n$ ($n \leq 3$) with a locally Lipschitz continuous boundary $\partial\mathcal{O}$. We consider the stochastic three-component reversible Gray-Scott system with multiplicative noise

$$\frac{\partial \tilde{u}}{\partial t} = d_1 \triangle \tilde{u} - (F + k)\tilde{u} + \tilde{u}^2 \tilde{v} - G\tilde{u}^3 + N\tilde{w} + \sigma \tilde{u} \circ \frac{dB_t}{dt},$$

$$\frac{\partial \tilde{v}}{\partial t} = d_2 \triangle \tilde{v} + F(1 - \tilde{v}) - \tilde{u}^2 \tilde{v} + G\tilde{u}^3 + \sigma \tilde{v} \circ \frac{dB_t}{dt}, \qquad (1.1)$$

$$\frac{\partial \tilde{w}}{\partial t} = d_3 \triangle \tilde{w} + k\tilde{u} - (F + N)\tilde{w} + \sigma \tilde{w} \circ \frac{dB_t}{dt},$$

where $\tilde{u} = \tilde{u}(x, t)$, $\tilde{v} = \tilde{v}(x, t)$, and $\tilde{w} = \tilde{w}(x, t)$ are real-valued functions on $\mathcal{O} \times [\tau, +\infty)$, $\tau \in \mathbf{R}$; all the parameters are arbitrarily given positive constants; B_t is a Brownian motion

and $\circ$ denotes the Stratonovich sense of the stochastic term. In this work, we consider the homogenous Neumann boundary condition

$$\frac{\partial \widetilde{u}}{\partial \boldsymbol{v}}(x,t) = \frac{\partial \widetilde{v}}{\partial \boldsymbol{v}}(x,t) = \frac{\partial \widetilde{w}}{\partial \boldsymbol{v}}(x,t) = 0, \quad x \in \partial \mathcal{O}, \tag{1.2}$$

where $\partial/\partial \boldsymbol{v}$ is the outward normal derivative, and with an initial condition

$$\widetilde{u}(x,0) = \widetilde{u}_0(x), \qquad \widetilde{v}(x,0) = \widetilde{v}_0(x), \qquad \widetilde{w}(x,0) = \widetilde{w}_0(x), \quad x \in \mathcal{O}. \tag{1.3}$$

The three-component reversible Gray-Scott model was firstly introduced by Mahara et al. [1]. Recently in [2], You gave the existence of global attractor for system (1.1) when $\sigma = 0$ with Neumann boundary condition (1.2) on a bounded domain of space dimension $n \leq 3$ by the method of the rescaling and grouping estimate. However, the reactions and diffusions are often affected by stochastic factors then it is important and meaningful to take the asymptotic behavior of solutions to consideration. Particularly, the dynamics of certain systems frequently follows some self-organization process where the development of new, complex structures takes place primarily in and through the system itself. This self-organization is normally triggered by internal variation processes, which are usually called fluctuations or noise, that have a positive influence on the system. For instance, recent theoretical studies and experiments with cultured glial cells and the Belousov-Zhabotinsky reaction have shown that noise may play a constructive role on the dynamical behavior of spatially extended systems [3–5]. Therefore, one cannot ignore the role of noise in chemical and biological self-organization and its relationship with the environmental selection of emergent patterns [6]. In [7–9], the influence of additive noise on Gray-Scott systems was discussed. As pointed in [10, 11], the effects of additive and multiplicative noises are fundamentally different in nonlinear systems. While the effect of additive noise does not depend on the state of the system, the effect of multiplicative noise is state dependent. Natural systems in which the effect of noise on the system's dynamics does depend on the recent state are autocatalytic chemical reactions or growth processes in developmental biology. More generally speaking, in each system whose dynamics shows some degree of self-referentiality, the effect of exogenous noise will depend on the recent system's state. If noise is multiplicative, "new" phenomena can occur; that is, the noisy system can exhibit behavior, which is qualitatively different from that of the deterministic system, a phenomenon that has been coined noise-induced transitions.

A fundamental problem in the study of dynamics of a stochastic partial differential equation is to show that it generates a random dynamical system (or stochastic flow). One of the most interesting concepts of the theory of random dynamical systems is the random attractor, which was introduced in the 90s of the last century (see [12]). An attractor for an autonomous dynamical system is a compact set in the phase space, attracting the image of particular sets of initial states under the evolution of the dynamical system. However, the random case is more complicated, because random attractors depend on the random parameter and have their own temporal dynamics induced by the noise (cf. the definition in Section 3). Moreover, the existence of a random attractor to the stochastic reversible Gray-Scott system, especially of three components, is widely open to the best of our knowledge. According to methodology of [2] of nondissipative coupling of three variables and the coefficients barrier, we consider system (1.1)–(1.3), which gives partly an answer to the

problems of random perturbations proposed in [13]. In this paper, we use the notions and frameworks in [12, 14, 15] to study the stochastic three-component reversible Gray-Scott system with multiplicative white noise.

The paper is organized as follows. In Section 2, we give the existence and uniqueness of solution. Section 3 is devoted to the existence of a random attractor.

2. Existence and Uniqueness of Solutions

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{\theta_t : \Omega \to \Omega, t \in \mathbf{R}\}$ is a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t \omega$ is measurable, $\theta_0 = id$, and $\theta_{t+s} = \theta_t \theta_s$ for all $s, t \in \mathbf{R}$. The flow θ_t together with the probability space $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbf{R}})$ is called as a measurable dynamical system.

A random dynamical system (RDS) on a Polish space (X, d) with Borel σ-algebra $\mathcal{B}$ over θ on $(\Omega, \mathcal{F}, P)$ is a measurable map

$$\varphi : \mathbf{R}^+ \times \Omega \times X \longrightarrow X, \qquad (t, \omega, x) \longmapsto \varphi(t, \omega)x, \tag{2.1}$$

such that P-almost surely (P-a.s.) we have

(i) $\varphi(0, \omega) = id$ on X;

(ii) (cocycle property) $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega)\varphi(s, \omega)$ for all $s, t \geq 0$.

An RDS is continuous or differentiable if $\varphi(t, \omega) : X \to X$ is continuous or differentiable.

A map $B : \Omega \to 2^X$ is said to be a closed (compact) random set if $B(\omega)$ is closed (compact) for P-a.s. $\omega \in \Omega$ and if $\omega \mapsto d(x, B(\omega))$ is P-a.s. measurable for all $x \in X$.

Consider the product Hilbert spaces $L^2(\mathcal{O})$, $H = [L^2(\mathcal{O})]^3$, and $H^1(\mathcal{O})$, $E = [H^1(\mathcal{O})]^3$ with the usual inner products and norms

$$(u, v) = \int_\Omega uv dx, \qquad |u| = (u, u)^{1/2} \quad \forall u, v \in L^2(\mathcal{O}),$$

$$((u, v)) = \sum_{i=1}^{n} (D_i u, D_i v) + F(u, v), \qquad \|u\| = ((u, u))^{1/2} \quad \forall u, v \in H^1(\mathcal{O}). \tag{2.2}$$

Obviously, for fixed F the scalar product and norm defined above are equivalent to the usual scalar product and norm in $H^1(\mathcal{O})$. And the norm of $L^p(\mathcal{O})$ will be denoted by $|\cdot|_{L^p}$ if $p \neq 2$, $X = (L^6(\mathcal{O}))^3$.

Define the unbounded positive linear operator

$$A \, (D(A) \longrightarrow H) = \tilde{\Delta} - FI, \tag{2.3}$$

where

$$\tilde{\Delta} = \begin{pmatrix} d_1\Delta & 0 & 0 \\ 0 & d_2\Delta & 0 \\ 0 & 0 & d_3\Delta \end{pmatrix},$$

$$D(A) = \left\{ (\phi,\varphi,\psi) \in \left[H^2(\mathcal{O})\right]^3 : \frac{\partial\phi}{\partial\boldsymbol{\nu}} = \frac{\partial\varphi}{\partial\boldsymbol{\nu}} = \frac{\partial\psi}{\partial\boldsymbol{\nu}} = 0 \text{ on } \partial\mathcal{O} \right\}. \tag{2.4}$$

By the Lumer-Phillips theorem and the generation theorem for analytic semigroup [16], the operator A in (2.3) is linear, sectorial, closed, and defined and is the generator of an analytic C_0-semigroup $\{e^{At}, t \geq 0\}$ on the Hilbert space H. Its spectral set consists of only nonnegative eigenvalues, denoted by $\mu_i = \lambda_i + F$, $i \geq 0$, where λ_i are the corresponding eigenvalues of $\tilde{\Delta}$ satisfying

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \quad (\lambda_i \longrightarrow \infty \text{ as } i \longrightarrow \infty). \tag{2.5}$$

By the fact that $H^1(\mathcal{O}) \hookrightarrow L^6(\mathcal{O})$ is a continuous embedding for $n \leq 3$ and by the generalized Hölder inequality, one has

$$\left|u^2 v\right| \leq |u|_{L^6}^2 |v|_{L^6}, \quad \text{for } u, v \in L^6(\mathcal{O}). \tag{2.6}$$

Therefore, the nonlinear mapping defined on E,

$$f(\tilde{u}, \tilde{v}, \tilde{w}) = \begin{pmatrix} -k\tilde{u} + \tilde{u}^2\tilde{v} - G\tilde{u}^3 + N\tilde{w} \\ F - \tilde{u}^2\tilde{v} + G\tilde{u}^3 \\ k\tilde{u} - N\tilde{w} \end{pmatrix} : E \longrightarrow H, \tag{2.7}$$

is locally Lipschitz continuous. Thus, the initial boundary problem (1.1)–(1.3) is formulated as an initial value problem of the stochastic three-component reversible Gray-Scott system with multiplicative noise

$$\frac{d\tilde{g}}{dt} = A\tilde{g} + f(\tilde{g}) + \sigma\tilde{g} \circ \frac{dB_t}{dt}, \tag{2.8}$$

and an initial condition

$$\tilde{g}(0) = \tilde{g}_0 = (\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \in H, \tag{2.9}$$

where $\tilde{g}(t) = (\tilde{u}(t,\cdot), \tilde{v}(t,\cdot), \tilde{w}(t,\cdot))$. B_t is a one-dimensional two-sided Wiener process on a probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbf{R}, \mathbf{R}^m) : \omega(0) = 0\}, \tag{2.10}$$

the Borel σ-algebra $\mathcal{F}$ on Ω is generated by the compact open topology, and P is the corresponding Wiener measure on $\mathcal{F}$. We can define a family of measure-preserving and ergodic transformations (a flow) $\{\theta_t\}_{t\in\mathbf{R}}$ by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t). \tag{2.11}$$

By means of the change of variables

$$\begin{aligned}
u(t) &= \alpha(t)\tilde{u}(t), \\
v(t) &= \alpha(t)\tilde{v}(t) \quad \text{with } \alpha(t) = e^{-\sigma B_t}, \\
w(t) &= \alpha(t)\tilde{w}(t),
\end{aligned} \tag{2.12}$$

system (1.1) can be written as

$$\frac{\partial u}{\partial t} = d_1\triangle u - (F + k)u + \alpha^{-2}(t)u^2 v - G\alpha^{-2}(t)u^3 + Nw, \tag{2.13}$$

$$\frac{\partial v}{\partial t} = d_2\triangle v + F(\alpha(t) - v) - \alpha^{-2}(t)u^2 v + G\alpha^{-2}(t)u^3, \tag{2.14}$$

$$\frac{\partial w}{\partial t} = d_3\triangle w + ku - (F + N)w. \tag{2.15}$$

That is $g(t,\cdot) = (u(t,\cdot), v(t,\cdot), w(t,\cdot))$ satisfies

$$\frac{dg}{dt} = Ag + \tilde{f}(g,\omega), \tag{2.16}$$

with initial condition

$$g(0) = \tilde{g}(0) = g_0 = (u_0, v_0, w_0) \in H, \tag{2.17}$$

where

$$\tilde{f}(g,\omega) = \begin{pmatrix} -ku + \alpha^{-2}(t)u^2 v - G\alpha^{-2}(t)u^3 + Nw \\ F - \alpha^{-2}(t)u^2 v + G\alpha^{-2}(t)u^3 \\ ku - Nw \end{pmatrix}. \tag{2.18}$$

Due to the fact that $H^1(\mathcal{O}) \hookrightarrow L^6(\mathcal{O})$ and (2.6), we know that $\tilde{f}(g,\omega)$ is locally Lipschitz continuous with respect to g and bounded for every $\omega \in \Omega$. By the same method in [17, Chapters II and XV], we can prove for P-a.s. every $\omega \in \Omega$ the local existence and uniqueness of the weak solution $g(t)$, $t \in [\tau, T]$ for some $\tau < T$, of (2.16) with $g(\tau) = g_0$, which is continuously depending on the initial data $g_0 \in H$ and turns out to be a strong solution

on $(\tau, T]$ by [16, Theorem 48.5]. One can show that for P-a.s. every $\omega \in \Omega$, the following statements hold for all $\tau < T$.

(i) If $\widetilde{g}(0, \omega) \in H$, then $\widetilde{g}(t, \omega)$ lies in

$$C([\tau, T); H) \cap C^1((\tau, T); H) \cap L^2([\tau, T); E). \tag{2.19}$$

(ii) $\widetilde{g}(t, \widetilde{g}(0, \omega))$ is jointly continuous in t and $\widetilde{g}(0, \omega)$ in $[\tau, T) \times H$.

(iii) The solution mapping of (2.16) satisfies the property of an RDS.

This system has a unique solution for every $\omega \in \Omega$. Hence the solution mapping

$$\widetilde{S}(t, \omega) : \widetilde{g}(\tau, \omega) \longmapsto \widetilde{g}(t, \omega) \tag{2.20}$$

generates an RDS. So the transformation

$$\widehat{S}(t, \omega) : \alpha^{-1}(t)\widetilde{g}(\tau, \omega) \longmapsto \alpha^{-1}(t)\widetilde{g}(t, \omega) \tag{2.21}$$

also determines an RDS corresponding to system (1.1).

We will prove the existence of a nonempty compact random attractor for the RDS $\widehat{S}(t, \omega)$.

3. Existence of a Random Attractor

A random set $\mathcal{K}(\omega)$ is said to absorb the set $B \subset X$ for an RDS φ if P-a.s. there exists $t_B(\omega)$ such that

$$\varphi(t, \theta_{-t}\omega)B \subset \mathcal{K}(\omega) \quad \forall t \geq t_B(\omega). \tag{3.1}$$

A random set $\mathcal{A}(\omega)$ is said to be a random attractor associated to the RDS φ if P-a.s.:

(i) $\mathcal{A}(\omega)$ is a random compact set, that is, P-a.s. $\omega \in \Omega$, $\mathcal{A}(\omega)$ is compact, and for all $x \in X$ and P-a.s. the map $x \mapsto \text{dist}(x, \mathcal{A}(\omega))$ is measurable.

(ii) $\varphi(t, \omega)\mathcal{A}(\omega) = A(\theta_t\omega)$ for all $t \geq 0$ (invariance).

(iii) For all bounded $B \subset X$,

$$\lim_{t \to \infty} \text{dist}(\varphi(t, \theta_{-t}\omega)B, \mathcal{A}(\omega)) = 0, \tag{3.2}$$

where $\text{dist}(\cdot, \cdot)$ denotes the Hausdorff semidistance:

$$\text{dist}(Y, Z) = \sup_{x \in Y} \inf_{y \in Z} d(x, y), \quad Y, Z \in X. \tag{3.3}$$

Proposition 3.1 (see [14, 15]). *Let ϕ be an RDS on a Polish space (X, d) with Borel σ-algebra $\mathcal{B}$ over the flow $\{\theta_t\}_{t \in \mathbf{R}}$ on a probability space $(\Omega, \mathcal{F}, P)$. Suppose there exists a random compact set $\mathcal{K}(\omega)$ such that for any bounded nonrandom set $B \subset X$ P-a.s*

$$\operatorname{dist}(\varphi(t, \theta_{-t}\omega)B, \mathcal{K}(\omega)) \longrightarrow 0 \quad as \ t \longrightarrow +\infty. \tag{3.4}$$

Then the set

$$\mathcal{A}(\omega) = \overline{\bigcup_{B \subset X} \Lambda_B(\omega)} \tag{3.5}$$

is a unique random attractor for ϕ, where the union is taken over all bounded $B \subset X$ and $\Lambda_B(\omega)$ is the omega-limit set of B given by

$$\Lambda_B(\omega) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \phi(t, \theta_{-t}\omega)B}. \tag{3.6}$$

Now, we will show the existence of a random attractor for the RDS (2.16).

Lemma 3.2. *There exists a random variable $r_1(\omega) > 0$, depending on F, G, σ, and μ, such that for all $\rho > 0$ there exists $t(\omega) \leq -1$ such that the following holds P-a.s. For all $t_0 \leq t(\omega)$, and for all $\widetilde{g}_0 \in H$ with $|\widetilde{g}_0| \leq \rho$, the solution $g(t, \omega; t_0, \alpha(t_0)\widetilde{g}_0)$ of system (2.16) over $[t_0, \infty)$, with $g(t_0) = \alpha(t_0, \omega)\widetilde{g}_0$, satisfies the inequality*

$$\left| g(-1, \omega; t_0, \alpha(t_0, \omega)\widetilde{g}_0) \right|^2 \leq r_1^2(\omega). \tag{3.7}$$

Proof. Define

$$W(t, x) = \frac{N}{k} w(t, x), \qquad \mu = \frac{k}{N}. \tag{3.8}$$

Then (2.13)–(2.15) can be written as

$$\frac{\partial u}{\partial t} = d_1 \triangle u - (F + k)u + \alpha^{-2}(t)u^2 v - G\alpha^{-2}(t)u^3 + kW, \tag{3.9}$$

$$\frac{\partial v}{\partial t} = d_2 \triangle v + F(\alpha(t) - v) - \alpha^{-2}(t)u^2 v + G\alpha^{-2}(t)u^3, \tag{3.10}$$

$$\mu \frac{\partial W}{\partial t} = \mu d_3 \triangle W + ku - (\mu F + k)W. \tag{3.11}$$

Taking the inner products of (3.9)–(3.11) with $Gu(t)$, $v(t)$, and $GW(t)$, respectively. Then sum up the resulting equalities. By the Neumann boundary condition (1.2), we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(G|u|^2+|v|^2+\mu G|W|^2\right)+G(F+k)|u|^2+F|v|^2+G(\mu F+k)|W|^2+d_1G\|u\|^2+d_2\|v\|^2+\mu Gd_3\|W\|^2\\
&\leq \int_\Omega F\alpha(t)v\,dx+2kG\int_\Omega uW\,dx-\alpha^{-2}(t)\int_\Omega\left(Gu^2-uv\right)^2dx\\
&\leq kG|u|^2+kG|W|^2+\frac{F}{2}|v|^2+\frac{F\alpha^2(t)}{2}|\mathcal{O}|,
\end{aligned}
\tag{3.12}
$$

where $|\mathcal{O}|$ denotes the volume of $\mathcal{O}$. Set

$$
d=\min\{d_1,d_2,d_3\},\qquad C_1=\frac{F|\mathcal{O}|}{\min\{1,G,G/\mu\}}.
\tag{3.13}
$$

Then (3.12) yields

$$
\frac{d}{dt}\left|g(t)\right|^2+d\left\|g(t)\right\|^2+F\left|g(t)\right|^2\leq C_1\alpha^2(t).
\tag{3.14}
$$

Applying Gronwall's inequality to (3.14) and then integrating in $[t_0,-1]$, with $t_0\leq -1$ we have

$$
\begin{aligned}
\left|g(-1)\right|^2&\leq\left|\alpha(t_0)g_0\right|^2e^{-F(-1-t_0)}+C_1\int_{t_0}^{-1}e^{-F(-1-s)}\alpha^2(s)ds\\
&\leq e^F\left(e^{Ft_0}\left|\alpha(t_0)g_0\right|^2+C_1\int_{t_0}^{-1}e^{Fs}\alpha^2(s)ds\right).
\end{aligned}
\tag{3.15}
$$

Consequently, give $B(0,\rho)\subset H$, P-a.s. there exists $t(\omega,\rho)\leq -1$ such that for $t_0\leq t(\omega,\rho)$ and all $g_0\in B(0,\rho)$,

$$
\left|g\left(-1,\omega;t_0,\alpha(t_0)\widetilde{g}_0\right)\right|^2\leq r_1^2(\omega),
\tag{3.16}
$$

with

$$
r_1^2(\omega)=e^F\left(1+C_1\int_{-\infty}^{-1}e^{Fs}\alpha^2(s)ds\right).
\tag{3.17}
$$

Indeed, it is enough to choose $t(\omega,\rho)$ such that

$$
e^{Ft_0}\alpha^2(t_0)\rho^2\leq 1,
\tag{3.18}
$$

and take into account (3.15) and the fact that P-a.s. $e^{Fs}\alpha^2(s)=e^{Fs}e^{-2\sigma W_s}\to 0$ as $s\to -\infty$. $\quad\square$

If we now return to (3.14) and integrate for $t \in [-1, 0]$, we have

$$|g(t)|^2 \leq |g(-1)|^2 e^{-F(t+1)} + C_1 \int_{-1}^{t} e^{-F(t-s)} \alpha^2(s) ds,$$

$$d \int_{-1}^{0} \|g(s)\|^2 ds \leq |g(-1)|^2 + C_1 \int_{-1}^{0} \alpha^2(s) ds. \tag{3.19}$$

Thus, we can conclude that given $B(0, \rho) \subset H$ and P-a.s. there exists $t(\omega, \rho) \leq -1$ such that for $t_0 \leq t(\omega, \rho)$ and for all $g_0 \in B(0, \rho)$,

$$|g(t, \omega; t_0, g_0)|^2 \leq e^{-F(t+1)} r_1^2(\omega) + C_1 \int_{-1}^{t} e^{-F(t-s)} \alpha^2(s) ds,$$

$$d \int_{-1}^{0} \|g(s, \omega; t_0, g_0)\|^2 ds \leq r_1^2(\omega) + C_1 \int_{-1}^{0} \alpha^2(s) ds. \tag{3.20}$$

To prove the absorption at time $t = 0$, we need the following proposition.

Proposition 3.3. *There exists a random variable $r_3(\omega) > 0$, depending on λ_1, σ, and d, such that for all $\rho' > 0$ there exists $t(\omega) \leq -1$ such that the following holds P-a.s. For all $t_0 \leq t(\omega)$ and for all $\tilde{g}_0 \in H$ with $|\tilde{g}_0| \leq \rho'$, the solution $g(t, \omega; t_0, \alpha(t_0) g_0)$ of system (2.16) over $[t_0, \infty)$, with $g(t_0) = \alpha(t_0, \omega) \tilde{g}_0$, satisfies the inequality*

$$|g(-1, \omega; t_0, \alpha(t_0, \omega) \tilde{g}_0)|_X^6 \leq r_3^2(\omega). \tag{3.21}$$

Also, for $t \in [-1, 0]$ one has

$$|g(t)|_X^6 \leq e^{-F(t+1)} r_3^2(\omega) + C_4 \int_{-1}^{t} e^{-F(t-\tau)} \alpha^6(\tau) d\tau. \tag{3.22}$$

Proof. Letting $V(t, x) = v(t, x)/G$, (3.9)–(3.11) can be written as

$$\frac{\partial u}{\partial t} = d_1 \triangle u - (F + k) u + G\alpha^{-2}(t) u^2 V - G\alpha^{-2}(t) u^3 + kW, \tag{3.23}$$

$$\frac{\partial V}{\partial t} = d_2 \triangle v + \frac{F}{G} \alpha(t) - FV - \alpha^{-2}(t) u^2 V + \alpha^{-2}(t) u^3, \tag{3.24}$$

$$\mu \frac{\partial W}{\partial t} = \mu d_3 \triangle W + ku - (\mu F + k) W. \tag{3.25}$$

Take the inner products $((3.23), u^5(t))$, $((3.24), GV^5(t))$ and $((3.25), W^5(t))$ and sum up the resulting equalities. By the Neumann boundary condition, we get

$$\frac{1}{6}\frac{d}{dt}\left(|u|_{L^6}^6 + G|V|_{L^6}^6 + \mu|W|_{L^6}^6\right) + 5\left(d_1\left\|u^2\nabla u\right\|^2 + d_2 G\left\|V^2\nabla V\right\|^2 + \mu d_3 G\left\|W^2\nabla W\right\|^2\right)$$

$$= -(F+k)\int_\Omega u^6 ds + F\int_\Omega \alpha(t)V^5 dx - GF\int_\Omega V^6 dx$$

$$- (\mu F + k)\int_\Omega W^6 dx + k\int_\Omega u^5 W dx + k\int_\Omega uW^5 dx$$

$$- G\alpha^{-2}(t)\int_\Omega \left(u^8 - u^7 V - u^3 V^5 + u^2 V^6\right)dx. \tag{3.26}$$

By using Young's inequality, we obtain

$$-G\alpha^{-2}(t)\int_\Omega \left(u^8 - u^7 V - u^3 V^5 + u^2 V^6\right)dx \le 0,$$

$$k\int_\Omega u^5 W dx + k\int_\Omega uW^5 dx \le k|u|_{L^6}^6 + k|W|_{L^6}^6. \tag{3.27}$$

From (3.27), (3.26) yields

$$\frac{d}{dt}\left(|u|_{L^6}^6 + G|V|_{L^6}^6 + \mu|W|_{L^6}^6\right) + F\left(|u|_{L^6}^6 + G|V|_{L^6}^6 + \mu|W|_{L^6}^6\right) \le \frac{F|\mathcal{O}|}{G^5}\alpha^6(t), \tag{3.28}$$

that is,

$$\frac{d}{dt}\left(|u|_{L^6}^6 + G^{-5}|v|_{L^6}^6 + \mu^{-5}|w|_{L^6}^6\right) + F\left(|u|_{L^6}^6 + G^{-5}|v|_{L^6}^6 + \mu^{-5}|w|_{L^6}^6\right) \le \frac{F|\mathcal{O}|}{G^5}\alpha^6(t). \tag{3.29}$$

By denoting

$$C_4 = \frac{F|\mathcal{O}|}{G^5 \min\{1, G^{-5}, \mu^{-5}\}}, \tag{3.30}$$

then (3.29) implies that

$$\frac{d}{dt}|g(t)|_X^6 + F|g(t)|_X^6 \le C_4\alpha^6(t). \tag{3.31}$$

Integrating in $[t_0, -1]$, with $t_0 \le -1$ we have

$$|g(-1)|_X^6 \le |\alpha(t_0)g_0|_X^6 e^{-F(-1-t_0)} + C_4\int_{t_0}^{-1} e^{-F(-1-s)}\alpha^6(s)ds. \tag{3.32}$$

Consequently, given $B(0, \rho') \subset H$, P-a.s. there exists $t(\omega, \rho') \leq -1$ such that for all $t_0 \leq t(\omega, \rho')$ and for all $g_0 \in B(0, \rho')$,

$$|g(-1, \omega, t_0, \alpha(t_0)g_0)|_X^6 \leq r_3^2(\omega), \tag{3.33}$$

with

$$r_3^2(\omega) = e^F \left(1 + C_4 \int_{-\infty}^{-1} e^{Fs} \alpha^6(s) ds \right). \tag{3.34}$$

In fact, it is enough to choose $t(\omega, \rho')$ to satisfy

$$e^{Ft_0} \alpha^6(t_0) \rho'^6 \leq 1, \tag{3.35}$$

and take into (3.32) and the fact that P-a.s. $e^{Fs} \alpha^6(s) = e^{Fs} e^{-6\sigma W_s} \to 0$ as $s \to -\infty$. Also, from (3.32) and for $t \in [-1, 0]$ we get

$$|g(t)|_X^6 \leq e^{-F(t+1)} r_3^2(\omega) + C_4 \int_{-1}^t e^{-F(t-\tau)} \alpha^6(\tau) d\tau. \tag{3.36}$$

$\square$

Lemma 3.4. *There exists a random variable $r_2(\omega) > 0$, depending on F, G, N, d, k, and σ, such that for all $\rho > 0$ there exists $t(\omega) \leq -1$ such that the following holds P-a.s. For all $t_0 \leq t(\omega)$ and for all $\tilde{g}_0 \in H$ with $|\tilde{g}_0| \leq \rho$, there exists a unique solution $g(t, \omega; t_0, \alpha(t_0)g_0)$ of system (2.16) over $[t_0, \infty)$, with $g(t_0) = \alpha(t_0, \omega)\tilde{g}_0$, and put $\tilde{g}(t, \omega; t_0, \tilde{g}_0) := \alpha^{-1}(t, \omega)g(t, \omega; t_0, \tilde{g}_0)$. Then*

$$\left\| \tilde{g}(0, \omega; t_0, \tilde{g}_0) \right\|^2 \leq r_2^2(\omega). \tag{3.37}$$

Proof. To get a bound in E, we multiply (2.13)–(2.15) by $-\Delta u$, $-\Delta v$, and $--\Delta w$, respectively. Add up the three equalities, and due to the Neumann boundary condition, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + \|v\|^2 + \|w\|^2\right) + d_1|\Delta u|^2 + d_2|\Delta v|^2 + d_3|\Delta w|^2 + (F+k)\|u\|^2 + F\|v\|^2 + (F+N)\|w\|^2$$

$$= -\alpha^{-2}(t)\int_\Omega u^2 v \Delta u dx + G\alpha^{-2}(t)\int_\Omega u^3 \Delta u dx - N\int_\Omega w\Delta u dx$$

$$- F\alpha(t)\int_\Omega \Delta v dx + \alpha^{-2}(t)\int_\Omega u^2 v \Delta v dx - G\alpha^{-2}(t)\int_\Omega u^3 \Delta v dx$$

$$- k\int_\Omega u\Delta w dx$$

$$\leq \frac{d_1}{2}|\Delta u|^2 + \frac{d_2}{2}|\Delta v|^2 + \frac{d_3}{2}|\Delta w|^2 + \alpha^{-4}(t)\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\int_\Omega u^4 v^2 dx$$

$$+ \frac{N^2}{d_1}\int_\Omega w^2 dx + \frac{k^2}{2d_3}\int_\Omega u^2 dx + \frac{G^2 \alpha^{-4}(t)}{d_2}\int_\Omega u^6 dx, \tag{3.38}$$

that is,

$$\frac{d}{dt}\left(\|u\|^2 + \|v\|^2 + \|w\|^2\right) + F\left(\|u\|^2 + \|v\|^2 + \|w\|^2\right)$$

$$\leq \alpha^{-4}(t)\left(\frac{4}{3d_1} + \frac{4}{3d_2} + \frac{2G^2}{d_2}\right)\int_\Omega \left(u^6 + v^6\right)dx + \frac{2N^2}{d_1}|w|^2 + \frac{k^2}{d_3}|u|^2 \tag{3.39}$$

$$\leq C_2\alpha^{-4}(t)|g|_X^6 + C_3|g|^2.$$

Here $d = \min\{d_1, d_2, d_3\}$, $C_2 = (1/d)(8/3 + 2G^2)$, $C_3 = (1/d)\max\{2N^2, k^2\}$. Then from (3.39) we have

$$\frac{d}{dt}\|g\|^2 + F\|g\|^2 \leq C_2\alpha^{-4}(t)|g|_X^6 + C_3|g|^2. \tag{3.40}$$

Integrating (3.40) in $[s, 0]$, $s \in [-1, 0]$, we obtain

$$\|g(0)\|^2 \leq \|g(s)\|^2 + C_2\int_s^0 \alpha^{-4}(\tau)|g(\tau)|_X^6 d\tau + C_3\int_s^0 |g(\tau)|^2 d\tau. \tag{3.41}$$

Integrating (3.41) in $[-1, 0]$ and by (3.20),

$$\|g(0)\|^2 \leq \int_{-1}^0 \|g(\tau)\|^2 d\tau + C_2\int_{-1}^0 \alpha^{-4}(\tau)|g(\tau)|_X^6 d\tau + C_3\int_{-1}^0 |g(\tau)|^2 d\tau$$

$$\leq \left(\frac{1}{d} + \frac{C_3}{F}\right)\left(r_1^2(\omega) + C_1\int_{-1}^0 \alpha^2(\tau)d\tau\right) + C_2\sup_{-1\leq t\leq 0}\alpha^{-4}(t)\int_{-1}^0 |g(\tau)|_X^6. \tag{3.42}$$

It is now straightforward from (3.31) and (3.36) that

$$\|g(0)\|^2 = \|\widetilde{g}(0)\|^2 \leq \left(\frac{1}{d} + \frac{C_3}{F}\right)\left(r_1^2(\omega) + C_1\int_{-1}^0 \alpha^2(\tau)d\tau\right)$$

$$+ \frac{C_2}{F}\sup_{-1\leq t\leq 0}\alpha^{-4}(t)\left(r_3^2(\omega) + C_4\int_{-1}^0 \alpha^6(\tau)d\tau\right). \tag{3.43}$$

Consequently, P-a.s. there exists $r^2(\omega)$ such that given $\rho > 0$, there exists $\widetilde{t}(\omega) \leq -1$ such that for all $t_0 \leq \widetilde{t}(\omega)$ and $\widetilde{g}_0 \in H$ with $|\widetilde{g}_0| \leq \rho$,

$$\|\widetilde{g}(0, \omega; t_0, \widetilde{g}_0)\|^2 \leq r_2^2(\omega), \tag{3.44}$$

where

$$r_2^2(\omega) = (C_5 + C_6)e^F + C_1C_5 \int_{-\infty}^{-1} e^{F(s+1)}\alpha^2(s)ds + C_1C_5 \int_{-1}^{0} \alpha^2(s)ds$$

$$+ C_4C_6 \int_{-\infty}^{-1} e^{F(s+1)}\alpha^6(s)ds + C_4C_6 \int_{-1}^{0} \alpha^6(s)ds, \qquad (3.45)$$

$$C_5 = \left(\frac{1}{d} + \frac{C_3}{F}\right), \qquad C_6 = \frac{C_2}{F} \sup_{-1 \le t \le 0} \alpha^{-4}(t). \qquad \square$$

Thus, we can have the main result.

Theorem 3.5. *The RDS $\hat{S}(t,\omega)$ has a nonempty compact random attractor $\mathcal{A}(\omega)$.*

Proof. This follows from Lemma 3.2 and Lemma 3.4 combined with the embedding of $E \hookrightarrow H$ and Proposition 3.1. $\square$

Remark 3.6. It is necessary and interesting for us to consider the Hausdorff dimension of the random attractor which is generated by the stochastic three-component reversible Gray-Scott system with multiplicative white noise, but it seems impossibile to apply the results in [18, 19] directly because of the higher-order terms. In order to verify the differentiability properties for the cocycle generated by the random system, we need to check condition (2.7)–(2.13) in [19]. Considering the linearized equation (2.13)–(2.15), we have

$$\frac{\partial \tilde{U}}{\partial t} = d_1 \triangle \tilde{U} - (F+k)\tilde{U} + 2uve^{2\sigma W_t}\tilde{U} + u^2\tilde{V}e^{2\sigma W_t} - 3Ge^{2\sigma W_t}u^2\tilde{U} + N\widetilde{W},$$

$$\frac{\partial \tilde{V}}{\partial t} = d_1 \triangle \tilde{V} - F\tilde{V} - 2uve^{2\sigma W_t}\tilde{U} - u^2\tilde{V}e^{2\sigma W_t} + 3Gu^2e^{2\sigma W_t}\tilde{U}, \qquad (3.46)$$

$$\frac{\partial \widetilde{W}}{\partial t} = d_1 \triangle \widetilde{W} + k\tilde{U} - (F+N)\widetilde{W}.$$

Letting $\Phi(t) = (\tilde{U}, \tilde{V}, \widetilde{W})$, $q(t) = \tilde{g}(t) - \overline{g}(t) - \Phi(t)$, then

$$q(t) = (q_1(t), q_2(t), q_3(t)) = \left(\tilde{u} - \overline{u} - \tilde{U}, \tilde{v} - \overline{v} - \tilde{V}, \tilde{w} - \overline{w} - \widetilde{W}\right) \qquad (3.47)$$

satisfies

$$\frac{\partial q_1}{\partial t} = d_1 \triangle q_1 - (F+k)q_1 + \alpha(t)\left(\tilde{u}^2\tilde{v} - \overline{u}^2\overline{v}\right) - 2\tilde{u}\tilde{v}e^{2\sigma W_t}(t)\tilde{U} - \tilde{u}^2 e^{2\sigma W_t}\tilde{V}$$

$$+ 3G\tilde{u}^2 e^{2\sigma W_t}\tilde{U} - G\alpha(t)\left(\tilde{u}^3 - \overline{u}^3\right) + Nq_3,$$

$$\frac{\partial q_2}{\partial t} = d_1 \triangle q_2 - Fq_2 - \alpha(t)\left(\tilde{u}^2\tilde{v} - \overline{u}^2\overline{v}\right) + 2\tilde{u}\tilde{v}e^{2\sigma W_t}\tilde{U} + \tilde{u}^2 e^{2\sigma W_t}\tilde{V} \qquad (3.48)$$

$$- 3G\tilde{u}^2 e^{2\sigma W_t}\tilde{U} + G\alpha(t)\left(\tilde{u}^3 - \overline{u}^3\right),$$

$$\frac{\partial q_3}{\partial t} = d_1 \triangle q_3 + kq_1 - (F+N)q_3.$$

Here, let

$$\widetilde{g}(t) = \widetilde{g}(t,\omega;0,\widetilde{g}_0) = (\widetilde{u}(t,\omega;0,\widetilde{u}_0),\widetilde{v}(t,\omega;0,\widetilde{v}_0),\widetilde{w}(t,\omega;0,\widetilde{w}_0)),$$
$$\overline{g}(t) = \overline{g}(t,\omega;0,\overline{g}_0) = (\overline{u}(t,\omega;0,\overline{u}_0),\overline{v}(t,\omega;0,\overline{v}_0),\overline{w}(t,\omega;0,\overline{w}_0))$$

$$(3.49)$$

be two solutions of system (1.1). From (3.48), it seems hard to get the conclusion.

Acknowledgments

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