

Research Article

Solutions for p -Laplacian Dynamic Delay Differential Equations on Time Scales

Hua Su,¹ Lishan Liu,² and Xinjun Wang³

¹ School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Shandong Jinan, 250014, China

² School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

³ School of Economics, Shandong University, Jinan 250014, China

Correspondence should be addressed to Hua Su, jnsuhua@163.com

Received 9 December 2011; Accepted 19 January 2012

Academic Editor: Rudong Chen

Copyright © 2012 Hua Su et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let T be a time scale. We study the existence of positive solutions for the nonlinear four-point singular boundary value problem with p -Laplacian dynamic delay differential equations on time scales, subject to some boundary conditions. By using the fixed-point index theory, the existence of positive solution and many positive solutions for nonlinear four-point singular boundary value problem with p -Laplacian operator is obtained.

1. Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [1] and is a new area of still fairly theoretical exploration in mathematics. Boundary value problems for delay differential equations arise in a variety of areas of applied mathematics, physics, and variational problems of control theory (see [2, 3]). In recent years, many authors have begun to pay attention to the study of boundary-value problems or with p -Laplacian equations or with p -Laplacian dynamic equations on time scales (see [4–14] and the references therein).

In [7], Sun and Li considered the existence of positive solution of the following dynamic equations on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(t, u(t)) = 0, \quad t \in (0, T), \quad (1.1)$$

$$\beta u(0) - \gamma u^\Delta(0) = 0, \quad \alpha u(\eta) = u(T), \quad (1.2)$$

where $\beta, \gamma \geq 0$, $\beta + \gamma > 0$, $\eta \in (0, \rho(T))$, $0 < \alpha < T/\eta$. They obtained the existence of single

and multiple positive solutions of the problem (1.1) and (1.2) by using fixed-point theorem and Leggett-Williams fixed-point theorem (see [15]), respectively.

In [4], Anderson discussed the following dynamic equation on time scales:

$$\begin{aligned} u^{\Delta\nabla}(t) + a(t)f(u(t)) &= 0, \quad t \in (0, T), \\ u(0) = 0, \quad \alpha u(\eta) &= u(T). \end{aligned} \quad (1.3)$$

He obtained some results for the existence of one positive solution of the problem (1.3) based on the limits $f_0 = \lim_{u \rightarrow 0^+} f(u)/u$ and $f_\infty = \lim_{u \rightarrow \infty} f(u)/u$.

In [5], Kaufmann studied the problem (1.3) and obtained the existence results of at least two positives solutions.

In [14], Wang et al. discussed the following dynamic equation on time scales by using Avery-Peterson fixed theorem (see [14]):

$$(\phi_p(u'))' + q(t)f(t, u(t), u(t-1), u'(t)) = 0, \quad t \in (0, 1), \quad (1.4)$$

$$u(t) = \xi(t), \quad -1 \leq t \leq 0, \quad u(1) = 0, \quad (1.5)$$

$$u(t) = \xi(t), \quad -1 \leq t \leq 0, \quad u'(1) = 0. \quad (1.5')$$

They obtained some results for the existence three positive solutions of the problem (1.4), (1.5) and (1.4), and (1.5'), respectively.

In [15], Lee and Sim discussed the following equation:

$$\begin{aligned} (\phi_p(u'))' + \lambda h(t)f(u(t)) &= 0, \quad \text{a. e. } t \in (0, 1), \\ u(0) = u(1) &= 0. \end{aligned} \quad (1.6)$$

By applying the global bifurcation theorem and figuring the shape of unbounded subcontinua of solutions, they obtain many different types of global existence results of positive solutions.

However, there are not many concerning the p -Laplacian problems on time scales. Especially, for the singular multipoint boundary value problems for p -Laplacian dynamic delay differential equations on time scales, with the author's acknowledge, no one has studied the existence of positive solutions in this case.

Recently, in [16], we have studied the existence of positive solutions for the following nonlinear two-point singular boundary value problem with p -Laplacian operator:

$$\begin{aligned} (\phi_p(u'))' + a(t)f(u(t)) &= 0, \quad 0 < t < 1, \\ \alpha\phi_p(u(0)) - \beta\phi_p(u'(0)) &= 0, \quad \gamma\phi_p(u(1)) + \delta\phi_p(u'(1)) = 0. \end{aligned} \quad (1.7)$$

By using the fixed-point theorem of cone expansion and compression of norm type, the existence of positive solution and infinitely many positive solutions for nonlinear singular boundary value problem (1.7) with p -Laplacian operator is obtained.

Now, motivated by the results mentioned above, in this paper, we study the existence of positive solutions for the following nonlinear four-point singular boundary value problem

with higher-order p -Laplacian dynamic delay differential equations operator on time scales (SBVP):

$$\left(\phi_p\left(u^\Delta(t)\right)\right)^\nabla + g(t)f(u(t-\tau), u(t)) = 0, \quad 0 < t < T, \tau > 0, \quad (1.8)$$

$$u(t) = \zeta(t), \quad -\tau \leq t \leq 0,$$

$$\alpha\phi_p(u(0)) - \beta\phi_p\left(u^\Delta(\xi)\right) = 0, \quad (1.9)$$

$$\gamma\phi_p(u(T)) + \delta\phi_p\left(u^\Delta(\eta)\right) = 0,$$

or

$$u(t) = \zeta(t), \quad -\tau \leq t \leq 0,$$

$$u(0) - B_0\left(u^\Delta(\xi)\right) = 0, \quad (1.10)$$

$$u(T) + B_1\left(u^\Delta(\eta)\right) = 0,$$

where $\phi_p(s)$ is p -Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = \phi_p^{-1}$, $1/p + 1/q = 1$. $\xi, \eta \in (0, T)$, $\tau \in [0, T]$ is prescribed and $\xi < \eta$, $g : (0, T) \rightarrow [0, \infty)$, $\alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0$ and B_0, B_1 are both nondecreasing continuous odd functions defined on $(-\infty, +\infty)$.

In this paper, by constructing one integral equation which is equivalent to the problem (1.8), (1.9) and (1.8), and (1.10), we research the existence of positive solutions for nonlinear singular boundary value problem (1.8), (1.9) and (1.8), and (1.10) when g and f satisfy some suitable conditions.

Our main tool of this paper is the following fixed point index theory.

Theorem 1.1 (see [17, 18]). *Suppose that E is a real Banach space, $K \subset E$ is a cone, let $\Omega_r = \{u \in K : \|u\| \leq r\}$. Let operator $T : \Omega_r \rightarrow K$ be completely continuous and satisfy $Tx \neq x, \forall x \in \partial\Omega_r$. Then*

$$(i) \text{ if } \|Tx\| \leq \|x\|, \forall x \in \partial\Omega_r, \text{ then } i(T, \Omega_r, K) = 1;$$

$$(ii) \text{ if } \|Tx\| \geq \|x\|, \forall x \in \partial\Omega_r, \text{ then } i(T, \Omega_r, K) = 0.$$

This paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we discuss the existence of single solution of the systems (1.8) and (1.9). In Section 4, we study the existence of at least two solutions of the systems (1.8) and (1.9). In Section 5, we discuss the existence of single and many solutions of the systems (1.8) and (1.10). In Section 6, we give two examples as the application.

2. Preliminaries and Lemmas

For convenience, we can found some basic definitions in [1, 19, 20].

In the rest of this paper, \mathbf{T} is closed subset of R with $0 \in \mathbf{T}_k, T \in \mathbf{T}^k$. And let $B = \{u \in C_{ld}[-\tau, T]\}$, then B is a Banach space with the norm $\|u\| = \max_{t \in [-\tau, T]} |u(t)|$. And let

$$K = \{u \in B : u(t) \geq 0, u(t) \text{ is concave function}, t \in [0, T]\}. \quad (2.1)$$

Obviously, K is a cone in B . Set $K_r = \{u \in K : \|u\| \leq r\}$.

Definition 2.1. $u(t)$ is called a solution of SBVP (1.8) and (1.9) if it satisfies the following:

- (1) $u \in C[-\tau, 0] \cap C_{ld}(0, T)$,
- (2) $u(t) > 0$ for all $t \in (0, T)$ and satisfies conditions (1.9),
- (3) $(\phi_p(u^\Delta(t)))^\nabla = -g(t)f(u(t-\tau), u(t))$ holds for $t \in (0, T)$.

In the rest of the paper, we also make the following assumptions:

- (H₁) $f \in C_{ld}([0, +\infty)^2, [0, +\infty))$,
- (H₂) $g(t) \in C_{ld}((0, T), [0, +\infty))$ and there exists $t_0 \in (0, T)$, such that

$$g(t_0) > 0, \quad 0 < \int_0^T g(s) \nabla s < +\infty, \quad (2.2)$$

- (H₃) $\zeta(t) \in C([-\tau, 0], \zeta(t) > 0$ on $[-\tau, 0)$ and $\zeta(0) = 0$,
- (H₄) B_0, B_1 are both increasing, continuous, odd functions defined on $(-\infty, +\infty)$, and at least one of them satisfies the condition that there exists one $b > 0$ such that

$$0 < B_i(v) \leq bv, \quad \forall v \geq 0, i = 0 \text{ or } 1. \quad (2.3)$$

It is easy to check that condition (H₂) implies that

$$0 < \int_0^T \phi_q \left(\int_0^s g(s_1) \nabla s_1 \right) \Delta s < +\infty. \quad (2.4)$$

We can easily get the following Lemmas.

Lemma 2.2. *Suppose that condition (H₂) holds. Then there exists a constant $\theta \in (0, 1/2)$ that satisfies*

$$0 < \int_\theta^{T-\theta} g(t) \nabla t < \infty. \quad (2.5)$$

Furthermore, the function

$$A(t) = \int_\theta^t \phi_q \left(\int_s^t g(s_1) \nabla s_1 \right) \Delta s + \int_t^{T-\theta} \phi_q \left(\int_t^s g(s_1) \nabla s_1 \right) \nabla s, \quad t \in [\theta, T-\theta] \quad (2.6)$$

is positive continuous functions on $[\theta, T-\theta]$; therefore, $A(t)$ has minimum on $[\theta, T-\theta]$. Hence, we suppose that there exists $L > 0$ such that $A(t) \geq L, t \in [\theta, T-\theta]$.

Proof. At first, it is easily seen that $A(t)$ is continuous on $[\theta, T - \theta]$. Next, let

$$A_1(t) = \int_{\theta}^t \phi_q \left(\int_s^t g(s_1) \nabla s_1 \right) \Delta s, \quad A_2(t) = \int_t^{T-\theta} \phi_q \left(\int_t^s g(s_1) \nabla s_1 \right) \Delta s. \quad (2.7)$$

Then, from condition (H_2) , we have the function $A_1(t)$ is strictly monotone nondecreasing on $[\theta, T - \theta]$ and $A_1(\theta) = 0$, the function $A_2(t)$ is strictly monotone nonincreasing on $[\theta, T - \theta]$ and $A_2(T - \theta) = 0$, which implies $L = \min_{t \in [\theta, T - \theta]} A(t) > 0$. The proof is complete. \square

Lemma 2.3 (see [16]). *Let $u \in K$ and θ of Lemma 2.2, then*

$$u(t) \geq \theta \|u\|, \quad t \in [\theta, T - \theta]. \quad (2.8)$$

Lemma 2.4. *Suppose that conditions (H_1) , (H_2) , (H_3) , and (H_4) hold, $u(t) \in B \cap C_{ld}(0, 1)$ is a solution of the following boundary value problems:*

$$\left(\phi_p \left(u^\Delta(t) \right) \right)^\nabla + g(t) f(u(t - \tau) + h(t - \tau), u(t)) = 0, \quad 0 < t < T, \quad (2.9)$$

$$u(t) = 0, \quad -\tau \leq t \leq 0,$$

$$\alpha \phi_p(u(0)) - \beta \phi_p \left(u^\Delta(\xi) \right) = 0, \quad (2.10)$$

$$\gamma \phi_p(u(T)) + \delta \phi_p \left(u^\Delta(\eta) \right) = 0,$$

or

$$u(t) = 0, \quad -\tau \leq t \leq 0,$$

$$u(0) - B_0 \left(u^\Delta(\xi) \right) = 0, \quad (2.10')$$

$$u(T) + B_1 \left(u^\Delta(\eta) \right) = 0,$$

where

$$h(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ 0, & 0 \leq t \leq T. \end{cases} \quad (2.11)$$

Then, $\bar{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8) and (1.9) or (1.8) and (1.10).

Proof. It is easy to check that $\bar{u}(t)$ satisfies (1.8) and (1.9) or (1.8) and (1.10). \square

So in the rest section of this paper, we focus on SBVP (2.9), (2.10), and (2.9), (2.10').

Lemma 2.5. *Suppose that conditions (H_1) , (H_2) , (H_3) , or (H_1) , (H_2) , (H_3) , (H_4) hold, $u(t) \in B \cap C_{ld}(0, 1)$ is a solution of boundary value problems (2.9), (2.10) or (2.9), (2.10'), respectively, if and*

only if $u(t) \in B$ is a solution of the following integral equation, respectively:

$$u(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \int_0^t \omega(s) \Delta s, & 0 \leq t \leq T, \end{cases} \quad (2.12)$$

$$u(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \int_0^t \bar{\omega}(s) \Delta s, & 0 \leq t \leq T, \end{cases}$$

where

$$\omega(t) = \begin{cases} \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ \quad + \int_0^t \phi_q \left(\int_s^{\sigma} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s, & 0 \leq t \leq \sigma, \\ \phi_q \left(\frac{\delta}{\gamma} \int_{\sigma}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ \quad + \int_t^T \phi_q \left(\int_{\sigma}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s, & \sigma \leq t \leq T, \end{cases} \quad (2.13)$$

$$\bar{\omega}(t) = \begin{cases} B_0 \circ \phi_q \left(\int_{\xi}^{\varrho} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ \quad + \int_0^t \phi_q \left(\int_s^{\varrho} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s, & 0 \leq t \leq \varrho, \\ B_1 \circ \phi_q \left(\int_{\varrho}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ \quad + \int_t^T \phi_q \left(\int_{\varrho}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s, & \varrho \leq t \leq T. \end{cases} \quad (2.13')$$

Here σ, ϱ is unique solution of the equation, respectively,

$$g_1(t) = g_2(t), \quad \bar{g}_1(t) = \bar{g}_2(t), \quad (2.14)$$

where

$$g_1(t) = \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ \quad + \int_0^t \phi_q \left(\int_s^{\sigma} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s, \\ g_2(t) = \phi_q \left(\frac{\delta}{\gamma} \int_{\sigma}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ \quad + \int_t^T \phi_q \left(\int_{\sigma}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s,$$

$$\begin{aligned}
\bar{g}_1(t) &= B_0 \circ \phi_q \left(\int_{\xi}^{\varrho} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\
&\quad + \int_0^t \phi_q \left(\int_s^{\varrho} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s, \\
\bar{g}_2(t) &= B_1 \circ \phi_q \left(\int_{\varrho}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\
&\quad + \int_t^T \phi_q \left(\int_{\varrho}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s.
\end{aligned} \tag{2.15}$$

Equation $g_1(t) = g_2(t)$, $\bar{g}_1(t) = \bar{g}_2(t)$ has unique solution in $(0, T)$. Because $g_1(t)$, $\bar{g}_1(t)$ is strictly monotone increasing on $[0, T)$, and $g_1(0) = 0$, $\bar{g}_1(0) = 0$, $g_2(t)$, $\bar{g}_2(t)$ is strictly monotone decreasing on $(0, T]$, and $g_2(T) = 0$, $\bar{g}_2(T) = 0$.

Proof. We only proof the first section of the results.

Necessity. Obviously, for $t \in (-\tau, 0)$, we have $u(t) = \zeta(t)$.

If $t \in (0, T)$, by the equation of the boundary condition and we have $u^\Delta(\xi) \geq 0$, $u^\Delta(\eta) \leq 0$, then there exist is a constant $\sigma \in [\xi, \eta] \subset (0, T)$ such that $u^\Delta(\sigma) = 0$.

Firstly, by integrating the equation of the problems (2.9) on (σ, t) , we have

$$\phi_p(u^\Delta(t)) = \phi_p(u^\Delta(\sigma)) - \int_{\sigma}^t g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s, \tag{2.16}$$

then

$$u^\Delta(t) = -\phi_q \left(\int_{\sigma}^t g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right), \tag{2.17}$$

thus

$$u(t) = u(\sigma) - \int_{\sigma}^t \phi_q \left(\int_{\sigma}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s. \tag{2.18}$$

By $u^\Delta(\sigma) = 0$ and condition (2.16), $t = \eta$ on (2.16), we have

$$\phi_p(u^\Delta(\eta)) = - \int_{\sigma}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s. \tag{2.19}$$

By the equation of the boundary condition (2.10), we have

$$\phi_p(u(T)) = -\frac{\delta}{\gamma} \phi_p(u^\Delta(\eta)), \tag{2.20}$$

then

$$u(T) = \phi_q \left(\frac{\delta}{\gamma} \int_{\sigma}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right). \quad (2.21)$$

Then, by (2.18) and let $t = T$ on (2.18), we have

$$\begin{aligned} u(\sigma) &= \phi_q \left(\frac{\delta}{\gamma} \int_{\sigma}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ &\quad + \int_{\sigma}^T \phi_q \left(\int_{\sigma}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s. \end{aligned} \quad (2.22)$$

Then

$$\begin{aligned} u(t) &= \phi_q \left(\frac{\delta}{\gamma} \int_{\sigma}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ &\quad + \int_t^T \phi_q \left(\int_{\sigma}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s. \end{aligned} \quad (2.23)$$

Similarly, for $t \in (0, \sigma)$, by integrating the equation of problems (2.9) on $(0, \sigma)$, we have

$$\begin{aligned} u(t) &= \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \Delta s \\ &\quad + \int_0^t \phi_q \left(\int_s^{\sigma} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s. \end{aligned} \quad (2.24)$$

Therefore, for any $t \in [0, T]$, $u(t)$ can be expressed as equation

$$u(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \int_0^t w(s) \Delta s, & 0 \leq t \leq T, \end{cases} \quad (2.25)$$

where $w(t)$ is expressed as (2.13).

Sufficiency. Suppose that $u(t) = \int_0^t w(s) \Delta s_{n-2} \Delta s$, $0 \leq t \leq T$. Then by (2.13), we have

$$u^{\Delta}(t) = \begin{cases} \phi_q \left(\int_t^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \geq 0, & 0 \leq t \leq \sigma, \\ -\phi_q \left(\int_{\sigma}^t g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \leq 0, & \sigma \leq t \leq T, \end{cases} \quad (2.26)$$

So, $(\phi_p(u^{\Delta})^{\nabla} + g(t) f(u(t-\tau) + h(t-\tau), u(t))) = 0$, $0 < t < T$. These imply that (2.9) holds.

Furthermore, by letting $t = 0$ and $t = T$ on (2.13) and (2.26), we can obtain the boundary value equations of (2.10). The proof is complete. \square

Now, we define an operator equation T given by

$$\begin{aligned} (Tu)(t) &= \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \int_0^t w(s) \Delta s, & 0 \leq t \leq T, \end{cases} \\ (\bar{T}u)(t) &= \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \int_0^t \bar{w}(s) \Delta s, & 0 \leq t \leq T, \end{cases} \end{aligned} \quad (2.27)$$

where $w(t)$, $\bar{w}(t)$ is given by (2.13) and (2.13').

From the definition of T , \bar{T} and above discussion, we deduce that for each $u \in K$, Tu , $\bar{T}u \in K$. Moreover, we have the following Lemma.

Lemma 2.6. $T, \bar{T} : K \rightarrow K$ is completely continuous.

Proof. We only proof the completely continuous of T .

Because

$$(Tu)^\Delta(t) = w^\Delta(t) = \begin{cases} \phi_q \left(\int_t^\sigma g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \geq 0, & 0 \leq t \leq \sigma, \\ -\phi_q \left(\int_\sigma^t g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \leq 0, & \sigma \leq t \leq T, \end{cases} \quad (2.28)$$

is continuous, decreasing on $[0, T]$, and satisfies that $(Tu)^\Delta(\sigma) = 0$, then, $Tu \in K$ for each $u \in K$ and $(Tu)(\sigma) = \max_{t \in [0, T]} (Tu)(t)$. This shows that $TK \subset K$. Furthermore, it is easy to check by Arzela-ascoli Theorem that $T : K \rightarrow K$ is completely continuous. \square

Lemma 2.7. Suppose that conditions (H_1) , (H_2) , and (H_3) hold, the solution $u(t) \in K$ of problem (2.9) and (2.10) satisfy

$$\max_{0 \leq t \leq T} |u(t-\tau) + h(t-\tau)| \leq \max_{-\tau \leq t \leq 0} |\zeta(t)|. \quad (2.29)$$

Proof. Firstly, we can have

$$\begin{aligned} \max_{0 \leq t \leq T} |u(t-\tau) + h(t-\tau)| &\leq \max_{0 \leq t \leq T} |u(t-\tau)| + \max_{0 \leq t \leq T} |h(t-\tau)| \\ &= \max_{-\tau \leq t \leq T-\tau} |u(t)| + \max_{-\tau \leq t \leq T-\tau} |h(t)| \\ &= \max_{-\tau \leq t \leq 0} |\zeta(t)|. \end{aligned} \quad (2.30)$$

The proof is complete. \square

For convenience, we set

$$\begin{aligned} H &= \max_{-\tau \leq t \leq 0} |\zeta(t)|, & \theta^* &= \frac{2}{L}, \\ \theta_* &= \frac{1}{(T + \phi_q(\beta/\alpha))\phi_q\left(\int_0^T g(r)\nabla r\right)}, & \theta_{**} &= \frac{1}{(b+1)\phi_q\left(\int_0^T g(r)\nabla r\right)}, \end{aligned} \quad (2.31)$$

where L is the constant from Lemma 2.2. By Lemma 2.5, we can also set

$$\begin{aligned} f^0 &= \lim_{u_2 \rightarrow 0} \max_{0 \leq u_1 \leq H} \frac{f(u_1, u_2)}{u_2^{p-1}}, & f^\infty &= \lim_{u_2 \rightarrow \infty} \max_{0 \leq u_1 \leq H} \frac{f(u_1, u_2)}{u_2^{p-1}}, \\ f_0 &= \lim_{u_2 \rightarrow 0} \min_{0 \leq u_1 \leq H} \frac{f(u_1, u_2)}{u_2^{p-1}}, & f_\infty &= \lim_{u_2 \rightarrow \infty} \min_{0 \leq u_1 \leq H} \frac{f(u_1, u_2)}{u_2^{p-1}}. \end{aligned} \quad (2.32)$$

3. The Existence of Single Positive Solution to (1.8) and (1.9)

In this section, we present our main results.

Theorem 3.1. *Suppose that condition (H_1) , (H_2) , and (H_3) hold. Assume that f also satisfies*

$$(A_1): f(u_1, u_2) \geq (mr)^{p-1}, \text{ for } \theta r \leq u_2 \leq r, 0 \leq u_1 \leq H,$$

$$(A_2): f(u_1, u_2) \leq (MR)^{p-1}, \text{ for } 0 \leq u_2 \leq R, 0 \leq u_1 \leq H,$$

where $m \in (\theta^*, \infty)$, $M \in (0, \theta_*)$.

Then, the SBVP (2.9), (2.10) has a solution u such that $\|u\|$ lies between r and R . Furthermore by Lemma 2.4, $\bar{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8) and (1.9).

Proof of Theorem 3.1. Without loss of generality, we suppose that $r < R$. For any $u \in K$, by Lemma 2.3, we have

$$u(t) \geq \theta \|u\|, \quad t \in [\theta, T - \theta]. \quad (3.1)$$

We define two open subset Ω_1 and Ω_2 of E :

$$\Omega_1 = \{u \in K : \|u\| < r\}, \quad \Omega_2 = \{u \in K : \|u\| < R\}. \quad (3.2)$$

For any $u \in \partial\Omega_1$, by (3.1), we have

$$r = \|u\| \geq u(t) \geq \theta \|u\| = \theta r, \quad t \in [\theta, T - \theta]. \quad (3.3)$$

For $t \in [\theta, T - \theta]$ and $u \in \partial\Omega_1$, we shall discuss it from three perspectives.

(i) If $\sigma \in [\theta, T - \theta]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$\begin{aligned} 2\|Tu\| &= 2(Tu)(\sigma) \\ &\geq \int_0^\sigma \phi_q\left(\int_s^\sigma g(r)f(u(r-\tau) + h(r-\tau), u(r))\nabla r\right)\Delta s \end{aligned}$$

$$\begin{aligned}
& + \int_{\sigma}^T \phi_q \left(\int_{\sigma}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\
& \geq \int_{\theta}^{\sigma} \phi_q \left(\int_s^{\sigma} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\
& \quad + \int_{\sigma}^{T-\theta} \phi_q \left(\int_{\sigma}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\
& \geq mrA(\sigma) \geq mrL \geq 2r = 2\|u\|.
\end{aligned} \tag{3.4}$$

(ii) If $\sigma \in (T - \theta, T]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$\begin{aligned}
\|Tu\| &= (Tu)(\sigma) \\
&\geq \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\
&\quad + \int_0^{\sigma} \phi_q \left(\int_s^{\sigma} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\
&\geq \int_{\theta}^{T-\theta} \phi_q \left(\int_s^{T-\theta} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\
&\geq mrA(T-\theta) \geq mrL \geq 2r > r = \|u\|.
\end{aligned} \tag{3.5}$$

(iii) If $\sigma \in (0, \theta)$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$\begin{aligned}
\|Tu\| &= (Tu)(\sigma) \\
&\geq \phi_q \left(\frac{\delta}{\gamma} \int_{\sigma}^{\eta} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\
&\quad + \int_{\sigma}^T \phi_q \left(\int_{\sigma}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\
&\geq \int_{\theta}^{T-\theta} \phi_q \left(\int_{\theta}^s g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\
&\geq mrA(\theta) \geq mrL \geq 2r > r = \|u\|.
\end{aligned} \tag{3.6}$$

Therefore, no matter under which condition, we all have

$$\|Tu\| > \|u\|, \quad \forall u \in \partial\Omega_1. \tag{3.7}$$

Then by Theorem 1.1, we have

$$i(T, \Omega_1, K) = 0. \tag{3.8}$$

On the other hand, for $u \in \partial\Omega_2$, we have $u(t) \leq \|u\| = R$, and by (A_2) , we know that

$$\begin{aligned} \|Tu\| &= (Tu)(\sigma) \\ &\leq \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\ &\quad + \int_0^T \phi_q \left(\int_s^{\sigma} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\ &\leq \left(T + \phi_q \left(\frac{\beta}{\alpha} \right) \right) MR \phi_q \left(\int_0^T g(r) \nabla r \right) \leq R = \|u\|, \end{aligned} \quad (3.9)$$

thus

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_2. \quad (3.10)$$

Then, by Theorem 1.1, we have

$$i(T, \Omega_2, K) = 1. \quad (3.11)$$

Therefore, by (3.8), and (3.11), $r < R$, we have

$$i(T, \Omega_2 \setminus \bar{\Omega}_1, K) = 1. \quad (3.12)$$

Then operator T has a fixed point $u \in (\Omega_1 \setminus \bar{\Omega}_2)$, and $r \leq \|u\| \leq R$. This completes the proof of Theorem 3.1. \square

Theorem 3.2. *Suppose that condition (H_1) , (H_2) , and (H_3) hold. Assume that f also satisfies*

$$\begin{aligned} (A_3) : f^0 &= \varphi \in [0, (\theta_*/4)^{p-1}), \\ (A_4) : f_{\infty} &= \lambda \in ((2\theta^*/\theta)^{p-1}, \infty). \end{aligned}$$

Then, the SBVP (2.9), (2.10) has a solution u which is bounded in $\|\cdot\|$. Furthermore, by Lemma 2.4, $\bar{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.9).

Proof of Theorem 3.2. First, by $f^0 = \varphi \in [0, (\theta_*/4)^{p-1})$, for $\epsilon = (\theta_*/4)^{p-1} - \varphi$, there exists an adequately small positive number ρ , as $0 \leq u_2 \leq \rho$, $u_2 \neq 0$, $u_1 \leq H$, we have

$$f(u_1, u_2) \leq (\varphi + \epsilon)(u_2)^{p-1} \leq \left(\frac{\theta_*}{4} \right)^{p-1} \rho^{p-1} = \left(\frac{\theta_*}{4} \rho \right)^{p-1}. \quad (3.13)$$

Then let $R = \rho$, $M = \theta_*/4 \in (0, \theta_*)$, thus by (3.13),

$$f(u_1, u_2) \leq (MR)^{p-1}, \quad 0 \leq u_2 \leq R. \quad (3.14)$$

So condition (A_2) holds.

Next, by condition (A_4) , $f_\infty = \lambda \in ((2\theta^*/\theta)^{p-1}, \infty)$, then for $\epsilon = \lambda - (2\theta^*/\theta)^{p-1}$, there exists an appropriately big positive number $r \neq R$, as $u_2 \geq \theta r$, $u_1 \leq H$, we have

$$f(u_1, u_2) \geq (\lambda - \epsilon)(u_2)^{p-1} \geq \left(\frac{2\theta^*}{\theta}\right)^{p-1} (\theta r)^{p-1} = (2\theta^* r)^{p-1}. \quad (3.15)$$

Let $m = 2\theta^* > \theta^*$, thus by (3.15), condition (A_1) holds. Therefore, by Theorem 3.1, we know that the results of Theorem 3.2 hold. The proof of Theorem 3.2 is complete. \square

Theorem 3.3. *Suppose that conditions (H_1) , (H_2) , (H_3) hold. Assume that f also satisfies*

$$(A_5): f^\infty = \lambda \in [0, (\theta_*/4)^{p-1}),$$

$$(A_6): f_0 = \varphi \in ((2\theta^*/\theta)^{p-1}, \infty).$$

Then, the SBVP (2.9), (2.10) has a solution u which is bounded in $\|\cdot\|$. Furthermore by Lemma 2.4, $\bar{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.9).

Proof of Theorem 3.3. First, by condition (A_6) , $f_0 = \varphi \in ((2\theta^*/\theta)^{p-1}, \infty)$, then for $\epsilon = \varphi - (2\theta^*/\theta)^{p-1}$, there exists an adequately small positive number r , as $0 \leq u_2 \leq r$, $u_2 \neq 0$, $u_1 \leq H$, we have

$$f(u_1, u_2) \geq (\varphi - \epsilon)(u_2)^{p-1} = \left(\frac{2\theta^*}{\theta}\right)^{p-1} (u_2)^{p-1}, \quad (3.16)$$

thus when $\theta r \leq u_2 \leq r$, $u_1 \leq H$, we have

$$f(u_1, u_2) \geq \left(\frac{2\theta^*}{\theta}\right)^{p-1} (\theta r)^{p-1} = (2\theta^* r)^{p-1}. \quad (3.17)$$

Let $m = 2\theta^* > \theta^*$, so by (3.17), condition (A_1) holds.

Next, by condition (A_5) : $f^\infty = \lambda \in [0, (\theta_*/4)^{p-1})$, then for $\epsilon = (\theta_*/4)^{p-1} - \lambda$, there exists a suitably big positive number $\rho \neq r$, as $u_2 \geq \rho$, $u_1 \leq H$, we have

$$f(u_1, u_2) \leq (\lambda + \epsilon)(u_2)^{p-1} \leq \left(\frac{\theta_*}{4}\right)^{p-1} (u_2)^{p-1}. \quad (3.18)$$

If f is unbounded, by the continuity of f on $[0, \infty)^2$, then exists constant $R (\neq r) \geq \rho$, and a point $(u_{01}, u_{02}) \in [0, \infty)^2$ such that

$$\begin{aligned} \rho &\leq u_{02} \leq R, \\ f(u_1, u_2) &\leq f(u_{01}, u_{02}), \quad 0 \leq u_2 \leq R, \quad u_1 \leq H. \end{aligned} \quad (3.19)$$

Thus, by $\rho \leq u_{02} \leq R$, $u_1 \leq H$, we know

$$f(u_1, u_2) \leq f(u_{01}, u_{02}) \leq \left(\frac{\theta_*}{4}\right)^{p-1} (u_{02})^{p-1} \leq \left(\frac{\theta_*}{4} R\right)^{p-1}. \quad (3.20)$$

Choose $M = \theta_*/4 \in (0, \theta_*)$. Then, we have

$$f(u_1, u_2) \leq (MR)^{p-1}, \quad 0 \leq u_2 \leq R, \quad u_1 \leq H. \quad (3.21)$$

If f is bounded, we suppose $f(u_1, u_2) \leq \overline{M}^{p-1}$, $u_2 \in [0, \infty)$, $\overline{M} \in R_+$, there exists an appropriately big positive number $R > (4/\theta_*)\overline{M}$, then choose $M = \theta_*/4 \in (0, \theta_*)$, we have

$$f(u_1, u_2) \leq \overline{M}^{p-1} \leq \left(\frac{\theta_*}{4}R\right)^{p-1} = (MR)^{p-1}, \quad 0 \leq u_2 \leq R, \quad u_1 \leq H. \quad (3.22)$$

Therefore, condition (A_2) holds. Therefore, by Theorem 3.1, we know that the results of Theorem 3.3 holds. The proof of Theorem 3.3 is complete. \square

4. The Existence of Many Positive Solutions to (1.8) and (1.9)

Next, we will discuss the existence of many positive solutions.

Theorem 4.1. *Suppose that conditions (H_1) , (H_2) , (H_3) , and (A_2) in Theorem 3.1 hold. Assume that f also satisfies*

$$(A_7): f_0 = +\infty,$$

$$(A_8): f_\infty = +\infty.$$

Then, the SBVP (2.9), (2.10) has at last two solutions u_1, u_2 such that

$$0 < \|u_1\| < R < \|u_2\|. \quad (4.1)$$

Furthermore, by Lemma 2.4, $\bar{u}_1(t) = u_1(t) + h(t)$, $\bar{u}_2(t) = u_2(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.9).

Proof of Theorem 4.1. First, by condition (A_7) , for any $N > 2/\theta L$, there exists a constant $\rho_* \in (0, R)$ such that

$$f(u_1, u_2) \geq (Nu_2)^{p-1}, \quad 0 < u_2 \leq \rho_*, \quad u_1 \leq H. \quad (4.2)$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (4.2) and Lemma 2.3, similar to the previous proof of Theorem 3.1, we can have from three perspectives

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}. \quad (4.3)$$

Then by Theorem 1.1, we have

$$i(T, \Omega_{\rho_*}, K) = 0. \quad (4.4)$$

Next, by condition (A_8) , for any $\overline{N} > 2/\theta L$, there exists a constant $\rho_0 > 0$ such that

$$f(u_1, u_2) \geq (\overline{N}u_2)^{p-1}, \quad u_2 > \rho_0, \quad u_1 \leq H. \quad (4.5)$$

We choose a constant $\rho^* > \max\{R, \rho_0/\theta\}$, obviously $\rho_* < R < \rho^*$. Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$. For any $u \in \partial\Omega_{\rho^*}$, by Lemma 2.3, we have

$$u(t) \geq \theta\|u\| = \theta\rho^* > \rho_0, \quad t \in [\theta, T - \theta]. \quad (4.6)$$

Then by (4.5) and also similar to the previous proof of Theorem 3.1, we can also have from three perspectives

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_{\rho^*}. \quad (4.7)$$

Then by Theorem 1.1, we have

$$i(T, \Omega_{\rho^*}, K) = 0. \quad (4.8)$$

Finally, set $\Omega_R = \{u \in K : \|u\| < R\}$, For any $u \in \partial\Omega_R$, by (A₂), Lemma 2.3 and also similar to the latter proof of Theorem 3.1, we can also have

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_R. \quad (4.9)$$

Then by Theorem 1.1, we have

$$i(T, \Omega_R, K) = 1. \quad (4.10)$$

Therefore, by (4.4), (4.8), (4.10), $\rho_* < R < \rho^*$, we have

$$i(T, \Omega_R \setminus \overline{\Omega_{\rho_*}}, K) = 1, \quad i(T, \Omega_{\rho^*} \setminus \overline{\Omega_R}, K) = -1. \quad (4.11)$$

Then T has fixed-point $u_1 \in \Omega_R \setminus \overline{\Omega_{\rho_*}}$, and fixed-point $u_2 \in \Omega_{\rho^*} \setminus \overline{\Omega_R}$. Obviously, u_1, u_2 are all positive solutions of problem (2.9), (2.10) and $\rho_* < \|u_1\| < R < \|u_2\| < \rho^*$. The proof of Theorem 4.1 is complete. \square

Theorem 4.2. *Suppose that conditions (H₁), (H₂), (H₃), and (A₁) in Theorem 3.1 hold. Assume that f also satisfies*

$$(A_9): f^0 = 0,$$

$$(A_{10}): f^\infty = 0.$$

Then, the SBVP (2.9), (2.10) has at least two solutions u_1, u_2 such that $0 < \|u_1\| < r < \|u_2\|$. Furthermore, by Lemma 2.4, $\bar{u}_1(t) = u_1(t) + h(t)$, $\bar{u}_2(t) = u_2(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

Proof of Theorem 4.2. First, by $f^0 = 0$, for $\epsilon_1 \in (0, \theta_*)$, there exists a constant $\rho_* \in (0, r)$ such that

$$f(u_1, u_2) \leq (\epsilon_1 u_2)^{p-1}, \quad 0 < u_2 \leq \rho_*, \quad u_1 \leq H. \quad (4.12)$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (4.12), we have

$$\begin{aligned}
\|Tu\| &= (Tu)(\sigma) \\
&\leq \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\
&\quad + \int_0^T \phi_q \left(\int_s^{\sigma} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s \\
&\leq \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\
&\quad + T \phi_q \left(\int_0^T g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \\
&\leq \left(T + \phi_q \left(\frac{\beta}{\alpha} \right) \right) \epsilon_1 \rho_* \phi_q \left(\int_0^T g(r) \nabla r \right) \leq \rho_* = \|u\|,
\end{aligned} \tag{4.13}$$

that is

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}. \tag{4.14}$$

Then by Theorem 1.1, we have

$$i(T, \Omega_{\rho_*}, K) = 1. \tag{4.15}$$

Next, let $f^*(x) = \max_{0 \leq u_2 \leq x, u_1 \leq H} f(u_1, u_2)$, and note that $f^*(x)$ is monotone increasing with respect to $x \geq 0$. Then from $f^\infty = 0$, it is easy to see that

$$\lim_{x \rightarrow \infty} \frac{f^*(x)}{x^{p-1}} = 0. \tag{4.16}$$

Therefore, for any $\epsilon_2 \in (0, \theta_*)$, there exists a constant $\rho^* > r$ such that

$$f^*(x) \leq (\epsilon_2 x)^{p-1}, \quad x \geq \rho^*. \tag{4.17}$$

Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$, for any $u \in \partial\Omega_{\rho^*}$, by (4.17), we have

$$\begin{aligned}
\|Tu\| &= (Tu)(\sigma) \\
&\leq \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\
&\quad + \int_0^T \phi_q \left(\int_s^{\sigma} g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \Delta s
\end{aligned}$$

$$\begin{aligned}
&\leq \phi_q \left(\frac{\beta}{\alpha} \int_{\xi}^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s)) \nabla s \right) \\
&\quad + T \phi_q \left(\int_0^T g(r) f(u(r-\tau) + h(r-\tau), u(r)) \nabla r \right) \\
&\leq \left(T + \phi_q \left(\frac{\beta}{\alpha} \right) \right) \phi_q \left(\int_0^T g(r) f^*(\rho^*) \nabla r \right) \\
&\leq \left(T + \phi_q \left(\frac{\beta}{\alpha} \right) \right) \epsilon_2 \rho^* \phi_q \left(\int_0^T g(r) \nabla r \right) \leq r^* = \|u\|,
\end{aligned} \tag{4.18}$$

that is

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{\rho^*}. \tag{4.19}$$

Then by Theorem 1.1, we have

$$i(T, \Omega_{\rho^*}, K) = 1. \tag{4.20}$$

Finally, set $\Omega_r = \{u \in K : \|u\| < r\}$. For any $u \in \partial\Omega_r$, by (A_1) , Lemma 2.3 and also similar to the previous proof of Theorem 3.1, we can also have

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_r. \tag{4.21}$$

Then by Theorem 1.1, we have

$$i(T, \Omega_r, K) = 0. \tag{4.22}$$

Therefore, by (4.15), (4.20), (4.22), $\rho_* < r < \rho^*$, we have

$$i(T, \Omega_r \setminus \overline{\Omega}_{\rho_*}, K) = -1, \quad i(T, \Omega_{\rho^*} \setminus \overline{\Omega}_r, K) = 1. \tag{4.23}$$

Then T have fixed point $u_1 \in \Omega_r \setminus \overline{\Omega}_{\rho_*}$, and fixed point $u_2 \in \Omega_{\rho^*} \setminus \overline{\Omega}_r$. Obviously, u_1, u_2 are all positive solutions of problem (1.8), (1.9) and $\rho_* < \|u_1\| < r < \|u_2\| < \rho^*$. The proof of Theorem 4.2 is complete. \square

Similar to Theorem 3.1, we also obtain the following Theorems.

Theorem 4.3. *Suppose that conditions $(H_1), (H_2), (H_3)$ and (A_2) in Theorem 3.1, (A_4) in Theorem 3.2 and (A_6) in Theorem 3.3 hold. Then, the SBVP (2.9), (2.10) has at last two solutions u_1, u_2 such that $0 < \|u_1\| < R < \|u_2\|$. Furthermore by Lemma 2.4, $\bar{u}_1(t) = u_1(t) + h(t)$, $\bar{u}_2(t) = u_2(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).*

Theorem 4.4. *Suppose that conditions $(H_1), (H_2), (H_3)$ and (A_1) in Theorem 3.1, (A_3) in Theorem 3.2 and (A_5) in Theorem 3.3 hold. Then, the SBVP (2.9), (2.10) have at last two solutions*

u_1, u_2 such that $0 < \|u_1\| < r < \|u_2\|$. Furthermore by Lemma 2.4, $\bar{u}_1(t) = u_1(t) + h(t)$, $\bar{u}_2(t) = u_2(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

5. The Existence of Many Positive Solutions to (1.8) and (1.10)

In the following, we will deal with problem (1.8), (1.10), the method is similar to that in Sections 3 and 4, so we omit many proof in this section.

Theorem 5.1. *Suppose that condition $(H_1), (H_2), (H_3), (H_4)$ hold. Assume that f also satisfies*

$$(A'_1): f(u_1, u_2) \geq (mr)^{p-1}, \text{ for } \theta r \leq u_2 \leq r, 0 \leq u_1 \leq H,$$

$$(A'_2): f(u_1, u_2) \leq (MR)^{p-1}, \text{ for } 0 \leq u_2 \leq R, 0 \leq u_1 \leq H,$$

where $m \in (\theta^*, \infty)$, $M \in (0, \theta_{**})$. Then, the SBVP (2.9), (2.13) has a solution u such that $\|u\|$ lies between r and R . Furthermore by Lemma 2.4, $\bar{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

Theorem 5.2. *Suppose that condition $(H_1), (H_2), (H_3), (H_4)$ hold. Assume that f also satisfies*

$$(A'_3): f^0 = \varphi \in [0, (\theta_{**}/4)^{p-1}),$$

$$(A'_4): f_\infty = \lambda \in ((2\theta^*/\theta)^{p-1}, \infty).$$

Then, the SBVP (2.9), (2.13) has a solution u which is bounded in $\|\cdot\|$. Furthermore by Lemma 2.4, $\bar{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

Theorem 5.3. *Suppose that condition $(H_1), (H_2), (H_3), (H_4)$ hold. Assume that f also satisfies*

$$(A'_5): f^\infty = \lambda \in [0, (\theta_{**}/4)^{p-1}),$$

$$(A'_6): f_0 = \varphi \in ((2\theta^*/\theta)^{p-1}, \infty).$$

Then, the SBVP (2.9), (2.13) has a solution u which is bounded in $\|\cdot\|$. Furthermore by Lemma 2.4, $\bar{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

Theorem 5.4. *Suppose that conditions $(H_1), (H_2), (H_3), (H_4)$ and (A'_2) in Theorem 5.1 hold. Assume that f also satisfies*

$$(A'_7): f_0 = +\infty,$$

$$(A'_8): f_\infty = +\infty.$$

Then, the SBVP (2.9), (2.13) has at least two solutions u_1, u_2 such that

$$0 < \|u_1\| < R < \|u_2\|. \quad (5.1)$$

Furthermore by Lemma 2.4, $\bar{u}_1(t) = u_1(t) + h(t)$, $\bar{u}_2(t) = u_2(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

Theorem 5.5. *Suppose that conditions $(H_1), (H_2), (H_3), (H_4)$ and (A'_1) in Theorem 5.1 hold. Assume that f also satisfies*

$$(A'_9): f^0 = 0,$$

$$(A'_{10}): f^\infty = 0.$$

Then, the SBVP (2.9), (2.13) has at least two solutions u_1, u_2 such that $0 < \|u_1\| < r < \|u_2\|$. Furthermore by Lemma 2.4, $\bar{u}_1(t) = u_1(t) + h(t)$, $\bar{u}_2(t) = u_2(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

Theorem 5.6. Suppose that conditions $(H_1), (H_2), (H_3), (H_4)$ and (A_2) in Theorem 5.1, (A_4) in Theorem 5.2 and (A'_6) in Theorem 3.3 hold. Then, the SBVP (2.9), (2.13) has at least two solutions u_1, u_2 such that $0 < \|u_1\| < R < \|u_2\|$. Furthermore by Lemma 2.4, $\bar{u}_1(t) = u_1(t) + h(t)$, $\bar{u}_2(t) = u_2(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

Theorem 5.7. Suppose that conditions $(H_1), (H_2), (H_3), (H_4)$ and (A'_1) in Theorem 5.1, (A'_3) in Theorem 5.2 and (A'_5) in Theorem 3.3 hold. Then, the SBVP (2.9), (2.13) has at least two solutions u_1, u_2 such that $0 < \|u_1\| < r < \|u_2\|$. Furthermore by Lemma 2.4, $\bar{u}_1(t) = u_1(t) + h(t)$, $\bar{u}_2(t) = u_2(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

6. Application

In the section, we present two simple examples to explain our result.

Example 6.1. Let $\mathbf{T} = \{1 - (1/2)^N\} \cup \{1\}$, where \mathbf{N} denotes the set of all nonnegative integers. Consider the following 3-order singular boundary value problem (SBVP) with p -Laplacian

$$\begin{aligned} \left(\phi_p(u^\Delta)\right)^\nabla(t) + \frac{1}{20}t^{-1/2}u^{1/2}(t) \cdot \left[\frac{1}{5} + \frac{(94/5)e^{2u(t)}}{120u(t-1) + 7e^{u(t)} + e^{2u(t)}}\right] &= 0, \quad 0 < t < 1, \\ u(t) &= t^2 - 1, \quad -1 \leq t \leq 0, \end{aligned} \quad (6.1)$$

$$\phi_p(u(0)) - \phi_p\left(u^\Delta\left(\frac{1}{4}\right)\right) = 0, \quad \phi_p(u(1)) + \delta\phi_p\left(u^\Delta\left(\frac{1}{2}\right)\right) = 0,$$

where

$$\begin{aligned} \alpha = \gamma = 1, \quad \beta = 1, \quad p = \frac{3}{2}, \quad \delta \geq 0, \quad \xi = \frac{1}{4}, \quad \eta = \frac{1}{2}, \quad \theta = \frac{1}{4}, \\ \tau = T = 1. \end{aligned} \quad (6.2)$$

So, by Lemma 2.4, we discuss the following SBVP:

$$\begin{aligned} \left(\phi_p(u^\Delta)\right)^\nabla(t) + \frac{1}{20}t^{-1/2}u^{1/2}(t) \cdot \left[\frac{1}{5} + \frac{(94/5)e^{2u(t)}}{120[u(t-1) + h(t-1)] + 7e^{u(t)} + e^{2u(t)}}\right] &= 0, \quad 0 < t < 1, \\ u(t) &= 0, \quad -1 \leq t \leq 0, \\ \phi_p(u(0)) - \phi_p\left(u^\Delta\left(\frac{1}{4}\right)\right) &= 0, \quad \phi_p(u(1)) + \delta\phi_p\left(u^\Delta\left(\frac{1}{2}\right)\right) = 0, \end{aligned} \quad (6.3)$$

where

$$h(t) = \begin{cases} t^2 - 1, & -1 \leq t \leq 0, \\ 0, & 0 \leq t \leq 1, \end{cases} \quad g(t) = \frac{1}{20}t^{-1/2}, \quad \zeta(t) = t^2 - 1, \quad (6.4)$$

$$f(u_1, u_2) = (u_2)^{1/2} \left[\frac{1}{5} + \frac{(94/5)e^{2u_2}}{120u_1 + 7e^{u_2} + e^{2u_2}} \right].$$

Then obviously,

$$q = 3, \quad H = \max_{-1 \leq t \leq 0} |\zeta(t)| = 1, \quad f^0 = \varphi = \lim_{u_2 \rightarrow 0^+} \max_{0 \leq u_1 \leq 1} \frac{f(u_1, u_2)}{u_2^{p-1}} = \frac{51}{20}, \quad (6.5)$$

$$f_\infty = \lambda = \lim_{u_2 \rightarrow \infty} \min_{0 \leq u_1 \leq 1} \frac{f(u_1, u_2)}{u_2^{p-1}} = \frac{95}{5}, \quad \int_0^T g(t) \nabla t = \frac{1}{10},$$

so conditions $(H_1), (H_2), (H_3)$ hold.

Next,

$$\theta_* = \frac{1}{(T + \phi_q(\beta/\alpha))\phi_q\left(\int_0^T g(r) \nabla r\right)} = 50, \quad (6.6)$$

then $(\theta_*/4)^{p-1} = 5\sqrt{2}/2 > 51/20$, that is, $\varphi \in [0, (\theta_*/4)^{p-1})$, so condition (A_3) holds.

For $\theta = 1/4$, it is easy to see by calculating that

$$L = \min_{t \in [\theta, T-\theta]} A(t) = \frac{1}{16} \left(\frac{7}{36} + \frac{\sqrt{3}}{3} \right). \quad (6.7)$$

Because of

$$\left(\frac{2\theta^*}{\theta} \right)^{p-1} = 96 \times \left(\frac{1}{7 + 12\sqrt{3}} \right)^{1/2} < \frac{95}{5}, \quad (6.8)$$

then

$$\lambda \in \left(\left(\frac{2\theta^*}{\theta} \right)^{p-1}, \infty \right), \quad (6.9)$$

so condition (A_4) holds. Then by Theorem 3.2, SBVP (6.3) has at least a positive solution $u(t)$. So, $\bar{u}(t) = u(t) + h(t)$, $-1 < t < 1$ is the positive solution of SBVP (6.1).

Example 6.2. Consider the following 3-order singular boundary value problem (SBVP) with p -Laplacian:

$$\begin{aligned} (\phi_p(u^\Delta))^\nabla(t) + \frac{1}{64\pi^4} t^{-1/2} (1-t) [u(t-1) + u^2(t) + u^4(t)] &= 0, \quad 0 < t < 1, \\ u(t) &= -te^t, \quad -1 \leq t \leq 0, \end{aligned} \quad (6.10)$$

$$2\phi_p(u(0)) - \phi_p\left(u^\Delta\left(\frac{1}{4}\right)\right) = 0, \quad \phi_p(u(1)) + \delta\phi_p\left(u^\Delta\left(\frac{1}{2}\right)\right) = 0,$$

where

$$\begin{aligned} \beta = \gamma = 1, \quad \alpha = 2, \quad p = 4, \quad \delta \geq 0, \quad p = 4, \quad \xi = \frac{1}{4}, \quad \eta = \frac{1}{3}, \\ \theta = \frac{1}{4}, \quad \tau = T = 1. \end{aligned} \quad (6.11)$$

So, by Lemma 2.4, we discuss the following SBVP:

$$\begin{aligned} (\phi_p(u^\Delta))^\nabla(t) + \frac{1}{64\pi^4} t^{-1/2} (1-t) [[u(t-1) + h(t-1)] + u^2(t) + u^4(t)] &= 0, \quad 0 < t < 1, \\ u(t) &= 0, \quad -1 \leq t \leq 0, \\ 2\phi_p(u(0)) - \phi_p\left(u^\Delta\left(\frac{1}{4}\right)\right) &= 0, \quad \phi_p(u(1)) + \delta\phi_p\left(u^\Delta\left(\frac{1}{2}\right)\right) = 0, \end{aligned} \quad (6.12)$$

where

$$\begin{aligned} h(t) &= \begin{cases} \zeta(t), & -1 \leq t \leq 0, \\ 0, & 0 \leq t \leq 1, \end{cases} \quad \zeta(t) = -te^t, \\ g(t) &= \frac{1}{64\pi^4} t^{-1/2} (1-t), \quad f(u_1, u_2) = u_1 + u_2^2 + u_2^4. \end{aligned} \quad (6.13)$$

Then obviously,

$$q = \frac{4}{3}, \quad \int_0^1 g(t) \nabla t = \frac{1}{64\pi^3}, \quad H = \max_{-1 \leq t \leq 0} |\zeta(t)| = e, \quad f_\infty = +\infty, \quad f_0 = +\infty, \quad (6.14)$$

so conditions $(H_1), (H_2), (H_3), (A_7), (A_8)$ hold.

Next,

$$\phi_q\left(\int_0^1 a(t) \nabla t\right) = \frac{1}{4\pi}, \quad \theta_* = \frac{4\pi}{1 + \sqrt[3]{4}}, \quad (6.15)$$

we choose $R = 3$, $M = 2$, and for $\theta = 1/4$, because of the monotone increasing of $f(u_1, u_2, u_3)$ on $[0, \infty)^3$, then

$$f(u_1, u_2) \leq f(e, 3) = e + 90, \quad 0 \leq u_2 \leq 3, \quad 0 \leq u_1 \leq e. \quad (6.16)$$

Therefore, by

$$M \in (0, \theta_*), \quad (MR)^{p-1} = (6)^3 = 216, \quad (6.17)$$

we know that

$$f(u_1, u_2, u_3) \leq (MR)^{p-1}, \quad 0 \leq u_2 \leq 3, \quad 0 \leq u_1 \leq e, \quad (6.18)$$

so condition (A_2) holds. Then by Theorem 4.1, SBVP (6.12) has at least two positive solutions v_1 , v_2 and $0 < \|v_1\| < 3 < \|v_2\|$. Then, by Lemma 2.4, $\bar{v}_1(t) = v_1(t) + h(t)$, $\bar{v}_2(t) = v_2(t) + h(t)$, $t \in (-1, 1)$ are the positive solutions of the SBVP (6.10).

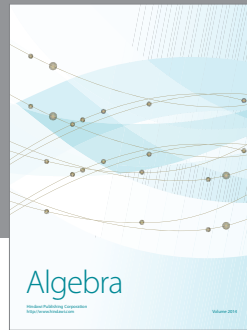
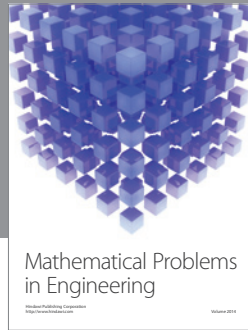
Acknowledgments

The first the second authors were supported financially by Shandong Province Natural Science Foundation (ZR2009AQ004), NSFC (11026108, 11071141), and the third author was supported by Shandong Province planning Foundation of Social Science (09BJGJ14), and Shandong Province Natural Science Foundation (Z2007A04).

References

- [1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.
- [2] G. B. Gustafson and K. Schmitt, "Nonzero solutions of boundary value problems for second order ordinary and delay-differential equations," *Journal of Differential Equations*, vol. 12, pp. 129–147, 1972.
- [3] L. H. Erbe and Q. Kong, "Boundary value problems for singular second-order functional-differential equations," *Journal of Computational and Applied Mathematics*, vol. 53, no. 3, pp. 377–388, 1994.
- [4] D. R. Anderson, "Solutions to second-order three-point problems on time scales," *Journal of Difference Equations and Applications*, vol. 8, no. 8, pp. 673–688, 2002.
- [5] E. R. Kaufmann, "Positive solutions of a three-point boundary-value problem on a time scale," *Electronic Journal of Differential Equations*, vol. 82, 11 pages, 2003.
- [6] F. M. Atici and G. Sh. Guseinov, "On Green's functions and positive solutions for boundary value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 75–99, 2002.
- [7] H. R. Sun and W.-T. Li, "Positive solutions for nonlinear three-point boundary value problems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 508–524, 2004.
- [8] H. Su, Z. Wei, and B. Wang, "The existence of positive solutions for a nonlinear four-point singular boundary value problem with a p -Laplacian operator," *Nonlinear Analysis*, vol. 66, no. 10, pp. 2204–2214, 2007.
- [9] J. W. Lee and D. O'Regan, "Existence results for differential delay equations-I," *Journal of Differential Equations*, vol. 102, no. 2, pp. 342–359, 1993.
- [10] H. Su, B. Wang, Z. Wei, and X. Zhang, "Positive solutions of four-point boundary value problems for higher-order p -Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 2, pp. 836–851, 2007.

- [11] H. Su, "Positive solutions for n -order m -point p -Laplacian operator singular boundary value problems," *Applied Mathematics and Computation*, vol. 199, no. 1, pp. 122–132, 2008.
- [12] H. Su, B. Wang, and Z. Wei, "Positive solutions of four-point boundary-value problems for four-order p -Laplacian dynamic equations on time scales," *Electronic Journal of Differential Equations*, 13 pages, 2006.
- [13] R. I. Avery and D. R. Anderson, "Existence of three positive solutions to a second-order boundary value problem on a measure chain," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 65–73, 2002.
- [14] Y. Wang, W. Zhao, and W. Ge, "Multiple positive solutions for boundary value problems of second order delay differential equations with one-dimensional p -Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 641–654, 2007.
- [15] Y.-H. Lee and I. Sim, "Global bifurcation phenomena for singular one-dimensional p -Laplacian," *Journal of Differential Equations*, vol. 229, no. 1, pp. 229–256, 2006.
- [16] H. Su, Z. Wei, and F. Xu, "The existence of positive solutions for nonlinear singular boundary value system with p -Laplacian," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 826–836, 2006.
- [17] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Sandiego, Calif, USA, 1988.
- [18] D. Guo, V. Lakshmikantham, and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, vol. 373 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1996.
- [19] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, vol. 370 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Boston, Mass, USA, 1996.
- [20] Bohner M. and Peterson A., *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston Inc., Boston, Mass, USA, 2003.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

