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Research Article

On a Generalized Hyers-Ulam Stability of Trigonometric Functional Equations

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Let G be an Abelian group, let $\mathbb{C}$ be the field of complex numbers, and let $f, g : G \to \mathbb{C}$. We consider the generalized Hyers-Ulam stability for a class of trigonometric functional inequalities, $|f(x-y)-f(x)g(y)+g(x)f(y)| \le \psi(y)$, $|g(x-y)-g(x)g(y)-f(x)f(y)| \le \psi(y)$, where $\psi : G \to \mathbb{R}$ is an arbitrary nonnegative function.

1. Introduction

The Hyers-Ulam stability problems of functional equations go back to 1940 when S. M. Ulam proposed a question concerning the approximate homomorphisms from a group to a metric group (see [1]). A partial answer was given by Hyers et al. [2, 3] under the assumption that the target space of the involved mappings is a Banach space. After the result of Hyers, Aoki [4], and Bourgin [5, 6] dealt with this problem, however, there were no other results on this problem until 1978 when Rassias [7] dealt again with the inequality of Aoki [4]. Following the Rassias' result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [2, 7–21]. The following four functional equations are called *trigonometric functional equations*.

$$f(x+y) - f(x)g(y) - g(x)f(y) = 0 \tag{1.1}$$

$$g(x+y) - g(x)g(y) + f(x)f(y) = 0 \tag{1.2}$$

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0 \tag{1.3}$$

$$g(x-y) - g(x)g(y) - f(x)f(y) = 0 \tag{1.4}$$

The four functional equations have been investigated separately. The general solutions and regular solutions of the above equations are introduced [22, 23]. In particular, the last equation (1.4) is most interesting in the sense that (1.4) alone characterizes the two trigonometric functions $f(x) = \cos(ax)$, $g(x) = \sin(ax)$ under some regularities of g, which none of the remaining equations are able to do.

In [19], Székelyhidi developed his idea of using invariant subspaces of functions defined on a group or semigroup to obtain the Hyers-Ulam stability of the trigonometric functional equations (1.1) and (1.2). As results, he obtained the Hyers-Ulam stability when for each fixed y the difference

$$T_1(x) := f(x+y) - f(x)g(y) - g(x)f(y) \tag{1.5}$$

is a bounded function of x and the Hyers-Ulam stability when for each fixed y the difference

$$T_2(x) := g(x+y) - g(x)g(y) + f(x)f(y) \tag{1.6}$$

is a bounded function of x, where f, g are mappings from an Abelian (amenable) group G to the field $\mathbb{C}$ of complex numbers.

In this paper, we complete the parallel Hyers-Ulam stability to that of [19] for the functional equations (1.3) and (1.4). As results, we obtained the Hyers-Ulam stability when for each fixed y the difference

$$T_3(x) := f(x-y) - f(x)g(y) + g(x)f(y) \tag{1.7}$$

is a bounded function of x and the Hyers-Ulam stability when for each fixed y the difference

$$T_4(x) := g(x-y) - g(x)g(y) - f(x)f(y) \tag{1.8}$$

is a bounded function of x.

In fact, the authors [10] obtained weaker versions of the Hyers-Ulam stability for the functional equations (1.3) and (1.4), that is, we proved the Hyers-Ulam stability of (1.3) when the difference

$$T_3(x,y) := f(x-y) - f(x)g(y) + g(x)f(y) \tag{1.9}$$

is uniformly bounded for all x and y, and we proved the Hyers-Ulam stability of (1.4) when the difference

$$T_4(x,y) := g(x-y) - g(x)g(y) - f(x)f(y) \tag{1.10}$$

is uniformly bounded for all x and y.

So, the results in this paper would be generalizations of those in [10]. We refer the reader to [9, 15, 16, 20, 21] for some related Hyers-Ulam stability of functional equations of trigonometric type.

2. Main Theorems

A function a from a semigroup $\langle S,+\rangle$ to the field $\mathbb{C}$ of complex numbers is said to be *an additive function* provided that $a(x+y)=a(x)+a(y)$ and $m:S\to\mathbb{C}$ is said to be *an exponential function* provided that $m(x+y)=m(x)m(y)$. Throughout this paper, we denote by G an Abelian group, $\mathbb{C}$ the set of complex numbers, and $\varphi:G\to\mathbb{R}$ a fixed nonnegative function. For the proof of stabilities of (1.3) and (1.4), we need the following.

Lemma 2.1 (see [2]). *Let S be a semigroup. Assume that $f,g:S\to\mathbb{C}$ satisfy the inequality; for each $y\in S$, there exists a positive constant M_y such that*

$$\left|f(x+y)-f(x)g(y)\right|\le M_y, \tag{2.1}$$

for all $x\in S$, then either f is a bounded function or g is an exponential function.

Proof. Suppose that g is not exponential, then there are $y,z\in S$ such that $g(y+z)\neq g(y)g(z)$. Now we have

$$\begin{aligned} f(x+y+z)-f(x+y)g(z)&=(f(x+y+z)-f(x)g(y+z))\\ &\quad-g(z)(f(x+y)-f(x)g(y))+f(x)(g(y+z)-g(y)g(z)), \end{aligned} \tag{2.2}$$

and hence,

$$\begin{aligned} f(x)=&\,(g(y+z)-g(y)g(z))^{-1}\\ &\times((f(x+y+z)-f(x+y)g(z))-(f(x+y+z)-f(x)g(y+z))\\ &\quad+g(z)(f(x+y)-f(x)g(y))). \end{aligned} \tag{2.3}$$

In view of (2.1), the right hand side of (2.3) is bounded as a function of x. Consequently, f is bounded. □

We discuss the general solutions $f,g:G\to\mathbb{C}$ of the corresponding trigonometric functional equations

$$f(x-y)-f(x)g(y)+g(x)f(y)=0, \tag{2.4}$$

$$g(x-y)-g(x)g(y)-f(x)f(y)=0. \tag{2.5}$$

Lemma 2.2 (see [22, 23]). *The general solutions (f,g) of the functional equation* (2.4) *are given by one of the following:*

(i) $f=0$ *and g is arbitrary,*

(ii) $f(x)=\lambda_1(m(x)-m(-x))$ *and* $g(x)=\lambda_2 f(x)+(1/2)(m(x)+m(-x))$ *for some $\lambda_1,\lambda_2\in\mathbb{C}$, where m is an exponential function,*

(iii) $f(x)=a(x)m(x)$, $g(x)=(1+\lambda a(x))m(x)$ *for some $\lambda\in\mathbb{C}$, where a is an additive function and m is an exponential function satisfying $m^2\equiv 1$.*

Also, the general solutions (g, f) of the functional equation (2.5) *are given by one of the following:*

(i) $g(x) = \lambda$ *and* $f(x) = \pm\sqrt{\lambda - \lambda^2}$ *for some* $\lambda \in \mathbb{C}$,

(ii) $g(x) = (1/2)(m(x) + m(-x))$ *and* $f(x) = (1/2i)(m(x) - m(-x))$, *where* m *is an exponential function.*

Proof. The solutions of the functional equation (2.4) are given in [23, p. 217, Theorem 11]. For the functional equation (2.5), combining the result of L. Vietoris [22, p. 177] and that of J. A. Baker [23, p. 220], we obtain that every nonconstant function g satisfying (2.5) has the form

$$g(x) = \frac{1}{2}(m(x) + m(-x)), \tag{2.6}$$

for some exponential function m. Thus, using (2.5), we have

$$f(x) = \frac{1}{2i}(m(x) - m(-x)). \tag{2.7}$$

This completes the proof. □

For the proof of the stability of (1.1), we need the following. Throughout this paper, we denote by φ an arbitrary nonnegative function on G.

Lemma 2.3. *Let* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \le \varphi(y), \tag{2.8}$$

for all $x, y \in G$, *then either there exist* $\lambda_1,\ \lambda_2 \in \mathbb{C}$, *not both zero, and* $M > 0$ *such that*

$$|\lambda_1 f(x) - \lambda_2 g(x)| \le M, \tag{2.9}$$

or else

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0, \tag{2.10}$$

for all $x, y \in G$.

Proof. Suppose that the inequality (2.9) holds only when $\lambda_1 = \lambda_2 = 0$. Let

$$k(x,y) = f(x+y) - f(x)g(-y) + g(x)f(-y), \tag{2.11}$$

and choose y_1 satisfying $f(-y_1) \neq 0$. Now it can be easily calculated that

$$g(x) = \lambda_0 f(x) + \lambda_1 f(x+y_1) - \lambda_1 k(x, y_1), \tag{2.12}$$

where $\lambda_0 = g(-y_1)/f(-y_1)$ and $\lambda_1 = -1/f(-y_1)$. By (2.11), we have

$$f(x+(y+z)) = f(x)g(-y-z) - g(x)f(-y-z) + k(x,y+z). \tag{2.13}$$

Also by (2.11) and (2.12), we have

$$\begin{aligned}
f((x+y)+z) &= f(x+y)g(-z) - g(x+y)f(-z) + k(x+y,z) \\
&= (f(x)g(-y) - g(x)f(-y) + k(x,y))g(-z) \\
&\quad - (\lambda_0 f(x+y) + \lambda_1 f(x+y+y_1) - \lambda_1 k(x+y,y_1))f(-z) + k(x+y,z) \\
&= (f(x)g(-y) - g(x)f(-y) + k(x,y))g(-z) \\
&\quad - \lambda_0(f(x)g(-y) - g(x)f(-y) + k(x,y))f(-z) \\
&\quad - \lambda_1(f(x)g(-y-y_1) - g(x)f(-y-y_1) + k(x,y+y_1))f(-z) \\
&\quad + \lambda_1 k(x+y,y_1)f(-z) + k(x+y,z).
\end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we have

$$\begin{aligned}
&(g(-y)g(-z) - \lambda_0 g(-y)f(-z) - \lambda_1 g(-y-y_1)f(-z) - g(-y-z))f(x) \\
&\qquad + (-g(-y)g(-z) + \lambda_0 f(-y)g(-z) + \lambda_1 f(-y-y_1)f(-z) + f(-y-z))g(x) \\
&\quad = -k(x,y)g(-z) + \lambda_0 k(x,y)f(-z) + \lambda_1 k(x,y+y_1)f(-z) \\
&\qquad - \lambda_1 k(x+y,y_1)f(-z) - k(x+y,z) + k(x,y+z).
\end{aligned} \tag{2.15}$$

Since $k(x,y)$ is bounded by $\varphi(-y)$, if we fix y, z, the right hand side of (2.15) is bounded by a constant M, where

$$\begin{aligned}
M = {}& \varphi(-y)|g(-z)| + \varphi(-y)|\lambda_0 f(-z)| + \varphi(-y-y_1)|\lambda_1 f(-z)| \\
& + \varphi(-y_1)|\lambda_1 f(-z)| + \varphi(-z) + \varphi(-y-z).
\end{aligned} \tag{2.16}$$

So by our assumption, the left hand side of (2.15) vanishes, so does the right hand side. Thus, we have

$$\begin{aligned}
&(-\lambda_0 k(x,y) - \lambda_1 k(x,y+y_1) + \lambda_1 k(x+y,y_1))f(-z) + k(x,y)g(-z) \\
&\quad = k(x,y+z) - k(x+y,z).
\end{aligned} \tag{2.17}$$

Now by the definition of k, we have

$$\begin{aligned}
k(x+y,z) - k(x,y+z) &= f(x+y+z) - f(x+y)g(-z) + g(x+y)f(-z) \\
&\quad - f(x+y+z) + f(x)g(-y-z) - g(x)f(-y-z) \\
&= f(-y-z-x) - f(-y-z)g(x) + g(-y-z)f(x) \\
&\quad - f(-z-x-y) + f(-z)g(x+y) - g(-z)f(x+y) \\
&= k(-y-z,-x) - k(-z,-x-y).
\end{aligned} \tag{2.18}$$

Hence, the right hand side of (2.17) is bounded by $\psi(x)+\psi(x+y)$. So if we fix x, y in (2.17), the left hand side of (2.17) is a bounded function of z. Thus, by our assumption, we conclude that $k(x,y)\equiv 0$. This completes the proof. □

In the following theorem, we assume that

$$\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k}\psi\left(-2^k x\right) < \infty, \tag{2.19}$$

or

$$\Phi_2(x) := \sum_{k=1}^{\infty} 2^{k}\psi\left(-2^{-k} x\right) < \infty. \tag{2.20}$$

For the proof, we discuss the following property.

Lemma 2.4. *Let $m : G \to \mathbb{C}$ be a bounded exponential function satisfying $m(x) \neq m(-x)$ for some $x \in G$, then there exists $y \in G$ such that*

$$\left|m(y) - m(-y)\right| \geq \sqrt{3}. \tag{2.21}$$

Furthermore, the constant $\sqrt{3}$ is the best one.

Proof. Since m is a bounded exponential, there exists $C > 0$ such that $|m(x)|^k = |m(kx)| \leq C$ for all $k \in \mathbb{Z}$ and $x \in G$, which implies $|m(x)| = 1$ for all $x \in G$. Assume that $m(x_0) \neq m(-x_0)$, then we have $m(x_0) \neq \pm 1$, and we may assume that $m(x_0) = e^{i\theta}$, $0 < \theta < \pi$. If $\theta \in [\pi/3, 2\pi/3]$, we have $|m(x_0) - m(-x_0)| = |e^{i\theta} - e^{-i\theta}| \geq \sqrt{3}$. If $\theta \in [0, \pi/3]$, there exists a positive integer k such that $k\theta \in [\pi/3, 2\pi/3]$, and we have $|m(kx_0) - m(-kx_0)| = |e^{ik_0\theta} - e^{-ik_0\theta}| \geq \sqrt{3}$. If $\theta \in [2\pi/3, 5\pi/6]$, then $2\theta \in [4\pi/3, 5\pi/3]$, and we have $|m(2x_0) - m(-2x_0)| = |e^{i2\theta} - e^{-i2\theta}| \geq \sqrt{3}$. Finally, if $\theta \in [5\pi/6, \pi]$, there exists a positive integer k such that $2k\theta \in [-\pi/3, -2\pi/3]$, and we have $|m(2kx_0) - m(-2kx_0)| = |e^{i2k\theta} - e^{-i2k\theta}| \geq \sqrt{3}$. Now define $m : \mathbb{Z} \to \mathbb{C}$ by $m(k) = e^{ik\pi/3}$. Then we have $|m(3k+1) - m(-3k-1)| = \sqrt{3}$ for all $k \in \mathbb{Z}$. Thus, $\sqrt{3}$ is the biggest one. This completes the proof. □

Theorem 2.5. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|f(x-y) - f(x)g(y) + g(x)f(y)\right| \leq \psi(y), \tag{2.22}$$

for all $x, y \in G$, then (f, g) satisfies one of the following:

(i) $f = 0$, *g is arbitrary,*

(ii) *f and g are bounded functions,*

(iii) $f(x) = \lambda_1(m(x) - m(-x))$ *and* $g(x) = \lambda_2 f(x) + (1/2)(m(x) + m(-x))$ *for some* $\lambda_1, \lambda_2 \in \mathbb{C}$, *where m is an exponential function,*

(iv) *there exist $\lambda \in \mathbb{C}$ and a bounded exponential function m such that*

$$g(x) = \lambda f(x) + m(x), \tag{2.23}$$

for all $x \in G$*, and* f *satisfies the condition; there exists* $d \geq 0$ *satisfying*

$$\left|f(x)\right| \leq \frac{2}{\sqrt{3}}(\psi(-x) + d), \tag{2.24}$$

for all $x \in G$*,*

(v) *there exist* $\lambda \in \mathbb{C}$ *and a bounded exponential function* m *satisfying* $m^2 \equiv 1$ *such that*

$$g(x) = \lambda f(x) + m(x), \tag{2.25}$$

for all $x \in G$*, and* f *satisfies one of the following conditions; there exists an additive function* $a_1 : G \to \mathbb{C}$ *such that*

$$\left|f(x) - \left(a_1(x) + f(0)\right)m(x)\right| \leq \Phi_1(x), \tag{2.26}$$

for all $x \in G$*, or there exists an additive function* $a_2 : G \to \mathbb{C}$ *such that*

$$\left|f(x) - \left(a_1(x) + f(0)\right)m(x)\right| \leq \Phi_2(x), \tag{2.27}$$

for all $x \in G$*, where* Φ_1 *and* Φ_2 *are the functions given in* (2.19) *and* (2.20).

Proof. In view of Lemma 2.3, we first consider the case when f, g satisfy (2.9). If $f = 0$, g is arbitrary which is the case (i). If f is a nontrivial bounded function, in view of (2.22), g is also bounded which gives the case (ii). If f is unbounded, it follows from (2.9) that $\lambda_2 \neq 0$ and

$$g(x) = \lambda f(x) + m(x), \tag{2.28}$$

for some $\lambda \in \mathbb{C}$ and a bounded function m. Putting (2.28) in (2.22), we have

$$\left|f(x-y) - f(x)m(y) + m(x)f(y)\right| \leq \psi(y), \tag{2.29}$$

for all $x, y \in G$. Replacing y by $-y$ and using the triangle inequality, we have, for some $C > 0$,

$$\left|f(x+y) - f(x)m(-y)\right| \leq C\left|f(-y)\right| + \psi(-y), \tag{2.30}$$

for all $x, y \in G$. By Lemma 2.1, m is an exponential function. If $m = 0$, putting $y = 0$ in (2.29), we have

$$\left|f(x)\right| \leq \psi(0). \tag{2.31}$$

Thus, we have $m \neq 0$ since f is unbounded. Since m is a nonzero bounded exponential function, it follows from the equalities

$$m(x) = m(x-y)m(y), \quad x, y \in G \tag{2.32}$$

that $m(0) = 1$ and $m(x) \neq 0$, for all $x \in G$. Putting $x = 0$ in (2.29) and replacing y by $-y$ multiplying $|m(x)|$ in the result, we have

$$\left|m(x)f(-y) + m(x)f(y) - f(0)m(x)m(-y)\right| \leq \varphi(-y), \tag{2.33}$$

for all $y \in G$. Replacing y by $-y$ in (2.29) and using (2.33), we have

$$\left|f(x+y) - f(x)m(-y) - m(x)f(y) + f(0)m(x)m(-y)\right| \leq 2\varphi(-y). \tag{2.34}$$

First we consider the case $m(x) \neq m(-x)$ for some $x \in G$. Replacing x by y and y by x in (2.34), we have

$$\left|f(y+x) - f(y)m(-x) - m(y)f(x) + f(0)m(y)m(-x)\right| \leq 2\varphi(-x), \tag{2.35}$$

for all $x, y \in G$. From (2.34) and (2.35), using the triangle inequality, putting $y = y_0$ such that $|m(y_0) - m(-y_0)| \geq \sqrt{3}$ and dividing $|m(y_0) - m(-y_0)|$, we have

$$\left|f(x)\right| \leq \frac{2}{\sqrt{3}}\left(\varphi(-x) + d\right), \tag{2.36}$$

for all $x \in G$, where $d = \varphi(-y_0) + |f(y_0)| + |f(0)|$, which gives (iv). Now we consider the case $m(x) = m(-x)$, for all $x \in G$. Dividing both the sides of (2.34) by $m(x)m(y)$, we have

$$\left|F(x+y) - F(x) - F(y)\right| \leq 2\varphi(-y), \tag{2.37}$$

for all $x, y \in G$, where $F(x) = f(x)/m(x) - f(0)$. By the well-known results in [4], there exists a unique additive function $a_1(x)$ given by

$$a_1(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{2.38}$$

such that

$$|F(x) - a_1(x)| \leq \Phi_1(x) \tag{2.39}$$

if $\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(-2^k x) < \infty$, and there exists a unique additive function $a_2(x)$ given by

$$a_2(x) = \lim_{n\to\infty} 2^{n} f(2^{-n} x) \tag{2.40}$$

such that

$$|F(x) - a_2(x)| \leq \Phi_2(x) \tag{2.41}$$

if $\Phi_2(x) := \sum_{k=1}^{\infty} 2^k \psi(-2^{-k}x) < \infty$. Multiplying $|m(x)|$ in both sides of (2.39) and (2.41), we get (v). Now we consider the case when f, g satisfy (2.10). In view of Lemma 2.2, the solutions of (2.10) are given by (i), (iii), or contained in the case (v). This completes the proof. □

Let X be a real normed space, and let $\psi : X \to \mathbb{R}$ be given by $\psi(x) = \epsilon \|x\|^p$, $p \geq 0$, $p \neq 1$, then ψ satisfies the conditions assumed in Theorem 2.5. In view of (2.19) and (2.20), we have

$$\Phi_1(x) = \frac{2\epsilon \|x\|^p}{2 - 2^p} \tag{2.42}$$

if $0 < p < 1$,

$$\Phi_2(x) = \frac{2\epsilon \|x\|^p}{2^p - 2} \tag{2.43}$$

if $p > 1$. Thus, as a direct consequence of Theorem 2.5, we have the following.

Corollary 2.6. *Let $f, g : X \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x-y) - f(x)g(y) + g(x)f(y) \right| \leq \epsilon \|y\|^p, \quad p \neq 1,\ p \geq 0, \tag{2.44}$$

for all $x, y \in X$, then (f, g) satisfies one of the following:

(i) *$f = 0$, g is arbitrary,*

(ii) *f and g are bounded functions,*

(iii) *$f(x) = \lambda_1(m(x) - m(-x))$ and $g(x) = \lambda_2 f(x) + (1/2)(m(x) + m(-x))$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$, where m is an exponential function,*

(iv) *there exist $\lambda \in \mathbb{C}$ and a bounded exponential function m such that*

$$g(x) = \lambda f(x) + m(x), \tag{2.45}$$

for all $x \in X$, and f satisfies the condition; there exists $d \geq 0$ satisfying

$$\left| f(x) \right| \leq \frac{2}{\sqrt{3}} (\psi(-x) + d), \tag{2.46}$$

for all $x \in X$,

(v) *there exist $\lambda \in \mathbb{C}$ and a bounded exponential function m satisfying $m^2 \equiv 1$ such that*

$$g(x) = \lambda f(x) + m(x), \tag{2.47}$$

for all $x \in X$, and f satisfies one of the following conditions; there exists an additive function $a : X \to \mathbb{C}$ such that

$$\left|f(x) - \left(a(x) + f(0)\right)m(x)\right| \leq \frac{2\epsilon\|x\|^p}{|2 - 2^p|}, \tag{2.48}$$

for all $x \in X$.

Now we prove the stability of (1.2). For the proof, we need the following.

Lemma 2.7. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|g(x - y) - g(x)g(y) - f(x)f(y)\right| \leq \varphi(y), \tag{2.49}$$

for all $x, y \in G$, then either there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$\left|\lambda_1 f(x) - \lambda_2 g(x)\right| \leq M, \tag{2.50}$$

or else

$$g(x - y) - g(x)g(y) - f(x)f(y) = 0, \tag{2.51}$$

for all $x, y \in G$.

Proof. Suppose that $\lambda_1 f(x) - \lambda_2 g(x)$ is bounded only when $\lambda_1 = \lambda_2 = 0$, and let

$$l(x, y) = g(x + y) - g(x)g(-y) - f(x)f(-y). \tag{2.52}$$

Since we may assume that f is nonconstant, we can choose y_1 satisfying $f(-y_1) \neq 0$. Now it can be easily get that

$$f(x) = \lambda_0 g(x) + \lambda_1 g(x + y_1) - \lambda_1 l(x, y_1), \tag{2.53}$$

where $\lambda_0 = -g(-y_1)/f(-y_1)$ and $\lambda_1 = 1/f(-y_1)$. From the definition of l and the use of (2.53), we have the following two equations:

$$\begin{aligned}
g((x + y) + z) &= g(x + y)g(-z) + f(x + y)f(-z) + l(x + y, z)\\
&= \left(g(x)g(-y) + f(x)f(-y) + l(x, y)\right)g(-z)\\
&\quad + \left(\lambda_0 g(x + y) + \lambda_1 g(x + y + y_1) - \lambda_1 l(x + y, y_1)\right)f(-z) + l(x + y, z)\\
&= \left(g(x)g(-y) + f(x)f(-y) + l(x, y)\right)g(-z)\\
&\quad + \lambda_0\left(g(x)g(-y) + f(x)f(-y) + l(x, y)\right)f(-z)\\
&\quad + \lambda_1\left(g(x)g(-y - y_1) + f(x)f(-y - y_1) + l(x, y + y_1)\right)f(-z)\\
&\quad - \lambda_1 l(x + y, y_1)f(-z) + l(x + y, z),
\end{aligned} \tag{2.54}$$

$$g(x + (y + z)) = g(x)g(-y - z) + f(x)f(-y - z) + l(x, y + z). \tag{2.55}$$

Equating (2.54) and (2.55), we have

$$\begin{aligned} g(x)\big(g(-y)g(-z) + \lambda_0 g(-y)f(-z) + \lambda_1 g(-y-y_1)f(-z) - g(-y-z)\big) \\ + f(x)\big(f(-y)g(-z) + \lambda_0 f(-y)f(-z) + \lambda_1 f(-y-y_1)f(-z) - f(-y-z)\big) \\ = -l(x,y)g(-z) - \lambda_0 l(x,y)f(-z) - \lambda_1 l(x,y+y_1)f(-z) \\ + \lambda_1 l(x+y,y_1)f(-z) - l(x+y,z) + l(x,y+z). \end{aligned} \tag{2.56}$$

In (2.56), when y, z are fixed, the right hand side is bounded, so by our assumption, we have

$$l(x,y)g(-z) + \big(\lambda_0 l(x,y) + \lambda_1 l(x,y+y_1) - \lambda_1 l(x+y,y_1)\big)f(-z) = l(x,y+z) - l(x+y,z). \tag{2.57}$$

Also we can write

$$\begin{aligned} l(x,y+z) - l(x+y,z) &= g(x+y+z) - g(x)g(y+z) - f(x)f(y+z) \\ &\quad - g(x+y+z) + g(x+y)g(z) + f(x+y)f(z) \\ &= l(y+z,x) - l(z,x+y) \\ &\le \psi(-x) + \psi(-x-y). \end{aligned} \tag{2.58}$$

Thus, if we fix x, y in (2.57), the right hand side of (2.57) is bounded. By our assumption, we have $l(x,y) \equiv 0$. This completes the proof. □

Theorem 2.8. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|g(x-y) - g(x)g(y) - f(x)f(y)\right| \le \psi(y), \tag{2.59}$$

for all $x, y \in G$, then (f, g) satisfies one of the following:

(i) *f and g are bounded functions,*

(ii) *$g(x) = (1/2)(m(x) + m(-x))$ and $f(x) = (1/2)(m(x) - m(-x))$, where m is an exponential function,*

(iii) *$f = \pm i(g - m)$ for a bounded exponential function m, and g satisfies*

$$\left|g(x) - \frac{1}{2}\big(g(0)m(-x) + m(x)\big)\right| \le \frac{1}{2}\psi(x), \tag{2.60}$$

for all $x \in G$. In particular if $\psi(0) = 0$, one has $g(0) = 1$, $f(0) = 0$.

Proof. In view of Lemma 2.7, we first consider the case when f, g satisfy (2.51). If f is bounded, then in view of the inequality (2.59), for each y, $g(x+y) - g(x)g(-y)$, is also bounded. It follows from Lemma 2.1 that g is bounded or a nonzero exponential function. If g is bounded, the case (i) follows. If g is a nonzero exponential function, from (2.59), using the triangle inequality, we have for some $d \ge 0$,

$$\left|g(x)\big(g(-y) - g(y)\big)\right| \le \psi(y) + d, \tag{2.61}$$

for all $x, y \in G$. Thus, it follows that

$$g(y) = g(-y), \tag{2.62}$$

for all $y \in G$, or else g is bounded, and equality (2.62) implies $g^2 \equiv 1$, which gives the case (i).

If f is unbounded, then in view of (2.59), g is also unbounded, and hence, $\lambda_1\lambda_2 \neq 0$ and

$$f(x) = \lambda g(x) + r(x), \tag{2.63}$$

for some $\lambda \neq 0$ and a bounded function r. Putting (2.63) in (2.59), replacing y by $-y$, and using the triangle inequality, we have

$$\left|g(x+y) - g(x)\left(\left(\lambda^2+1\right)g(-y) + \lambda r(-y)\right)\right| \le |(\lambda g(-y) + r(-y))r(x)| + \varphi(-y). \tag{2.64}$$

From Lemma 2.1, we have

$$\left(\lambda^2+1\right)g(y) + \lambda r(y) = m(y), \tag{2.65}$$

for some exponential function m. If $\lambda^2 \neq -1$, we have

$$f(x) = \frac{\lambda m(x) + r(x)}{\lambda^2+1}, \qquad g(x) = \frac{m(x) - \lambda r(x)}{\lambda^2+1}. \tag{2.66}$$

Putting (2.66) in (2.59), multiplying $|\lambda^2+1|$ in the result, and using the triangle inequality, we have for some $d \ge 0$,

$$|m(x)(m(-y) - m(y))| \le \left|\lambda^2+1\right|\varphi(y) + d, \tag{2.67}$$

for all $x, y \in G$. Since m is unbounded, we have

$$m(y) = m(-y), \tag{2.68}$$

for all $y \in G$, which implies $m^2 \equiv 1$, contradicting to the fact that m is unbounded. Thus, it follows that $\lambda^2 = -1$, and we have

$$f = \pm i(g - m), \tag{2.69}$$

where m is a bounded exponential function. Putting (2.69) in (2.59), we have

$$|g(x-y) - g(x)m(y) - g(y)m(x) + m(x)m(y)| \le \varphi(y), \tag{2.70}$$

for all $x,\ y \in G$. Replacing y by x in (2.70) and dividing the result by $2m(x)$, we have

$$\left| g(x) - \frac{1}{2}\left(g(0)m(-x) + m(x)\right) \right| \le \frac{1}{2}\psi(x), \tag{2.71}$$

for all $x \in G$. From (2.69), (2.71), we get (iii). Now we consider the case when f, g satisfy (2.51). In view of Lemma 2.2, the solutions of (2.51) are contained in (i) or given by (ii). Furthermore, if $\psi(0) = 0$, then putting $x = y = 0$ in (2.70), we have $g(0) = 1$, and from (2.69), we also have $f(0) = 0$. This completes the proof. □

In particular, if $f, g : \mathbb{R}^n \to \mathbb{C}$ is a continuous function and $\psi(x) = \epsilon|x|^p,\ p > 0,\ p \neq 1$, then Theorem 2.8 is reduced as follows.

Corollary 2.9. *Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be a continuous function satisfying* (2.59) *for $\psi(x) = \epsilon|x|^p$, then (f, g) satisfies one of the following:*

(i) *f and g are bounded functions,*

(ii) $g(x) = \cos(c \cdot x)$ *and* $f(x) = \sin(c \cdot x)$ *for some $c \in \mathbb{C}^n$,*

(iii) *there exists $a \in \mathbb{R}^n$ such that*

$$\begin{aligned} \left| f(x) - \sin(a \cdot x) \right| &\le \frac{\epsilon}{2}|x|^p, \\ \left| g(x) - \cos(a \cdot x) \right| &\le \frac{\epsilon}{2}|x|^p, \end{aligned} \tag{2.72}$$

for all $x \in \mathbb{R}^n$.

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