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Research Article

Global Existence and Boundedness of Solutions to a Second-Order Nonlinear Differential System

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We investigate the global existence and boundedness of solutions to a second-order nonlinear differential system.

1. Introduction

In this paper, we study the nonlinear system

$$\begin{aligned} x' &= \frac{1}{a(x)}\left[c(y) - b(x)\right], \\ y' &= -a(x)[h(x) - e(t)], \end{aligned} \tag{1.1}$$

where $a : R \to (0, \infty)$, $b, c, h : R \to R$, and $e : R \to R$ are continuous.

As a particular case of (1.1) we have well-known Liẽnard equation as follows:

$$x'' + f(x)x' + h(x) = e(t). \tag{1.2}$$

with $a(x) = 1$, $b(x) = \int_0^x f(s)ds$, $c(x) = x$, $x \in R$ and the second-order nonlinear differential equation as follows:

$$x'' + \left(f(x) + g(x)x'\right)x' + h(x) = e(t) \tag{1.3}$$

for $a(x) = \exp(\int_0^x g(s)ds)$, $b(x) = \int_0^x a(s)f(s)ds$, $c(x) = x$, $x \in R$.

System (1.1) can be regarded as a mathematical model for many phenomena in applied sciences (theory of feedback electronic circuits, motion of a mass-spring system). It has been investigated by several authors, compare [1–4] and the citations therein.

The purpose of this paper is to present new results on the global existence and boundedness of solutions for the system (1.1). The obtained results improve the recent results in [1, 5]. Our paper is divided into two section. In Section 2, we prove the global existence of solutions for (1.1). In Section 3, we get some new results on boundedness of solutions for the system (1.1).

2. Global Existence

In this section, we will present new results on the global existence of solutions to system (1.1) under general conditions on the nonlinearities.

Let us first define

$$C(y) = \int_0^y c(s)ds, \qquad H(x) = \int_0^x a^2(s)h(s)ds. \tag{2.1}$$

Then, we have the following.

Theorem 2.1. *Assume that*

(i) *there exists some* $K \geq 0$, *such that*

$$\begin{aligned} \operatorname{sgn}(x)H(x) + K \geq 0, \quad x \in R, \\ \operatorname{sgn}(y)C(y) + K \geq 0, \quad y \in R, \end{aligned} \tag{2.2}$$

(ii) *there exist some* $N \geq 0$ *and* $Q > 0$, *such that*

$$\begin{aligned} |H(x)| < Q, \quad |x| > N, \\ |C(y)| < Q, \quad |y| > N, \end{aligned} \tag{2.3}$$

(iii) $\lim_{|y|\to\infty} \operatorname{sgn}(y)C(y) = Q$, $\lim_{|x|\to\infty}[1/(Q - \operatorname{sgn}(x)H(x)) + \operatorname{sgn}(x)b(x)] = \infty$,

(iv) *there exist two positive functions* $\mu, \omega \in C([0, K + Q), (0, \infty))$ *such that*

$$\begin{aligned} a(x)|c(y)| \leq \min\{&\mu(\operatorname{sgn}(x)H(x) + K) + \omega(\operatorname{sgn}(x)H(x) + K), \\ &\mu(\operatorname{sgn}(y)C(y) + K) + \omega(\operatorname{sgn}(y)C(y) + K)\}, \\ &|x| > N, \ |y| > N, \end{aligned} \tag{2.4}$$

(v) $\operatorname{sgn}(x)a(x)b(x)h(x) \geq -[\mu(\operatorname{sgn}(x)H(x)+K)+\omega(\operatorname{sgn}(x)H(x)+K)]$, $|x| > N$ *and* $|h(x)| \leq M < \infty,\ x \in R$.

If

$$\int_0^{K+Q} \frac{ds}{\mu(s)+\omega(s)} = \infty, \tag{2.5}$$

then every solution of (1.1) *exists globally.*

Proof. Due that $a : R \to (0,\infty)$, b, c, $h : R \to R$ and $e : R \to R$ are continuous, by Peano's Existence Theorem [6], we have that the system (1.1) with any initial data (x_0, y_0) possesses a solution $(x(t), y(t))$ on $[0,T)$ for some maximal $T > 0$. If $T < \infty$, one has

$$\lim_{t\to T}\left(|x(t)| + \left|y(t)\right|\right) = \infty. \tag{2.6}$$

First, assume that $\lim_{t\to T}|y(t)| = \infty$.

Since $y(t)$ is continuous, there exists $0 \leq T_0 < T$ such that

$$\left|y(t)\right| > N, \quad t \in [T_0, T). \tag{2.7}$$

Take $V_1(t,x,y) = \operatorname{sgn}(y)C(y) + K,\ t \in R_+,\ x, y \in R$. Differentiating $V_1(t,x,y)$ with respect to t along solution $(x(t), y(t))$ of (1.1), we have

$$\begin{aligned} \frac{dV_1}{dt} &= \operatorname{sgn}(y)\left[-a(x)c(y)h(x) + a(x)c(y)e(t)\right] \\ &\leq (|h(x)| + |e(t)|)a(x)\left|c(y)\right| \\ &\leq (M + |e(t)|)\left[\mu(\operatorname{sgn}(y)C(y)+K) + \omega(\operatorname{sgn}(y)C(y)+K)\right], \\ &\qquad t \in [T_0, T). \end{aligned} \tag{2.8}$$

Since $0 \leq \operatorname{sgn}(y(t))C(y(t)) + K < Q + K$, $t \in [T_0, T)$, we obtain

$$\frac{dV_1(t)}{\mu(V_1(t)) + \omega(V_1(t))} \leq (M + |e(t)|)dt, \quad t \in [T_0, T). \tag{2.9}$$

We denote that $V_1(t) = V_1(t, x(t), y(t))$.

Since $\lim_{|y|\to\infty} \operatorname{sgn}(y)C(y) = Q$, $\int_0^{K+Q}(ds/(\mu(s)+\omega(s))) = \infty$, $y(t)$, $C(y)$ are continuous, there exists $T_0 \leq t_1 < t_2 < T$ such that

$$\int_{V_1(t_1)}^{V_1(t_2)} \frac{ds}{\mu(s)+\omega(s)} > M\int_0^T dt + \int_0^T |e(t)|dt. \tag{2.10}$$

Integrating (2.9) on $[t_1, t_2]$ with respect to t and using the above relation, we obtain the following contradiction:

$$M\int_0^T dt + \int_0^T |e(t)|dt < \int_{V_1(t_1)}^{V_1(t_2)} \frac{ds}{\mu(s)+\omega(s)} = \int_{t_1}^{t_2} \frac{dV_1(t)}{\mu(V_1(t))+\omega(V_1(t))}$$
$$\leq \int_{t_1}^{t_2} (M+|e(t)|)dt \leq M\int_0^T dt + \int_0^T |e(t)|dt. \tag{2.11}$$

Thus, there exists an $M > 0$ such that

$$\left|y(t)\right| \leq M, \quad t \in [0, T). \tag{2.12}$$

Second, by the result above, we have $\lim_{t\to T}|x(t)| = \infty$.

If $\lim_{|x|\to\infty}(1/(Q - \operatorname{sgn}(x)H(x))) = \infty$, that is, $\lim_{|x|\to\infty}\operatorname{sgn}(x)H(x) = Q$, we set

$$V_2(t, x, y) = \operatorname{sgn}(x)H(x) + K, \quad t \in R_+,\ x, y \in R. \tag{2.13}$$

Since $x(t)$ is continuous, there exists $0 \leq T_1 < T$ such that

$$|x(t)| > N, \quad t \in [T_1, T). \tag{2.14}$$

Differentiating $V_2(t, x, y)$ with respect to t along solution $(x(t), y(t))$ of (1.1), we have

$$\begin{aligned}\frac{dV_2}{dt} &= \operatorname{sgn}(x)\left[a(x)c(y)h(x) - a(x)b(x)h(x)\right]\\ &\leq (M+1)\left[\mu(\operatorname{sgn}(x)H(x)+K) + \omega(\operatorname{sgn}(x)H(x)+K)\right],\\ &\qquad t \in [T_1, T).\end{aligned} \tag{2.15}$$

Since $0 \leq \operatorname{sgn}(x(t))H(x(t)) + K < Q + K,\ t \in [T_1, T)$, we obtain

$$\frac{dV_2(t)}{\mu(V_2(t))+\omega(V_2(t))} \leq (M+1)dt, \quad t \in [T_1, T). \tag{2.16}$$

We denote $V_2(t) = V_2(t, x(t), y(t))$.

Since $\lim_{|x|\to\infty}\operatorname{sgn}(x)H(x) = Q$, $\int_0^{K+Q}(ds/(\mu(s)+\omega(s))) = \infty$, $x(t), H(x)$ are continuous, there exists $T_1 \leq t_3 < t_4 < T$ such that

$$\int_{V_2(t_3)}^{V_2(t_4)} \frac{ds}{\mu(s)+\omega(s)} > (M+1)\int_0^T dt. \tag{2.17}$$

Integrating (2.16) on $[t_3, t_4]$ with respect to t and using the above relation, we obtain the contradiction as follows:

$$(M+1)\int_0^T dt < \int_{V_2(t_3)}^{V_2(t_4)} \frac{ds}{\mu(s)+\omega(s)} = \int_{t_3}^{t_4} \frac{dV_2(t)}{\mu(V_2(t))+\omega(V_2(t))} \\ \leq \int_{t_3}^{t_4} (M+1)dt \leq (M+1)\int_0^T dt. \tag{2.18}$$

So consider $\lim_{|x|\to\infty}(1/(Q - \operatorname{sgn}(x)H(x))) < \infty$.

By (iii), we have $\lim_{|x|\to\infty} \operatorname{sgn}(x)b(x) = \infty$.

Set

$$W(t,x,y) = x, \quad t \in R_+, \ x,y \in R. \tag{2.19}$$

Then, along solutions to (1.1) we have

$$\frac{dW}{dt} = \frac{1}{a(x)}\left[c(y) - b(x)\right]. \tag{2.20}$$

If $\lim_{t\to T} x(t) = \infty$, we deduce that there exist x_1 and x_2 such that $x_0 < x_1 < x_2$ and

$$\frac{dW}{dt} < 0, \qquad x_1 \leq x \leq x_2, \qquad |y| \leq M. \tag{2.21}$$

Then, by the continuity of the solution, there exist $0 < t_1 < t_2 < T$ such that $x(t_1) = x_1$, $x(t_2) = x_2$. Integrating (2.21) on $[t_1, t_2]$, we have

$$W(t_1, x(t_1), y(t_1)) = x_1 > x_2 = W(t_2, x(t_2), y(t_2)). \tag{2.22}$$

This contradicts $x_1 < x_2$. Hence $x(t)$ is bounded from above.

Similarly, if $\lim_{t\to T} x(t) = -\infty$, we can obtain a contradiction by setting $W(t,x,y) = -x$. Thus, it follows that $x(t)$ is also bounded from above. This forces $T = \infty$ and completes the proof of Theorem 2.1. □

Example 2.2. Consider the following nonlinear system:

$$x' = \frac{\sqrt{1+x^2}}{1+y^2}, \\ y' = -\frac{1}{\sqrt{1+x^2}} + \frac{t^2}{\sqrt{1+x^2}}. \tag{2.23}$$

Set $a(x) = 1/\sqrt{1+x^2}$, $b(x) = 0$, $c(y) = 1/(1+y^2)$, $h(x) = 1$, $e(t) = t^2$. Then we have

$$C(y) = \arctan y, \qquad H(x) = \arctan x. \tag{2.24}$$

Take $K = N = 0$, $Q = \pi/2$, and $\mu(\theta) = \omega(\theta) = \cos\theta/2$, $\theta \in [0, \pi/2)$. Note that

$$\begin{aligned} a(x)\left|c(y)\right| &= \frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{1+y^2} \le \min\left\{ \frac{1}{\sqrt{1+x^2}}, \frac{1}{\sqrt{1+y^2}} \right\} \\ &= \min\{\mu(\operatorname{sgn}(x)\arctan x + K) + \omega(\operatorname{sgn}(x)\arctan x + K), \\ &\qquad \mu(\operatorname{sgn}(y)\arctan y + K) + \omega(\operatorname{sgn}(y)\arctan \ y + K)\} \end{aligned} \tag{2.25}$$

$\operatorname{sgn}(x)a(x)b(x)h(x) = 0$, $|h(x)| = 1$ and

$$\int_0^{\pi/2} \frac{d\theta}{\mu(\theta) + \omega(\theta)} = \int_0^{\pi/2} \frac{d\theta}{\cos\theta} \ge \frac{1}{2}\int_0^1 \frac{dx}{1-x} = \infty. \tag{2.26}$$

Applying Theorem 2.1, we know that every solution of (2.23) exists globally. Observe that the theorem and corollary in [5] cannot be used in the present case.

Theorem 2.3. *Assume that*

(i) *there exists some $K \ge 0$, such that*

$$\begin{aligned} H(x) + K \ge 0, &\quad x \in R, \\ C(y) + K \ge 0, &\quad y \in R, \end{aligned} \tag{2.27}$$

(ii) *there exist some $N \ge 0$ and $Q > 0$, such that*

$$\begin{aligned} |H(x)| < Q, &\quad |x| > N, \\ |C(y)| < Q, &\quad |y| > N, \end{aligned} \tag{2.28}$$

(iii) $\lim_{|y|\to\infty} C(y) = Q$, $\lim_{|x|\to\infty}[1/(Q - H(x)) + \operatorname{sgn}(x)b(x)] = \infty$,

(iv) *there exist two positive functions $\mu, \omega \in C([0, K+Q), (0, \infty))$ such that*

$$\begin{aligned} a(x)\left|c(y)\right| \le \min\{&\mu(H(x)+K) + \omega(H(x)+K), \\ &\mu(C(y)+K) + \omega(C(y)+K)\}, \\ &|x| > N, \ \left|y\right| > N, \end{aligned} \tag{2.29}$$

(v) $a(x)b(x)h(x) \ge -[\mu(H(x)+K) + \omega(H(x)+K)]$, $|x| > N$ *and* $|h(x)| \le M < \infty$, $x \in R$. *If*

$$\int_0^{K+Q} \frac{ds}{\mu(s) + \omega(s)} = \infty, \tag{2.30}$$

then every solution of (1.1) *exists globally.*

Proof. The proof of Theorem 2.3 is similar to that of Theorem 2.1, so we omit it. □

Example 2.4. Consider the following nonlinear system:

$$\begin{aligned} x' &= \frac{2y}{(1+y^2)^2} + \frac{2y\ \ln(1+x^2)}{(1+y^2)^2}, \\ y' &= -\frac{2x}{(1+x^2)[1+\ln(1+x^2)]} + \frac{t^3}{1+\ln(1+x^2)}. \end{aligned} \tag{2.31}$$

Set $a(x) = 1/(1+\ln(1+x^2))$, $b(x) = 0$, $c(y) = 2y/(1+y^2)^2$, $h(x) = 2x/(1+x^2)$, $e(t) = t^3$. Then we have $C(y) = 1-(1/(1+y^2))$, $H(x) = 1-(1/(1+\ln(1+x^2)))$. Take $K = N = 0$, $Q = 1$ and $\mu(t) = \omega(t) = (1-t)/2$, $t \in [0,1)$. Note that

$$\begin{aligned} a(x)|c(y)| &= \frac{1}{1+\ln(1+x^2)} \cdot \frac{2|y|}{(1+y^2)^2} \le \min\left\{\frac{1}{1+\ln(1+x^2)}, \frac{1}{1+y^2}\right\} \\ &= \min\{\mu(H(x)+K)+\omega(H(x)+K), \mu(C(y)+K)+\omega(C(y)+K)\}. \end{aligned} \tag{2.32}$$

$a(x)b(x)h(x) = 0$, $|h(x)| = 2|x|/(1+x^2) \le 1$ and

$$\int_0^1 \frac{ds}{\mu(s)+\omega(s)} = \int_0^1 \frac{ds}{1-s} = \infty. \tag{2.33}$$

Applying Theorem 2.3, we know that every solution of (2.31) exists globally. Observe that the theorem and corollary in [5] cannot be uses in the present case.

3. Boundedness

In this section, we will present some results on the boundedness of solutions to (1.1) under general conditions on the nonlinearities.

Theorem 3.1. *Assume that*

(i) *there exist functions* $f_1, f_2 \in C(R_+, R)$ *such that*

$$f_1(t) \le y'(t) = -a(x)[h(x)-e(t)] \le f_2(t), \quad x \in R,\ t \in R_+, \tag{3.1}$$

and $|\int_0^\infty f_1(t)dt| < \infty$, $|\int_0^\infty f_2(t)dt| < \infty$,

(ii) $\lim_{|x|\to\infty} \operatorname{sgn}(x)b(x) = \infty$.

If the solution $(x(t), y(t))$ *of* (1.1) *exists globally, then* $(x(t), y(t))$ *is bounded.*

Proof. By (i), we have

$$f_1(t) \le y'(t) \le f_2(t), \quad x \in R,\ t \in R_+. \tag{3.2}$$

Integrating (3.2) on $[0,t]$ with respect to t, we have

$$\int_0^t f_1(s)ds \leq y(t) - y_0 \leq \int_0^t f_2(s)ds. \tag{3.3}$$

Since $|\int_0^\infty f_1(t)dt| < \infty$, $|\int_0^\infty f_2(t)dt| < \infty$. Thus, there exists a $Y > 0$ such that

$$|y(t)| \leq Y, \quad t \geq 0. \tag{3.4}$$

Set

$$W(t,x,y) = -x, \quad t \in R_+, \ x,y \in R. \tag{3.5}$$

Then, along solutions to (1.1), we have

$$\frac{dW}{dt} = -\frac{1}{a(x)}\left[c(y) - b(x)\right]. \tag{3.6}$$

If $\lim_{t\to\infty} x(t) = -\infty$, we deduce that there exist x_1 and x_2 such that $x(0) > x_1 > x_2$ and

$$\frac{dW}{dt} < 0, \qquad x_2 \leq x \leq x_1, \qquad |y| \leq Y. \tag{3.7}$$

Then, by the continuity of the solution, we have that there exist $0 < t_1 < t_2 < \infty$ such that $x(t_1) = x_1$ and $x(t_2) = x_2$. Integrating (3.7) on $[t_1, t_2]$, we get

$$W(t_1, x(t_1), y(t_1)) = -x_1 > -x_2 = W(t_2, x(t_2), y(t_2)). \tag{3.8}$$

This contradicts $x_1 > x_2$. Hence $x(t)$ is bounded from below.

Similarly, if $\lim_{t\to T} x(t) = \infty$, we can obtain a contradiction by setting $W(t,x,y) = x$. Thus, it follows that $x(t)$ is also bounded from above. This completes the proof of Theorem 3.1. □

Example 3.2. Consider the following nonlinear system:

$$\begin{aligned} x' &= \frac{\sqrt{1+x^2}}{1+y^2} - x\sqrt{1+x^2}, \\ y' &= \frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{1+t^2}. \end{aligned} \tag{3.9}$$

Set $a(x) = 1/\sqrt{1+x^2}$, $b(x) = x$, $c(y) = 1/(1+y^2)$, $h(x) = 1$, $e(t) = 1 + (1/(1+t^2))$. Then we have $C(y) = \arctan\, y$ and $H(x) = \arctan\, x$. Take $K = N = 0$, $Q = \pi/2$ and $\mu(\theta) = \omega(\theta) = \cos\theta/2$, $\theta \in [0, \pi/2)$. Applying Theorem 2.1, we know that every solution of (3.9) exists globally.

Take $f_1(t) = 0$, $f_2(t) = 1/(1+t^2)$, we have

$$f_1(t) = 0 \le y'(t) \le f_2(t) = \frac{1}{1+t^2}, \quad x \in R,\ t \in R_+ \tag{3.10}$$

and $|\int_0^\infty f_2(t)dt| = \pi/2 < \infty$, $\lim_{|x|\to\infty} \operatorname{sgn}(x)b(x) = \lim_{|x|\to\infty} x \operatorname{sgn}(x) = \infty$. Applying Theorem 3.1, we know that every solution of (3.9) is bounded.

Theorem 3.3. *Assume that*

(i) *there exists some $K \ge 0$, such that*

$$\begin{aligned} &\operatorname{sgn}(x)H(x) + K \ge 0, \quad x \in R, \\ &\operatorname{sgn}(y)C(y) + K \ge 0, \quad y \in R, \end{aligned} \tag{3.11}$$

(ii) *there exist some $N \ge 0$ and $Q > 0$, such that*

$$\begin{aligned} &|H(x)| < Q, \quad |x| > N, \\ &|C(y)| < Q, \quad |y| > N, \end{aligned} \tag{3.12}$$

(iii) $\lim_{|y|\to\infty} \operatorname{sgn}(y)C(y) = Q$, $\lim_{|x|\to\infty}[1/(Q - \operatorname{sgn}(x)H(x)) + \operatorname{sgn}(x)b(x)] = \infty$,

(iv) *there exist two positive functions $\mu, \omega \in C([0, K+Q), (0,\infty))$ such that*

$$\begin{aligned} a(x)|c(y)| \le \min\{&\mu(\operatorname{sgn}(x)H(x) + K) + \omega(\operatorname{sgn}(x)H(x) + K), \\ &\mu(\operatorname{sgn}(y)C(y) + K) + \omega(\operatorname{sgn}(y)C(y) + K)\}, \\ &\qquad |x| > N,\ |y| > N, \end{aligned} \tag{3.13}$$

(v) $\operatorname{sgn}(x)a(x)b(x)h(x) \ge 0$, $|x| > N$ *and* $|h(x)| \le M < \infty$, $x \in R$,

(vi) $E = \int_0^\infty |e(t)|dt < \infty$, *and* $\int_0^{K+Q}(ds/(\mu(s)+\omega(s))) = \infty$.

If there exists $g(x)$ such that

$$0 < \frac{a(x)}{|c(y) - b(x)|} \le g(x), \quad x, y \in R \tag{3.14}$$

and $G = \int_{-\infty}^{\infty} g(x)|h(x)|dx < \infty$, then every solution of (1.1) *is bounded.*

Proof. Let $(x(t), y(t))$ be a solution to (1.1) with initial data (x_0, y_0). By Theorem 2.1, we have that $(x(t), y(t))$ exists globally. If $(x(t), y(t))$ is unbounded, we have

$$\lim_{t\to\infty}|x(t)| = \infty \quad \text{or} \quad \lim_{t\to\infty}|y(t)| = \infty. \tag{3.15}$$

First, assume that $\lim_{t\to T}|y(t)| = \infty$.

Since $y(t)$ is continuous, there exists $T_1 \geq 0$ such that

$$\left|y(t)\right| > N, \quad t \in [T_1, \infty). \tag{3.16}$$

Take $V_1(t,x,y) = \operatorname{sgn}(y)C(y) + K$, $t \in R_+$, $x, y \in R$. Differentiating $V_1(t,x,y)$ with respect to t along solution $(x(t), y(t))$ of (1.1), we have

$$\begin{aligned}\frac{dV_1}{dt} &= \operatorname{sgn}(y)\left[-a(x)c(y)h(x) + a(x)c(y)e(t)\right] \\ &\leq (|h(x)| + |e(t)|)a(x)\left|c(y)\right| \\ &\leq (|h(x)| + |e(t)|)\left[\mu(\operatorname{sgn}(y)C(y) + K) + \omega(\operatorname{sgn}(y)C(y) + K)\right], \\ &\qquad t \in [T_1, \infty).\end{aligned} \tag{3.17}$$

Since $0 \leq \operatorname{sgn}(y(t))C(y(t)) + K < Q + K$, $t \in [T_1, \infty)$, we obtain

$$\frac{dV_1(t)}{\mu(V_1(t)) + \omega(V_1(t))} \leq (|h[x(t)]| + |e(t)|)dt, \quad t \in [T_1, \infty). \tag{3.18}$$

We denote $V_1(t) = V_1(t, x(t), y(t))$.

By (vi), there exists $T_1 \leq t_1 < t_2 < \infty$ such that

$$\int_{V_1(t_1)}^{V_1(t_2)} \frac{ds}{\mu(s) + \omega(s)} > G + E. \tag{3.19}$$

Integrating (3.18) on $[t_1, t_2]$ with respect to t and using the above relation, we obtain the contradiction as follows:

$$\begin{aligned}G + E &< \int_{V_1(t_1)}^{V_1(t_2)} \frac{ds}{\mu(s) + \omega(s)} = \int_{V_1(t_1)}^{V_1(t_2)} \frac{dV_1(t)}{\mu(V_1(t)) + \omega(V_1(t))} \\ &\leq \int_{t_1}^{t_2} |h(x(t))|dt + \int_{t_1}^{t_2} |e(t)|dt \leq \int_{x(t_1)}^{x(t_2)} |h(x)|\frac{dx}{x'(t)} + E \\ &\leq \int_{-\infty}^{\infty} g(x)|h(x)|dx + E = G + E.\end{aligned} \tag{3.20}$$

Thus, there exists a $Y > 0$ such that

$$\left|y(t)\right| \leq Y, \quad t \in [0, \infty). \tag{3.21}$$

Second assume that $\lim_{t \to T}|x(t)| = \infty$.

If $\lim_{|x| \to \infty}(1/(Q - \operatorname{sgn}(x)H(x))) = \infty$, that is, $\lim_{|x| \to \infty} \operatorname{sgn}(x)H(x) = Q$, we set

$$V_2(t,x,y) = \operatorname{sgn}(x)H(x) + K, \quad t \in R_+, \ x, y \in R. \tag{3.22}$$

Since $x(t)$ is continuous, there exists $0 \le T_2 < \infty$ such that

$$|x(t)| > N, \quad t \in [T_2, \infty). \tag{3.23}$$

Differentiating $V_2(t, x, y)$ with respect to t along solution $(x(t), y(t))$ of (1.1), we have

$$\begin{aligned} \frac{dV_2}{dt} &= \operatorname{sgn}(x)\left[a(x)c(y)h(x) - a(x)b(x)h(x)\right] \\ &\le |h(x)|\left[\mu\left(\operatorname{sgn}(x)H(x) + K\right) + \omega\left(\operatorname{sgn}(x)H(x) + K\right)\right], \\ &\qquad t \in [T_2, \infty). \end{aligned} \tag{3.24}$$

Since $0 \le \operatorname{sgn}(x(t))H(x(t)) + K < Q + K$, $t \in [T_2, \infty)$, we obtain

$$\frac{dV_2(t)}{\mu(V_2(t)) + \omega(V_2(t))} \le |h[x(t)]|dt, \quad t \in [T_2, \infty). \tag{3.25}$$

We denote that $V_2(t) = V_2(t, x(t), y(t))$.

By (vi), there exists $T_2 \le t_3 < t_4 < T$ such that

$$\int_{V_2(t_3)}^{V_2(t_4)} \frac{ds}{\mu(s) + \omega(s)} > G. \tag{3.26}$$

Integrating (3.25) on $[t_3, t_4]$ with respect to t and using the above relation, we obtain the contradiction as follows:

$$\begin{aligned} G &< \int_{V_2(t_3)}^{V_2(t_4)} \frac{ds}{\mu(s) + \omega(s)} = \int_{t_3}^{t_4} \frac{dV_2(t)}{\mu(V_2(t)) + \omega(V_2(t))} \le \int_{t_3}^{t_4} |h(x(t))|dt \\ &= \int_{x(t_3)}^{x(t_4)} |h(x)| \frac{dx}{x'(t)} \le \int_{-\infty}^{\infty} g(x)|h(x)|dx = G. \end{aligned} \tag{3.27}$$

So consider $\lim_{|x| \to \infty}(1/((Q - \operatorname{sgn}(x)H(x)))) < \infty$.

By (iii), we have $\lim_{|x| \to \infty} \operatorname{sgn}(x)b(x) = \infty$. The proof of this condition is similar to that of Theorem 3.1, so we omit it. Thus, it follows that $x(t)$ is also bounded from above. Then every solution of (1.1) is bounded. This completes the proof of Theorem 3.3. □

Theorem 3.4. *Assume that*

(i) *there exists some $K \ge 0$, such that*

$$\begin{aligned} H(x) + K \ge 0, &\quad x \in R, \\ C(y) + K \ge 0, &\quad y \in R, \end{aligned} \tag{3.28}$$

(ii) *there exist some* $N \geq 0$ *and* $Q > 0$, *such that*

$$|H(x)| < Q, \quad |x| > N,$$
$$|C(y)| < Q, \quad |y| > N, \tag{3.29}$$

(iii) $\lim_{|y|\to\infty} C(y) = Q$, $\lim_{|x|\to\infty}[1/(Q - H(x)) + \operatorname{sgn}(x)b(x)] = \infty$,
(iv) *there exist two positive functions* $\mu, \omega \in C([0, K+Q), (0, \infty))$ *such that*

$$\begin{aligned} a(x)|c(y)| \leq \min\{\mu(H(x)+K) + \omega(H(x)+K), \\ \mu(C(y)+K) + \omega(C(y)+K)\}, \\ |x| > N,\ |y| > N, \end{aligned} \tag{3.30}$$

(v) $\operatorname{sgn}(x)a(x)b(x)h(x) \geq 0$, $|x| > N$ *and* $|h(x)| \leq M < \infty$, $x \in R$,
(vi) $E = \int_0^\infty |e(t)|dt < \infty$, *and* $\int_0^{K+Q}(ds/(\mu(s)+\omega(s))) = \infty$.

If there exists $g(x)$ *such that*

$$0 < \frac{a(x)}{|c(y) - b(x)|} \leq g(x), \quad x, y \in R \tag{3.31}$$

and $G = \int_{-\infty}^{\infty} g(x)|h(x)|dx < \infty$, *then every solution of* (1.1) *is bounded.*

Proof. The proof of Theorem 3.4 is similar to that of Theorem 3.3, so we omit it. □

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References

[1] A. Constantin, "A note on a second-order nonlinear differential system," *Glasgow Mathematical Journal*, vol. 42, no. 2, pp. 195–199, 2000.
[2] T. A. Burton, "On the equation $x'' + f(x)h(x')x' + g(x) = e(t)$," *Annali di Matematica Pura ed Applicata*, vol. 85, pp. 277–285, 1970.
[3] A. Constantin, "Global existence of solutions for perturbed differential equations," *Annali di Matematica Pura ed Applicata*, vol. 168, pp. 237–299, 1995.
[4] J. Kato, "On a boundedness condition for solutions of a generalized Liénard equation," *Journal of Differential Equations*, vol. 65, no. 2, pp. 269–286, 1986.
[5] Z. Yin, "Global existence and boundedness of solutions to a second order nonlinear differential system," *Studia Scientiarum Mathematicarum Hungarica*, vol. 41, no. 4, pp. 365–378, 2004.
[6] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities—Volume I: Ordinary Differential Equation*, Academic Press, London, UK, 1969.