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Research Article

Strong Convergence Theorems for the Generalized Split Common Fixed Point Problem

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We introduce the generalized split common fixed point problem (GSCFPP) and show that the GSCFPP for nonexpansive operators is equivalent to the common fixed point problem. Moreover, we introduce a new iterative algorithm for finding a solution of the GSCFPP and obtain some strong convergence theorems under suitable assumptions.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given intergers $p, r \geq 1$, let us recall that the multiple-set split feasibility problem (MSSFP) was recently introduced [1] and is to find a point:

$$x^* \in \bigcap_{i=1}^{p} C_i, \quad Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{1.1}$$

where $\{C_i\}_{i=1}^{p}$ and $\{Q_j\}_{j=1}^{r}$ are nonempty closed convex subsets of H_1 and H_2, respectively. If $p = r = 1$, the MSSFP (1.1) becomes the so-called split feasibility problem (SFP) [2] which is to find a point:

$$x^* \in C, \quad Ax^* \in Q, \tag{1.2}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively. Recently, the SFP (1.2) and MSSFP (1.1) have been investigated by many researchers; see, [3–10].

Since every closed convex subset in a Hilbert space is looked as the fixed point set of its associating projection, the MSSFP (1.1) becomes a special case of the split common fixed point problem (SCFPP), which is to find a point:

$$x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^{r} \mathrm{Fix}(T_j), \tag{1.3}$$

where $U_i : H_1 \to H_1$ $(i = 1, 2, \ldots, p)$ and $T_j : H_2 \to H_2$ $(j = 1, 2, \ldots, r)$ are nonlinear operators. If $p = r = 1$, the problem (1.3) reduces to the so-called two-set SCFPP, which is to find a point:

$$x^* \in \mathrm{Fix}(U), \quad Ax^* \in \mathrm{Fix}(T). \tag{1.4}$$

Censor and Segal in [11] firstly introduced the concept of SCFPP in finite-dimensional Hilbert spaces and considered the following iterative algorithm for the two-set SCFPP (1.4) for Class-$\mathfrak{T}$ operators:

$$x_{n+1} = U\left(x_n - \gamma A^*(I - T)Ax_n\right), \quad n \geq 0, \tag{1.5}$$

where $x_0 \in H_1$, $0 < \gamma < 2/\|A\|^2$ and I is the identity operator. They proved the convergence of the algorithm (1.5) to a solution of problem (1.4). Moreover, they introduced a parallel iterative algorithm, which converges to a solution of the SCFPP (1.3). However, the parallel iterative algorithm does not include the algorithm (1.5) as a special case.

Very recently, Wang and Xu in [12] considered the SCFPP (1.3) for Class-$\mathfrak{T}$ operators and introduced the following iterative algorithm for solving the SCFPP (1.3):

$$x_{n+1} = U_{[n]}\left(x_n - \gamma A^*\left(I - T_{[n]}\right)Ax_n\right), \quad n \geq 0. \tag{1.6}$$

Under some mild conditions, they proved some weak and strong convergence theorems. Their iterative algorithm (1.6) includes Censor and Segal's algorithm (1.5) as a special case for the two-set SCFPP (1.4). Moreover, they prove that the SCFPP (1.3) for the Class-$\mathfrak{T}$ operators is equivalent to a common fixed point problem. This is also a classical method. Many problems eventually converted to a common fixed point problem; see [13–15].

Motivated and inspired by the aforementioned research works, we introduce a generalized split common fixed point problem (GSCFPP) which is to find a point:

$$x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^{\infty} \mathrm{Fix}(T_j). \tag{1.7}$$

Then, we show that the GSCFPP (1.7) for nonexpansive operators is equivalent to the following common fixed point problem:

$$x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(U_i), \qquad x^* \in \bigcap_{j=1}^{\infty} \mathrm{Fix}(V_j), \tag{1.8}$$

where $V_j = I - \gamma A^*(I - T_j)A$ $(0 < \gamma \leq 1/\|A\|^2)$ for every $j \in \mathbb{N}$. Moreover, we give a new iterative algorithm for solving the GSCFPP (1.7) for nonexpansive operators and obtain some strong convergence theorems.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges weakly to x and converges strongly to x, respectively.

An operator $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is denoted by $F(T)$. It is known that $F(T)$ is closed and convex. An operator $f : H \to H$ is called contraction if there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in H$. Let C be a nonempty closed convex subset of H. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for every $y \in C$. P_C is called a metric projection of H onto C. It is known that for each $x \in H$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \tag{2.1}$$

for all $y \in C$.

Let $\{T_n\}$ be a sequence of operators of H into itself. The set of common fixed points of $\{T_n\}$ is denoted by $F(\{T_n\})$, that is, $F(\{T_n\}) = \bigcap_{n=1}^{\infty} F(T_n)$. A sequence $\{T_n\}$ is said to be strongly nonexpansive if each $\{T_n\}$ is nonexpansive and

$$x_n - y_n - (T_n x_n - T_n y_n) \longrightarrow 0 \tag{2.2}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0$; see [16, 17]. A sequence $\{z_n\}$ in H is said to be an approximate fixed point sequence of $\{T_n\}$ if $z_n - T_n z_n \to 0$. The set of all bounded approximate fixed point sequences of $\{T_n\}$ is denoted by $\widetilde{F}(\{T_n\})$; see [16, 17]. We know that if $\{T_n\}$ has a common fixed point, then $\widetilde{F}(\{T_n\})$ is nonempty; that is, every bounded sequence in the common fixed point set is an approximate fixed point sequence. A sequence $\{T_n\}$ with a common fixed point is said to satisfy the condition (Z) if every weak cluster point of $\{x_n\}$ is a common fixed point whenever $\{x_n\} \in \widetilde{F}(\{T_n\})$. A sequence $\{T_n\}$ of nonexpansive mappings of H into itself is said to satisfy the condition (R) if

$$\lim_{n\to\infty} \sup_{y\in D} \|T_{n+1}y - T_n y\| = 0 \tag{2.3}$$

for every nonempty bounded subset D of H; see [18].

In order to prove our main results, we collect the following lemmas in this section.

Lemma 2.1 (see [16]). *Let C be a nonempty subset of a Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into H. Let $\{\lambda_n\}$ be a sequence in $[0, 1]$ such that* $\liminf_{n\to\infty}\lambda_n > 0$. *Let $\{U_n\}$ be a sequence of mappings of C into H defined by $U_n = \lambda_n I + (1 - \lambda_n)T_n$ for $n \in \mathbb{N}$, where I is the identity mapping on C. Then $\{U_n\}$ is a strongly nonexpansive sequence.*

Lemma 2.2 (see [16]). *Let H be a Hilbert space, C a nonempty subset of H, and $\{S_n\}$ and $\{T_n\}$ sequences of nonexpansive self-mappings of C. Suppose that $\{S_n\}$ or $\{T_n\}$ is a strongly nonexpansive sequence and $\widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\})$ is nonempty. Then $\widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\}) = \widetilde{F}(\{S_n T_n\})$.*

Lemma 2.3 (see [17]). *Let H be a Hilbert space, and C a nonempty subset of H. Both $\{S_n\}$ and $\{T_n\}$ satisfy the condition (R) and $\{T_n y : n \in \mathbb{N},\ y \in D\}$ is bounded for any bounded subset D of C. Then $\{S_n T_n\}$ satisfies the condition (R).*

Lemma 2.4 (see [19]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)\, y_n + \beta_n\, x_n$ for all integers $n \ \ge\ 0$ and*

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| \ - \ \|x_{n+1} - x_n\|\right) \le 0. \tag{2.4}$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.5 (see [20]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \ge 0, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

Now we state and prove our main results of this paper.

Lemma 3.1. *Let $A : H_1 \ \to\ H_2$ be a given bounded linear operator and let $T_n : H_2 \ \to\ H_2$ be a sequence of nonexpansive operators. Assume*

$$A^{-1}(\mathrm{Fix}(\{T_n\})) = \{x \in H_1 : Ax \in \mathrm{Fix}(\{T_n\})\} \ne \emptyset. \tag{3.1}$$

For each constant $\gamma > 0$, V_n is defined by the following:

$$V_n = I - \gamma A^*(I - T_n)A. \tag{3.2}$$

Then $\mathrm{Fix}(\{V_n\}) = A^{-1}(\mathrm{Fix}(\{T_n\}))$. Moreover, for $0 < \gamma \le 1/\|A\|^2$, V_n is nonexpansive on H_1 for $n \in \mathbb{N}$.

Proof. Since the inclusion $A^{-1}(\mathrm{Fix}(\{T_n\})) \subseteq \mathrm{Fix}(\{V_n\})$ is evident, now we only need to show the converse inclusion. If $z \in \mathrm{Fix}(\{V_n\})$, then we have $A^*(I - T_n)Az = 0$. Since $A^{-1}(\mathrm{Fix}(\{T_n\})) \neq \emptyset$, we take an arbitrary $p \in A^{-1}(\mathrm{Fix}(\{T_n\}))$. Hence

$$\begin{aligned}
\|Az - T_nAz\|^2 &= \langle Az - T_nAz, Az - T_nAz\rangle \\
&= \big\langle Az - T_nAz, Az - Ap + Ap - T_nAz\big\rangle \\
&= \big\langle A^*(I - T_n)Az, z - p\big\rangle + \big\langle Az - T_nAz, Ap - T_nAz\big\rangle \\
&= -\frac{1}{2}\big\|Az - Ap\big\|^2 + \frac{1}{2}\|Az - T_nAz\|^2 + \frac{1}{2}\big\|Ap - T_nAz\big\|^2 \\
&\leq \frac{1}{2}\|Az - T_nAz\|^2.
\end{aligned} \tag{3.3}$$

It follows that $(1/2)\|Az - T_nAz\|^2 \leq 0$, then $Az = T_nAz$ for every $n \in \mathbb{N}$, hence $z \in A^{-1}(\mathrm{Fix}(\{T_n\}))$. Next we turn to show that V_n is a nonexpansive operator for $n \in \mathbb{N}$. Since T_n is nonexpansive, we have

$$\begin{aligned}
\big\|(I - T_n)Ax - (I - T_n)Ay\big\|^2 &= \big\|Ax - Ay\big\|^2 + \big\|T_nAx - T_nAy\big\|^2 - 2\big\langle Ax - Ay, T_nAx - T_nAy\big\rangle \\
&\leq 2\big\|Ax - Ay\big\|^2 - 2\big\langle Ax - Ay, T_nAx - T_nAy\big\rangle \\
&\leq 2\big\langle Ax - Ay, Ax - Ay - \big(T_nAx - T_nAy\big)\big\rangle.
\end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned}
\big\|V_nx - V_ny\big\|^2 &= \big\|(I - \gamma A^*(I - T_n)A)x - (I - \gamma A^*(I - T_n)A)y\big\|^2 \\
&= \big\|x - y\big\|^2 + \gamma^2\|A\|^2\big\|(I - T_n)Ax - (I - T_n)Ay\big\|^2 \\
&\quad - 2\gamma\big\langle Ax - Ay, (I - T_n)Ax - (I - T_n)Ay\big\rangle \\
&\leq \big\|x - y\big\|^2 + \gamma\Big(\gamma\|A\|^2 - 1\Big)\big\|(I - T_n)Ax - (I - T_n)Ay\big\|^2.
\end{aligned} \tag{3.5}$$

For $0 < \gamma \leq 1/\|A\|^2$, we can immediately obtain that V_n is a nonexpansive operator for every $n \in \mathbb{N}$. $\square$

From Lemma 3.1, we can obtain that the solution set of GSCFPP (1.7) is identical to the solution set of problem (1.8).

Theorem 3.2. *Let $\{U_n\}$ and $\{V_n\}$ be sequences of nonexpansive operators on Hilbert space H_1. Both $\{U_n\}$ and $\{V_n\}$ satisfy the conditions (R) and (Z). Let $f : H_1 \to H_1$ be a contraction with coefficient*

$\rho \in [0,1)$. *Suppose* $\Omega = \mathrm{Fix}(U_n) \bigcap \mathrm{Fix}(V_n) \neq \emptyset$. *Take an initial guess* $x_1 \in H_1$ *and define a sequence* $\{x_n\}$ *by the following algorithm:*

$$\begin{aligned} y_n &= \lambda_n x_n + (1-\lambda_n) V_n x_n, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n, \end{aligned} \tag{3.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *are sequences in* $[0,1]$. *If the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$;

(v) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

then $\{x_n\}$ *converges strongly to* $w \in \Omega$ *where* $w = P_\Omega f(w)$.

Proof. We proceed with the following steps.

Step 1. First show that there exists $w \in \Omega$ such that $w = P_\Omega f(w)$.

In fact, since f is a contraction with coefficient ρ, we have

$$\left\| P_\Omega f(x) - P_\Omega f(y) \right\| \leq \left\| f(x) - f(y) \right\| \leq \rho \left\| x - y \right\| \tag{3.7}$$

for every x, y. Hence $P_\Omega f$ is also a contraction. Therefore, there exists a unique $w \in \Omega$ such that $w = P_\Omega f(w)$.

Step 2. Now we show that $\{x_n\}$ is bounded.

Let $p \in \Omega$, then $p \in \mathrm{Fix}(\{U_n\})$ and $p \in \mathrm{Fix}(\{V_n\})$. Hence

$$\|U_n y_n - p\| \leq \|y_n - p\| \leq \lambda_n \|x_n - p\| + (1-\lambda_n) \|V_n x_n - p\| \leq \|x_n - p\|. \tag{3.8}$$

Then

$$\begin{aligned} \left\| x_{n+1} - p \right\| &\leq \alpha_n \left\| f(x_n) - p \right\| + \beta_n \left\| x_n - p \right\| + \gamma_n \left\| U_n y_n - p \right\| \\ &\leq \alpha_n \rho \left\| x_n - p \right\| + \alpha_n \left\| f(p) - p \right\| + \beta_n \left\| x_n - p \right\| + \gamma_n \left\| x_n - p \right\| \\ &\leq \left(1 - \alpha_n (1-\rho)\right) \left\| x_n - p \right\| + \alpha_n (1-\rho) \frac{1}{1-\rho} \left\| f(p) - p \right\| \\ &\leq \max\left\{ \left\| x_n - p \right\|, \frac{1}{1-\rho} \left\| f(p) - p \right\| \right\}. \end{aligned} \tag{3.9}$$

By induction on n,

$$\left\| V_n x_n - p \right\| \leq \left\| x_n - p \right\| \leq \max\left\{ \left\| x_1 - p \right\|, \frac{1}{1-\rho} \left\| f(p) - p \right\| \right\} \tag{3.10}$$

for every $n \in \mathbb{N}$. This shows that $\{x_n\}$ and $\{V_n x_n\}$ are bounded, and hence, $\{U_n y_n\}$, $\{y_n\}$, and $\{f(x_n)\}$ are also bounded.

Step 3. We claim that $\widetilde{F}(\{A_n\}) = \widetilde{F}(\{V_n\})$ and $\widetilde{F}(\{U_n A_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\})$, where $A_n = \lambda_n I + (1-\lambda_n)V_n$.

We first show the former equality. Let $\{z_n\}$ be a bounded sequence in H_1. If $\{z_n\} \in \widetilde{F}(\{V_n\})$, then

$$\|A_n z_n - z_n\| = \|\lambda_n z_n + (1-\lambda_n)V_n z_n - z_n\| = (1-\lambda_n)\|V_n z_n - z_n\| \longrightarrow 0. \tag{3.11}$$

Hence $\{z_n\} \in \widetilde{F}(\{A_n\})$. On the other hand, if $\{z_n\} \in \widetilde{F}(\{A_n\})$, combining (3.11) and $\limsup_{n\to\infty}\lambda_n < 1$, we obtain that $\|V_n z_n - z_n\| \to 0$. Hence $\{z_n\} \in \widetilde{F}(\{V_n\})$. Therefore, $\widetilde{F}(\{A_n\}) = \widetilde{F}(\{V_n\})$.

Next, we show the latter equality. Using Lemma 2.1, we know that $\{A_n\}$ is a strongly nonexpansive sequence. Thus, since $\widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{A_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\}) \neq \emptyset$, from Lemma 2.2 we have

$$\widetilde{F}(\{U_n A_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{A_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\}). \tag{3.12}$$

Step 4. $\{S_n\}$ satisfies the condition (R), where $S_n = U_n A_n$.

Let D be a nonempty bounded subset of H_1. From the definition of $\{A_n\}$, we have, for all $y \in D$,

$$\begin{aligned}
\left\|A_{n+1}y - A_n y\right\| &= \left\|\lambda_{n+1}y + (1-\lambda_{n+1})V_{n+1}y - \lambda_n y - (1-\lambda_n)V_n y\right\| \\
&\le |\lambda_{n+1}-\lambda_n|\left\|y\right\| + \left\|V_{n+1}y - V_n y\right\| + \left\|\lambda_{n+1}V_{n+1}y - \lambda_n V_n y\right\| \\
&\le |\lambda_{n+1}-\lambda_n|\left\|y\right\| + \left\|V_{n+1}y - V_n y\right\| + \left\|\lambda_{n+1}V_{n+1}y - \lambda_n V_{n+1} y\right\| \\
&\quad + \left\|\lambda_n V_{n+1}y - \lambda_n V_n y\right\| \\
&= |\lambda_{n+1}-\lambda_n|\left\|y\right\| + \left\|V_{n+1}y - V_n y\right\| + |\lambda_{n+1}-\lambda_n|\left\|V_{n+1}y\right\| \\
&\quad + \lambda_n\left\|V_{n+1}y - V_n y\right\| \\
&= |\lambda_{n+1}-\lambda_n|\left(\left\|y\right\| + \left\|V_{n+1}y\right\|\right) + (1+\lambda_n)\left\|V_{n+1}y - V_n y\right\|.
\end{aligned} \tag{3.13}$$

It follows that

$$\sup_{y\in D}\left\|A_{n+1}y - A_n y\right\| \le |\lambda_{n+1}-\lambda_n|\sup_{y\in D}\left(\left\|y\right\| + \left\|V_{n+1}y\right\|\right) + (1+\lambda_n)\sup_{y\in D}\left\|V_{n+1}y - V_n y\right\|. \tag{3.14}$$

Since $\{V_n\}$ satisfies the condition (R) and $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n| = 0$, we have

$$\lim_{n\to\infty}\sup_{y\in D}\left\|A_{n+1}y - A_n y\right\| = 0, \tag{3.15}$$

that is, $\{A_n\}$ satisfies the condition (R). Since $\{A_n y : n \in \mathbb{N},\ \ y \in D\}$ is bounded for any bounded subset D of H_1, by using Lemma 2.3, we have that $\{V_n A_n\}$ satisfies the condition (R), that is, $\{S_n\}$ satisfies the condition (R).

Step 5. We show $\|x_{n+1} - x_n\| \to 0$.

We can write (3.6) as $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$ where $z_n = (\alpha_n f(x_n) + \gamma_n S_n x_n)/1-\beta_n$. It follows that

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S_{n+1} x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(f(x_{n+1}) - f(x_n) \right) + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) f(x_n) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (S_{n+1} x_{n+1} - S_n x_n) + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) S_n x_n. \end{aligned} \tag{3.16}$$

From Step 2, we may assume that $\{x_n\} \subset D'$, where D' is a bounded set of H_1. Then from (3.16), we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| &\le \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \left(\|f(x_n)\| + \|S_n x_n\| \right) + \frac{\alpha_{n+1}}{1-\beta_{n+1}} \rho \|x_{n+1} - x_n\| \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|S_{n+1} x_{n+1} - S_n x_{n+1}\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|S_n x_{n+1} - S_n x_n\| \\ &\le \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \left(\|f(x_n)\| + \|S_n x_n\| \right) + \left[1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}} (1-\rho) \right] \|x_{n+1} - x_n\| \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \sup_{y \in D'} \|S_{n+1} y - S_n y\|. \end{aligned} \tag{3.17}$$

It follows that

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \left(\|f(x_n)\| + \|S_n x_n\| \right) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \sup_{y \in D'} \|S_{n+1} y - S_n y\| - \frac{\alpha_{n+1}}{1-\beta_{n+1}} (1-\rho) \|x_{n+1} - x_n\|. \end{aligned} \tag{3.18}$$

Since $\{S_n\}$ satisfies the condition (R), combining $\alpha_n \to 0$ as $n \to \infty$, we have

$$\limsup_{n \to \infty} \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) \le 0. \tag{3.19}$$

Hence by Lemma 2.4, we get $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1-\beta_n) \|z_n - x_n\| = 0. \tag{3.20}$$

Step 6. We claim that $\{x_n\} \in \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\})$.

From (3.6), we have

$$\begin{aligned}\|S_n x_n - x_n\| &\leq \|S_n x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &= \left\|S_n x_n - \alpha_n f(x_n) - \beta_n x_n - \gamma_n S_n x_n\right\| + \|x_{n+1} - x_n\| \\ &\leq \alpha_n \left\|S_n x_n - f(x_n)\right\| + \beta_n \|S_n x_n - x_n\| + \|x_{n+1} - x_n\|,\end{aligned} \tag{3.21}$$

and hence

$$(1 - \beta_n)\|S_n x_n - x_n\| \leq \alpha_n \left\|S_n x_n - f(x_n)\right\| + \|x_{n+1} - x_n\|. \tag{3.22}$$

Since $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ and $\limsup_{n\to\infty}\beta_n < 1$, we derive

$$\|S_n x_n - x_n\| \longrightarrow 0. \tag{3.23}$$

Thus (3.23) and Steps 2 and 3 imply that

$$\{x_n\} \in \widetilde{F}(\{S_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\}). \tag{3.24}$$

Step 7. Show $\limsup_{n\to\infty}\langle f(w) - w, x_n - w\rangle \leq 0$, where $w = P_\Omega f(w)$.

Since $\{x_n\}$ is bounded, there exist a point $v \in H_1$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle f(w) - w, x_n - w\right\rangle = \lim_{i\to\infty}\left\langle f(w) - w, x_{n_i} - w\right\rangle \tag{3.25}$$

and $x_{n_i} \rightharpoonup v$. Since $\{U_n\}$ and $\{V_n\}$ satisfy the condition (Z), from Step 6, we have $v \in F(\{U_n\}) \cap F(\{V_n\})$. Using (2.1), we get

$$\begin{aligned}\limsup_{n\to\infty}\left\langle f(w) - w, x_n - w\right\rangle &= \lim_{i\to\infty}\left\langle f(w) - w, x_{n_i} - w\right\rangle \\ &= \left\langle f(w) - w, v - w\right\rangle \leq 0.\end{aligned} \tag{3.26}$$

Step 8. Show $x_n \to w = P_\Omega f(w)$.

Since $w \in \Omega$, using (3.8), we have

$$\begin{aligned}
\|x_{n+1} - w\|^2 &= \langle \alpha_n (f(x_n) - w) + \beta_n (x_n - w) + \gamma_n (U_n y_n - w), x_{n+1} - w \rangle \\
&\le \alpha_n \langle f(x_n) - f(w), x_{n+1} - w \rangle + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\
&\quad + \beta_n \|x_n - w\| \cdot \|x_{n+1} - w\| + \gamma_n \|y_n - w\| \cdot \|x_{n+1} - w\| \\
&\le \frac{1}{2} \alpha_n \rho \left(\|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\
&\quad + \frac{1}{2} \beta_n \left(\|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) + \frac{1}{2} \gamma_n \left(\|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) \\
&\le \frac{1}{2} [1 - \alpha_n (1 - \rho)] \|x_n - w\|^2 + \frac{1}{2} \|x_{n+1} - w\|^2 + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle,
\end{aligned} \tag{3.27}$$

which implies that

$$\|x_{n+1} - w\|^2 \le [1 - \alpha_n (1 - \rho)] \|x_n - w\|^2 + 2\alpha_n (1 - \rho) \frac{1}{1 - \rho} \langle f(w) - w, x_{n+1} - w \rangle, \tag{3.28}$$

for every $n \in \mathbb{N}$. Consequently, according to Step 7, $\rho \in [0, 1)$, and Lemma 2.5, we deduce that $\{x_n\}$ converges strongly to $w = P_\Omega(w)$. This completes the proof. □

Combining Lemma 3.1 and Theorem 3.2, we can obtain the following strong convergence theorem for solving the GSCFPP (1.7).

Theorem 3.3. *Let $\{U_n\}$ and $\{T_n\}$ be sequences of nonexpansive operators on Hilbert space H_1 and H_2, respectively. Both $\{U_n\}$ and $\{T_n\}$ satisfy the conditions (R) and (Z). Let $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in [0, 1)$. Suppose that the solution set Ω of GSCFPP (1.7) is nonempty. Take an initial guess $x_1 \in H_1$ and define a sequence $\{x_n\}$ by the following algorithm:*

$$\begin{aligned}
y_n &= x_n - \gamma (1 - \lambda_n) A^* (I - T_n) A x_n, \\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n,
\end{aligned} \tag{3.29}$$

where $\gamma \in (0, 1/\|A\|^2)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $[0, 1]$. If the following conditions are satisfied:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \ge 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iv) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1$;

(v) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

then $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_\Omega f(w)$.

Proof. Set $V_n = I - \gamma A^*(I - T_n)A$. By Lemma 3.1, V_n is a nonexpansive operator for every $n \in \mathbb{N}$. We can rewrite (3.29) as

$$\begin{gathered} y_n = \lambda_n x_n + (1 - \lambda_n) V_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n. \end{gathered} \tag{3.30}$$

We only need to prove that $\{V_n\}$ satisfies the conditions (R) and (Z). Assume that D is a nonempty bounded subset of H_1. For every $y \in D$, we have

$$\begin{aligned} \left\| (I - \gamma A^*(I - T_{n+1})A)y - (I - \gamma A^*(I - T_n)A)y \right\| &\le \gamma \left\| A^*(I - T_{n+1})Ay - A^*(I - T_n)Ay \right\| \\ &\le \gamma \|A\| \left\| T_{n+1}(Ay) - T_n(Ay) \right\|. \end{aligned} \tag{3.31}$$

Since $\{T_n\}$ satisfies the condition (R), and $D' = \{Ay : y \in D\}$ is bounded, it follows from (3.31) that

$$\begin{aligned} \sup_{y \in D} \left\| (I - \gamma A^*(I - T_{n+1})A)y - (I - \gamma A^*(I - T_n)A)y \right\| &\le \gamma \|A\| \sup_{y \in D} \left\| T_{n+1}(Ay) - T_n(Ay) \right\| \\ &= \gamma \|A\| \sup_{z \in D'} \|T_{n+1}z - T_n z\| \longrightarrow 0. \end{aligned} \tag{3.32}$$

Therefore, $\{V_n\}$ satisfies the condition (R).

Assume that $x_n \rightharpoonup z$ and $x_n - V_n x_n \to 0$; we next show that $V_n z = z$. By using $x_n - V_n x_n \to 0$, we have $A^*(I - T_n)Ax_n \to 0$. Since $A^{-1}(\mathrm{Fix}(\{T_n\})) \neq \emptyset$, we choose an arbitrary point $p \in A^{-1}(\mathrm{Fix}(\{T_n\}))$; then for every $n \in \mathbb{N}$,

$$\begin{aligned} \|Ax_n - T_n Ax_n\|^2 &= \langle Ax_n - T_n Ax_n, Ax_n - Ap + Ap - T_n Ax_n \rangle \\ &= \langle A^*(I - T_n)Ax_n, x_n - p \rangle + \langle Ax_n - T_n Ax_n, Ap - T_n Ax_n \rangle \\ &= \langle A^*(I - T_n)Ax_n, x_n - p \rangle - \frac{1}{2}\|Ax_n - Ap\|^2 + \frac{1}{2}\|Ax_n - T_n Ax_n\|^2 \\ &\quad + \frac{1}{2}\|Ap - T_n Ax_n\|^2 \\ &\le \langle A^*(I - T_n)Ax_n, x_n - p \rangle + \frac{1}{2}\|Ax_n - T_n Ax_n\|^2. \end{aligned} \tag{3.33}$$

Hence

$$\frac{1}{2}\|Ax_n - T_n Ax_n\|^2 \le \langle A^*(I - T_n)Ax_n, x_n - p \rangle \longrightarrow 0. \tag{3.34}$$

Then we get $Ax_n \in \widetilde{F}(\{T_n\})$. Since $\{T_n\}$ satisfies the condition (Z) and $Ax_n \rightharpoonup Az$, we have $Az \in F(\{T_n\})$. From Lemma 3.1, we have $z \in \mathrm{Fix}(\{V_n\})$. □

Let $T : H \to H$ be a nonexpansive mapping with a fixed point, and define $T_n = T$ for all $n \in \mathbb{N}$. Then $\{T_n\}$ satisfies the conditions (R) and (Z). Thus, one obtains the algorithm for solving the two-set SCFPP (1.4).

Corollary 3.4. *Let U and T be nonexpansive operators on Hilbert space H_1 and H_2, respectively. Let $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in [0,1)$. Suppose that the solution set Ω of SCFPP* (1.4) *is nonempty. Take an initial guess $x_1 \in H_1$ and define a sequence $\{x_n\}$ by the following algorithm in* (3.29)*, where $\gamma \in (0, 1/\|A\|^2)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $[0,1]$. If the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 1$;

(v) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$.

Then $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_\Omega f(w)$.

Remark 3.5. By adding more operators to the families $\{U_n\}$ and $\{T_n\}$ by setting $U_i = I$ for $i \geq p+1$ and $T_j = I$ for $j \geq r+1$, the SCFPP (1.3) can be viewed as a special case of the GSCFPP (1.7).

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