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Research Article

On the Convergence of Iterative Processes for Generalized Strongly Asymptotically ϕ-Pseudocontractive Mappings in Banach Spaces

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We prove the equivalence and the strong convergence of iterative processes involving generalized strongly asymptotically ϕ-pseudocontractive mappings in uniformly smooth Banach spaces.

1. Introduction

Throughout this paper, we assume that X is a uniformly convex Banach space and X^* is the dual space of X. Let J denote the normalized duality mapping form X into 2^{X^*} given by $J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$ for all $x \in X$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if X is uniformly smooth, then J is single valued and is norm to norm uniformly continuous on any bounded subset of X. In the sequel, we will denote the single valued duality mapping by j.

In 1967, Browder [1] and Kato [2], independently, introduced accretive operators (see, for details, Chidume [3]). Their interest is connected with the existence of results in the theory of nonlinear equations of evolution in Banach spaces.

In 1972, Goebel and Kirk [4] introduced the class of *asymptotically nonexpansive mappings* as follows.

Definition 1.1. Let K be a subset of a Banach space X. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if for each $x, y \in K$

$$\left\|T^n x - T^n y\right\| \leq k_n \left\|x - y\right\|, \tag{1.1}$$

where $\{k_n\}_n \subset [1, \infty)$ is a sequence of real numbers converging to 1.

This class is more general than the class of nonexpansive mappings as the following example clearly shows.

Example 1.2 (see [4]). If B is the unit ball of l^2 and $T : B \to B$ is defined as

$$T(x_1, x_2, \ldots) = \left(0, x_1^2, a_2 x_2, a_3 x_3, \ldots\right), \tag{1.2}$$

where $\{a_i\}_{i\in\mathbb{N}} \subset (0,1)$ is such that $\prod_{i=2}^{\infty} a_i = 1/2$, it satisfies.

$$\|Tx - Ty\| \leq 2\|x - y\|, \qquad \|T^n x - T^n y\| \leq 2\prod_{j=2}^{n} a_i \|x - y\|. \tag{1.3}$$

In 1974, Deimling [5], studying the zeros of accretive operators, introduced the class of *φ-strongly accretive operators*.

Definition 1.3. An operator A defined on a subset K of a Banach space X is said, φ-strongly accretive if

$$\langle Ax - Ay, j(x - y)\rangle \geq \varphi(\|x - y\|)\|x - y\|, \tag{1.4}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $\varphi(0) = 0$.

Note that in the special case in which $\varphi(t) = kt$, $k \in (0,1)$, we obtain a strongly accretive operator.

Osilike [6], among the others, proved that $Ax = x - (x/(x+1))$ in $\mathbb{R}^+$ is φ-strongly accretive where $\varphi(t) = (t^2/(1+t))$ but not strongly accretive.

Since an operator A is a strongly accretive operator if and only if $(I - A)$ is a strongly pseudocontractive mapping (i.e., $\langle (I-A)x - (I-A)y,\ j(x-y)\rangle \leq k\|x-y\|^2$, $k < 1$), taking in to account Definition 1.3, it is natural to study the class of φ-pseudocontractive mappings, that is, the maps such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \varphi(\|x - y\|)\|x - y\|, \tag{1.5}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $\varphi(0) = 0$. Of course, the set of fixed points for these mappings contains, at most, only one point.

Recently, has been also studied the following class of maps.

Definition 1.4. A mapping T is a generalized ϕ-strongly pseudocontractive mapping if

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \phi(\|x - y\|), \tag{1.6}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $\phi(0) = 0$.

Choosing $\phi(t) = \varphi(t)t$, we obtain Definition 1.3. In [7], Xiang remarked that it is a open problem if every generalized ϕ-strongly pseudocontractive mapping is φ-pseudocontractive mapping. In the same paper, Xiang obtained a fixed-point theorem for continuous and generalized ϕ-strongly pseudocontractive mappings in the setting of the Banach spaces.

In 1991, Schu [8] introduced the class of *asymptotically pseudocontractive mappings*.

Definition 1.5 (see [8]). Let X be a normed space, $K \subset X$ and $\{k_n\}_n \subset [1, \infty)$. A mapping $T : K \to K$ is said to be asymptotically pseudocontractive with the sequence $\{k_n\}_n$ if and only if $\lim_{n\to\infty} k_n = 1$, and for all $n \in \mathbb{N}$ and all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x-y) \rangle \le k_n \|x - y\|^2, \tag{1.7}$$

where J is the normalized duality mapping.

Obviously every asymptotically nonexpansive mapping is asymptotically pseudocontractive, but the converse is not valid; it is well known that $T : [0,1] \to [0,1]$ defined by $Tx = (1 - x^{2/3})^{3/2}$ is not Lipschitz but asymptotically pseudocontractive [9].

In [8], Schu proved the following.

Theorem 1.6 (see [8]). *Let H be a Hilbert space and $A \subset H$ closed and convex; $L > 0$; $T : A \to A$ completely continuous, uniformly L-Lipschitzian, and asymptotically pseudocontractive with sequence $\{k_n\}_n \in [1, \infty)$; $q_n := 2k_n - 1$ for all $n \in \mathbb{N}$; $\sum_n (q_n^2 - 1) < \infty$; $\{\alpha_n\}_n, \{\beta_n\}_n \in [0,1]$; $\epsilon \le \alpha_n \le \beta_n \le b$ for all $n \in \mathbb{N}$, some $\epsilon > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$; $x_1 \in A$; for all $n \in \mathbb{N}$, define*

$$\begin{aligned} z_n &:= \beta_n T^n(x_n) + (1 - \beta_n) x_n, \\ x_{n+1} &:= \alpha_n T^n(z_n) + (1 - \alpha_n) x_n, \end{aligned} \tag{1.8}$$

then $\{x_n\}_n$ converges strongly to some fixed point of T.

Until 2009, no results on fixed-point theorems for asymptotically pseudocontractive mappings have been proved. First, Zhou in [10] completed this lack in the setting of Hilbert spaces proving a fixed-point theorem for an asymptotically pseudocontractive mapping that is also uniformly L-Lipschitzian and uniformly asymptotically regular and that the set of fixed points of T is closed and convex. Moreover, Zhou proved the strong convergence of a CQ-iterative method involving this kind of mappings.

In this paper, our attention is on the class of the *generalized strongly asymptotically ϕ-pseudocontraction* defined as follows.

Definition 1.7. If X is a Banach space and K is a subset of X, a mapping $T : K \to K$ is said to be a generalized asymptotically ϕ-strongly pseudocontraction if

$$\langle T^n x - T^n y, j(x-y) \rangle \le k_n \|x - y\|^2 - \phi(\|x - y\|), \tag{1.9}$$

where $\{k_n\}_n \subset [1, \infty)$ is converging to one and $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing and such that $\phi(0) = 0$.

One can note that

(i) if T has fixed points, then it is unique. In fact, if x, z are fixed points for T, then for every $n \in \mathbb{N}$,

$$\|x - z\|^2 = \langle T^n x - T^n z, j(x - z) \rangle \le k_n \|x - z\|^2 - \phi(\|x - z\|), \tag{1.10}$$

so passing n to $+\infty$, it results that

$$\|x - z\|^2 \le \|x - z\|^2 - \phi(\|x - z\|) \implies -\phi(\|x - z\|) \ge 0. \tag{1.11}$$

Since $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing and $\phi(0) = 0$, then $x = z$.

(ii) the mapping $Tx = x/(x+1)$, where $x \in [0, 1]$, is generalized asymptotically strongly ϕ-pseudocontraction with $k_n = 1$, for all $n \in \mathbb{N}$ and $\phi(t) = s^3/(1 + s)$. However, T is not strongly pseudocontractive; see [6].

We study the equivalence between three kinds of iterative methods involving the generalized asymptotically strongly ϕ-pseudocontractions.

Moreover, we prove that these methods are equivalent and strongly convergent to the unique fixed point of the generalized strongly asymptotically ϕ-pseudocontraction T, under suitable hypotheses.

We will briefly introduce some of the results in the same line of ours. In 2001, [11] Chidume and Osilike proved the strong convergence of the iterative method

$$\begin{aligned} y_n &= a_n x_n + b_n S x_n + c_n u_n, \\ x_{n+1} &= a'_n x_n + b'_n S y_n c'_n v_n, \end{aligned} \tag{1.12}$$

where $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, $Sx = x - Tx + f$ (T a ϕ-strongly accretive operator), and $f \in X$, to a solution of the equation $Tx = f$.

In 2003, Chidume and Zegeye [12] studied the following iterative method:

$$x_{n+1} = (1 - \lambda_n) x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \tag{1.13}$$

where T is a Lipschitzian pseudocontractive map with fixed points. The authors proved the strong convergence of the method to a fixed point of T under suitable hypotheses on the control sequences $(\theta_n)_n, (\lambda_n)_n$.

Taking in to account Chidume and Zegeye [12] and Chang [13], we introduce the modified Mann and Ishikawa iterative processes as follows: for any given $x_0 \in X$, the sequence $\{x_n\}_n$ is defined by

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n T^n x_n - \delta_n (x_n - v_n), \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T^n y_n - \gamma_n (x_n - u_n), \quad n \ge 0, \end{aligned} \tag{1.14}$$

where $\{\alpha_n\}_n$, $\{\gamma_n\}_n$, $\{\beta_n\}_n$, and $\{\delta_n\}_n$ are four sequences in $(0, 1)$ satisfying the conditions $\alpha_n + \gamma_n \le 1$ and $\beta_n + \delta_n \le 1$ for all $n \ge 0$.

In particular, if $\beta_n = \delta_n = 0$ for all $n \geq 0$, we can define a sequence $\{z_n\}_n$ by

$$\begin{aligned} &z_0 \in X, \\ &z_{n+1} = (1-\alpha_n)z_n + \alpha_n T^n z_n - \gamma_n(z_n - w_n), \quad n \geq 0, \end{aligned} \tag{1.15}$$

which is called *the modified Mann iteration sequence*.

We also introduce an implicit iterative process as follows:

$$z'_n = (1-\alpha_n)z'_{n-1} + \alpha_n T^n z'_n - \gamma_n\left(z'_{n-1} - w'_n\right), \quad n \geq 1, \tag{1.16}$$

where $\{\alpha_n\}_n, \{\gamma_n\}_n$ are two real sequences in $[0,1]$ satisfying $\alpha_n + \gamma_n \leq 1$ and $\alpha_n k_n < 1$ for all $n \geq 1$, $\{w'_n\}_n$ is a sequence in X, and z'_0 is an initial point.

The algorithm is well defined. Indeed, if T is a asymptotically strongly ϕ-pseudocontraction, one can observe that, for every fixed n, the mapping S_n defined by $S_n x := (1-\alpha_n-\gamma_n)z_{n-1} + \alpha_n T^n x + \gamma w_n$ is such that

$$\langle S_n x - S_n y, j(x-y)\rangle = \langle T^n x - T^n y, j(x-y)\rangle \leq \alpha_n k_n \|x-y\|^2, \tag{1.17}$$

that is, S_n is a strongly pseudocontraction, for every fixed n, then (see Theorem 13.1 in [14]) there exists a unique fixed point of S_n for each n.

These kind of iterative processes (also called by Chang *iterative processes with errors*) have been developed in [15–18], while equivalence theorem for Mann and Ishikawa methods has been studied, in [19, 20], among the others.

In [21], Huang established equivalences between convergence of the modified Mann iteration process with errors (1.15) and convergence of modified Ishikawa iteration process with errors (1.14) for strongly successively ϕ-pseudocontractive mappings in uniformly smooth Banach space.

In the next section, we prove that, in the setting of the uniformly smooth Banach space, if T is an asymptotically strongly ϕ-pseudocontraction, not only (1.14) and (1.15) are equivalent but also (1.16) is equivalent to the others. Moreover, we prove also that (1.14), (1.15), and (1.16) strongly converge to the unique fixed point of T, if it exists.

2. Preliminaries

We recall some definitions and conclusions.

Definition 2.1. X is said to be a uniformly smooth Banach space if the smooth module of X

$$\rho_X(t) = \sup\left\{\frac{1}{2}\left(\|x-y\| + \|x+y\|\right) - 1 : \|x\| \leq 1, \|y\| \leq t\right\} \tag{2.1}$$

satisfies $\lim_{t\to 0}\rho_X(t)/t = 0$.

Lemma 2.2 (see [22]). *Let X be a Banach space, and let $j : X \to 2^{X^*}$ be the normalized duality mapping, then for any $x, y \in X$, one has*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y). \tag{2.2}$$

The next lemma is one of the main tools for our proofs.

Lemma 2.3 (see [21]). *Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\phi(0) = 0$, and let $\{a_n\}_n, \{b_n\}_n, \{c_n\}_n$, and $\{e_n\}_n$ be nonnegative real sequences such that*

$$\lim_{n\to\infty} b_n = 0, \qquad c_n = o(b_n), \qquad \sum_{n=1}^{\infty} b_n = \infty, \qquad \lim_{n\to\infty} e_n = 0. \tag{2.3}$$

Suppose that there exists an integer $N_1 > 0$ such that

$$a_{n+1}^2 \leq a_n^2 - 2b_n\phi(|a_{n+1} - e_n|) + c_n, \quad \forall n \geq N_1, \tag{2.4}$$

then $\lim_{n\to\infty} a_n = 0$.

Proof. The proof is the same as in [21], but we substitute $(a_{n+1} - e_n)$ with $|a_{n+1} - e_n|$, in (2.4). □

Lemma 2.4 (see [23]). *Let $\{s_n\}_n$, $\{c_n\}_n \subset \mathbb{R}_+$, $\{a_n\}_n \subset (0,1)$, and $\{b_n\}_n \subset \mathbb{R}$ be sequences such that*

$$s_{n+1} \leq (1 - a_n)s_n + b_n + c_n, \tag{2.5}$$

for all $n \geq 0$. Assume that $\sum_n |c_n| < \infty$, then the following results hold:

(1) *if $b_n \leq \beta a_n$ (where $\beta \geq 0$), then $\{s_n\}_n$ is a bounded sequence;*

(2) *if one has $\sum_n a_n = \infty$ and $\limsup_n b_n/a_n \leq 0$, then $s_n \to 0$ as $n \to \infty$.*

Remark 2.5. If in Lemma 2.3 choosing $e_n = 0$, for all n, $\phi(t) = kt^2$ $(k < 1)$, then the inequality (2.4) becomes

$$\begin{aligned} &a_{n+1}^2 \leq a_n^2 - 2b_nka_{n+1}^2 + c_n \Longrightarrow \\ &a_{n+1}^2 \leq \frac{1}{1+2b_nk}a_n^2 + \frac{c_n}{1+2b_nk} \\ &\qquad = \left(1 - \frac{2b_nk}{1+2b_nk}\right)a_n^2 + \frac{c_n}{1+2b_nk}. \end{aligned} \tag{2.6}$$

Setting $\alpha_n := 2b_nk/(1+2b_nk)$ and $\beta_n := c_n/(1+2b_nk)$ and by the hypotheses of Lemma 2.3, we get $\alpha_n \to 0$ as $n \to \infty$, $\sum_n \alpha_n = \infty$, and $\limsup_n \beta_n/\alpha_n = 0$. That is, we reobtain Lemma 2.4 in the case of $c_n = 0$.

3. Main Results

The ideas of the proofs of our main Theorems take in to account the papers of Chang and Chidume et al. [11, 13, 24].

Theorem 3.1. *Let X be a uniformly smooth Banach space, and let $T : X \to X$ be generalized strongly asymptotically ϕ-pseudocontractive mapping with fixed point x^* and bounded range.*

Let $\{x_n\}$ and $\{z_n\}$ be the sequences defined by (1.14) *and* (1.15)*, respectively, where $\{\alpha_n\},\{\gamma_n\},\{\beta_n\},\{\delta_n\} \subset [0,1]$ satisfy*

(H1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = 0$ *and* $\gamma_n = o(\alpha_n)$,

(H2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

and the sequences $\{u_n\},\{v_n\},\{w_n\}$ are bounded in X, then for any initial point $z_0, x_0 \in X$, the following two assertions are equivalent:

(i) *the modified Ishikawa iteration sequence with errors* (1.14) *converges to x^*;*

(ii) *the modified Mann iteration sequence with errors* (1.15) *converges to x^*.*

Proof. First of all, we note that by boundedness of the range of T, of the sequences $\{w_n\}, \{u_n\}$ and by Lemma 2.4, it results that $\{z_n\}$ and $\{x_n\}$ are bounded sequences. So, we can set

$$M = \sup_n \left\{ \begin{array}{c} \left\|T^n z_n - T^n y_n\right\|,\ \|T^n x_n - x_n\|, \\ \left\|T^n y_n - x_n\right\|,\ \|T^n z_n - z_n\|, \\ \|z_n - x_n\|,\ \|u_n - x_n\|,\ \|v_n - x_n\|, \\ \|w_n - z_n\|,\ \|w_n - u_n\| \end{array} \right\}. \tag{3.1}$$

By Lemma 2.2, we have

$$\begin{aligned} \|z_{n+1} - x_{n+1}\|^2 &= \left\|(1-\alpha_n-\gamma_n)(z_n - x_n) + \alpha_n(T^n z_n - T^n y_n) + \gamma_n(w_n - u_n)\right\|^2 \\ &\le (1-\alpha_n-\gamma_n)^2\|z_n - x_n\|^2 + 2\langle \alpha_n(T^n z_n - T^n y_n) + \gamma_n(w_n - u_n), j(z_{n+1} - x_{n+1})\rangle \\ &\le (1-\alpha_n)^2\|z_n - x_n\|^2 + 2\alpha_n\langle T^n z_n - T^n y_n, j(z_n - y_n)\rangle \\ &\quad + 2\alpha_n\langle T^n z_n - T^n y_n, j(z_{n+1} - x_{n+1}) - j(z_n - y_n)\rangle + 2\gamma_n\langle w_n - u_n, j(z_{n+1} - x_{n+1})\rangle \\ &\le (1-\alpha_n)^2\|z_n - x_n\|^2 + 2\alpha_n k_n\left\|z_n - y_n\right\|^2 - 2\alpha_n\phi(\left\|z_n - y_n\right\|) \\ &\quad + 2\alpha_n\left\|T^n z_n - T^n y_n\right\|\left\|j(z_{n+1} - x_{n+1}) - j(z_n - y_n)\right\| + 2\gamma_n\|w_n - u_n\|\|z_{n+1} - x_{n+1}\| \\ &\le (1-\alpha_n)^2\|z_n - x_n\|^2 + 2\alpha_n k_n\left\|z_n - y_n\right\|^2 - 2\alpha_n\phi(\left\|z_n - y_n\right\|) \\ &\quad + 2\alpha_n\sigma_n M + 2\gamma_n M^2, \end{aligned} \tag{3.2}$$

where $\sigma_n = \|j(z_{n+1} - x_{n+1}) - j(z_n - y_n)\|$. Using (1.14) and (1.15), we have

$$\begin{aligned}
\left\|(z_{n+1} - x_{n+1}) - (z_n - y_n)\right\| &\le \left\|x_{n+1} - y_n\right\| + \|z_{n+1} - z_n\| \\
&= \left\|\alpha_n(T^n y_n - x_n) + \gamma_n(u_n - x_n) - \beta_n(T^n x_n - x_n) - \delta_n(v_n - x_n)\right\| \\
&\quad + \left\|\alpha_n(T^n z_n - z_n) + \gamma_n(w_n - z_n)\right\| \\
&\le 2M(\alpha_n + \gamma_n + \beta_n + \delta_n) \longrightarrow 0, \quad (n \to \infty).
\end{aligned} \tag{3.3}$$

In view of the uniformly continuity of j, we obtain that $\sigma_n \to 0$ as $n \to \infty$. Furthermore, it follows from the definition of $\{y_n\}$ that for all $n \ge 0$

$$\begin{aligned}
\left\|z_n - y_n\right\|^2 &= \left\|z_n - x_n + \beta_n(-T^n x_n + x_n) + \delta_n(-v_n + x_n)\right\|^2 \\
&\le \left[\|z_n - x_n\| + \beta_n\|T^n x_n - x_n\| + \delta_n\|v_n - x_n\|\right]^2 \\
&\le \left[\|z_n - x_n\| + (\beta_n + \delta_n)M\right]^2 \\
&= \|z_n - x_n\|^2 + (\beta_n + \delta_n)\left(2\|z_n - x_n\|M + (\beta_n + \delta_n)M^2\right) \\
&\le \|z_n - x_n\|^2 + 3(\beta_n + \delta_n)M^2,
\end{aligned} \tag{3.4}$$

$$\left\|y_n - x_n\right\| \le \beta_n\|T^n x_n - x_n\| + \delta_n\|v_n - x_n\| \le (\beta_n + \delta_n)M \longrightarrow 0, \quad (n \to \infty), \tag{3.5}$$

so

$$\begin{aligned}
\|z_{n+1} - x_{n+1}\| &= \left\|z_n - x_n - (\alpha_n + \gamma_n)(z_n - x_n) + \alpha_n(T^n z_n - T^n y_n) + \gamma_n(w_n - u_n)\right\| \\
&\le \|z_n - x_n\| + (\alpha_n + \gamma_n)\|z_n - x_n\| + \alpha_n\left\|T^n z_n - T^n y_n\right\| + \gamma_n\|w_n - u_n\| \\
&\le \left\|z_n - y_n\right\| + \left\|y_n - x_n\right\| + 2(\alpha_n + \gamma_n)M. \\
&\le \left\|z_n - y_n\right\| + (\beta_n + \delta_n)M + 2(\alpha_n + \gamma_n)M.
\end{aligned} \tag{3.6}$$

Therefore, we have

$$\left\|z_n - y_n\right\| \ge \|z_{n+1} - x_{n+1}\| - e_n, \tag{3.7}$$

where $e_n = (\beta_n + \delta_n)M + 2(\alpha_n + \gamma_n)M$. By (H1), we have that $e_n \to 0$ as $n \to \infty$. If $\|z_{n+1} - x_{n+1}\| - e_n \le 0$ for an infinite number of indices, we can extract a subsequence such that $\|z_{n_k} - x_{n_k}\| - e_{n_k - 1} \le 0$. For this subsequence, $\|z_{n_k} - x_{n_k}\| \to 0$, as $k \to \infty$.

In this case, we can prove that $\|z_n - x_n\| \to 0$, that is, the thesis.

Firstly, we note that substituting (3.4) into (3.2), we have

$$\begin{aligned}
\|z_{n+1}-x_{n+1}\|^2 &\le \|z_n-x_n\|^2+\alpha_n^2\|z_n-x_n\|^2+2\alpha_n(k_n-1)\|z_n-x_n\|^2 \\
&\quad + 6\alpha_n k_n(\beta_n+\delta_n)M^2-2\alpha_n\phi(\|z_n-y_n\|)+2\alpha_n\sigma_n M+2\gamma_n M^2 \\
&\le \|z_n-x_n\|^2-2\alpha_n\phi(\|z_n-y_n\|)+\alpha_n^2M^2+2\alpha_n(k_n-1)M^2 \\
&\quad + 6\alpha_n\overline{k}(\beta_n+\delta_n)M^2+2\alpha_n\sigma_n M+2\gamma_n M^2 \\
&= \|z_n-x_n\|^2-\alpha_n\phi(\|z_n-y_n\|)+2\gamma_n M^2 \\
&\quad - \alpha_n\Big[\phi(\|z_n-y_n\|)-\alpha_n M^2-2(k_n-1)M^2 \\
&\qquad -6\overline{k}(\beta_n+\delta_n)M^2-2\sigma_n M\Big],
\end{aligned} \tag{3.8}$$

where $\overline{k} := \sup_n(k_n)$.

Moreover, we observe that

$$\left\|z_{n_j}-y_{n_j}\right\| \le \left\|z_{n_j}-x_{n_j}\right\|+\left\|y_{n_j}-x_{n_j}\right\| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.9}$$

Thus, for every fixed $\epsilon > 0$, there exists j_1 such that for all $j > j_1$

$$\left\|z_{n_j}-y_{n_j}\right\| < 2\epsilon \qquad \left\|z_{n_j}-x_{n_j}\right\| < \epsilon. \tag{3.10}$$

Since $\{\alpha_n\}_n$, $\{(k_n-1)\}_n$, $\{(\beta_n+\delta_n)\}_n$, $\{\sigma_n\}_n$, and $\{\gamma_n\}_n$ are null sequences (and in particular $\gamma_n = o(\alpha_n)$), for the previous fixed $\epsilon > 0$, there exists an index N such that, for all $n \in N$,

$$\begin{gathered}
|\alpha_n| < \min\left\{\frac{\epsilon}{16M}, \frac{\phi(\epsilon/2)}{8M^2}\right\}, \\
|\gamma_n| < \frac{\epsilon}{16M}, \qquad \left|\frac{\gamma_n}{\alpha_n}\right| < \frac{\phi(\epsilon/2)}{4M^2}, \\
|k_n-1| < \frac{\phi(\epsilon/2)}{16M^2}, \\
|\beta_n+\delta_n| < \min\left\{\frac{\epsilon}{4M}, \frac{\phi(\epsilon/2)}{48\overline{k}M^2}\right\}, \\
|\sigma_n| < \frac{\phi(\epsilon/2)}{16M},
\end{gathered} \tag{3.11}$$

for all $n > N$.

Take $n^* > \max\{N, n_{j_1}\}$ such that $n^* = n_k$ for a certain k.

We prove, by induction, that $\|z_{n^*+i}-x_{n^*+i}\| < \epsilon$, for every $i \in \mathbb{N}$. Let $i = 1$. Suppose that $\|z_{n^*+1}-x_{n^*+1}\| \ge \epsilon$.

By (3.6), we have

$$\begin{aligned}\epsilon \le \|z_{n^*+1} - x_{n^*+1}\| &\le \|z_{n^*} - y_{n^*}\| + (\beta_{n^*} + \delta_{n^*})M + 2(\alpha_{n^*} + \gamma_{n^*})M \\ &\le \|z_{n^*} - y_{n^*}\| + M\frac{\epsilon}{4M} + 2M\left(\frac{\epsilon}{16M} + \frac{\epsilon}{16M}\right) \\ &= \|z_{n^*} - y_{n^*}\| + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \|z_{n^*} - y_{n^*}\| + \frac{\epsilon}{2}.\end{aligned} \tag{3.12}$$

Thus, $\|z_{n^*} - y_{n^*}\| \ge \epsilon/2$. Since ϕ is strictly increasing, $\phi(\|z_{n^*} - y_{n^*}\|) \ge \phi(\epsilon/2)$.

From (3.8), we obtain that

$$\begin{aligned}\|z_{n^*+1} - x_{n^*+1}\|^2 < \epsilon^2 &- \alpha_n^*\left(\phi\|z_{n^*} - y_{n^*}\|) - 2M^2\frac{\gamma_n^*}{\alpha_n^*}\right) \\ &- \alpha_n^*\left[\phi(\|z_{n^*} - y_{n^*}\|) - \alpha_n^* M^2 - 2(k_n^* - 1)M^2 - 6\overline{k}(\beta_n^* + \delta_n^*)M^2 - 2\sigma_{n^*}M\right].\end{aligned} \tag{3.13}$$

One can note that

$$\alpha_n^* M^2 + 2(k_n^* - 1)M^2 + 6\overline{k}(\beta_n^* + \delta_n^*)M^2 + 2\sigma_{n^*}M \le \frac{\phi(\epsilon/2)}{8} + \frac{\phi(\epsilon/2)}{8} + \frac{\phi(\epsilon/2)}{8} + \frac{\phi(\epsilon/2)}{8}. \tag{3.14}$$

hence

$$\phi(\|z_{n^*} - y_{n^*}\|) - \alpha_n^* M^2 - 2(k_{n^*} - 1)M^2 - 6\overline{k}(\beta_{n^*} + \delta_{n^*})M^2 - 2\sigma_{n^*}M \ge \phi\left(\frac{\epsilon}{2}\right) - \frac{\phi(\epsilon/2)}{2} > 0. \tag{3.15}$$

In the same manner,

$$\phi(\|z_{n^*} - y_{n^*}\|) - 2M^2\frac{\gamma_{n^*}}{\alpha_{n^*}} > \phi(\epsilon/2) - \frac{\phi(\epsilon/2)}{2} > 0. \tag{3.16}$$

Thus,

$$\|z_{n^*+1} - x_{n^*+1}\|^2 < \epsilon^2. \tag{3.17}$$

So we have $\|z_{n^*+1} - x_{n^*+1}\| < \epsilon$, which contradicts $\|z_{n^*+1} - x_{n^*+1}\| \ge \epsilon$. By the same idea, we can prove that $\|z_{n^*+2} - x_{n^*+2}\| < \epsilon$ and then, by inductive step, $\|z_{n^*+i} - x_{n^*+i}\| \le \epsilon$, for all i. This is enough to ensure that $\|z_n - x_n\| \to 0$.

If there are only finite indices for which $\|z_{n+1} - x_{n+1}\| - e_n \le 0$, then definitively $\|z_{n+1} - x_{n+1}\| - e_n \ge 0$. By the strict increasing function ϕ, we have definitively

$$\phi(\|z_n - y_n\|) \ge \phi(\|z_{n+1} - x_{n+1}\| - e_n). \tag{3.18}$$

Again substituting (3.4) and (3.18) into (3.2) and simplifying, we have

$$\begin{aligned}\|z_{n+1}-x_{n+1}\|^2 \le{}& \|z_n-x_n\|^2+\alpha_n^2\|z_n-x_n\|^2+2\alpha_n(k_n-1)\|z_n-x_n\|^2\\&+6\alpha_nk_n(\beta_n+\delta_n)M^2-2\alpha_n\phi(\|z_{n+1}-x_{n+1}\|-e_n)+2\alpha_n\sigma_nM+2\gamma_nM^2\\ \le{}& \|z_n-x_n\|^2-2\alpha_n\phi(\|z_{n+1}-x_{n+1}\|-e_n)+\alpha_n^2M^2+2\alpha_n(k_n-1)M^2\\&+6\alpha_nk_n(\beta_n+\delta_n)M^2+2\alpha_n\sigma_nM+2\gamma_nM^2.\end{aligned}\tag{3.19}$$

Suppose that $a_n=\|z_n-x_n\|, b_n=\alpha_n,$ and $c_n=\alpha_n^2M^2+2\alpha_n(k_n-1)M^2+6\alpha_nk_n(\beta_n+\delta_n)M^2+2\alpha_n\sigma_nM+2\gamma_nM^2$. It follows from $\lim_{n\to\infty}\sigma_n=0$, $\lim_{n\to\infty}e_n=0$, $\lim_{n\to\infty}k_n=1$, and the hypothesis that we have, $\sum_{n=1}^{\infty}b_n=\infty$ and $c_n=o(b_n)$, $e_n\to 0$ as $n\to\infty$. By virtue of Lemma 2.3, we obtain that $\lim_{n\to\infty}a_n=0$. Hence, $\lim_{n\to\infty}\|z_n-x_n\|=0$. □

Theorem 3.2. *Let X be a uniformly smooth Banach space, and let $T: X\to X$ be generalized strongly asymptotically ϕ-pseudocontractive mapping with fixed point x^* and bounded range.*

Let $\{z_n\}$ and $\{z'_n\}$ be the sequences defined by (1.15) *and* (1.16), *respectively, where $\{\alpha_n\},\{\gamma_n\}\subset[0,1]$ are null sequences satisfying*

(H1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\gamma_n=o(\alpha_n)$,

(H2) $\sum_{n=1}^{\infty}\alpha_n=\infty$,

of Theorem 3.1 and such that $\alpha_nk_n<1$, for every $n\in\mathbb{N}$.

Suppose moreover that the sequences $\{w_n\},\{w'_n\}$ are bounded in X, then for any initial point $z'_0, z_0\in X$, the following two assertions are equivalent:

(i) *the modified Mann iteration sequence with errors* (1.15) *converges to the fixed point x^*,*

(ii) *the implicit iteration sequence with errors* (1.16) *converges to the fixed point x^*.*

Proof. As in Theorem 3.1, by the boundedness of the range of T and by Lemma 2.4, one obtains that our schemes are bounded. We define

$$M=\sup_n\left\{\begin{matrix}\|T^nz_n-T^nz'_n\|, & \|T^nz_n-z_n\|, & \|T^nz'_n-z'_n\|, & \left\|T^nz'_n-z'_{n-1}\right\|\\ \|z_{n+1}-z'_n\|, & \|z_n-w_n\|, & \left\|w'_n-z'_{n-1}\right\|, & \|w_n-w'_n\|\end{matrix}\right\}.\tag{3.20}$$

By the iteration schemes (1.15) and (1.16), we have

$$\begin{aligned}\left\|z_{n+1}-z'_n\right\|^2 \le{}& \left\|(1-\alpha_n-\gamma_n)(z_n-z'_{n-1})+\alpha_n(T^nz_n-T^nz'_n)+\gamma_n(w_n-w'_n)\right\|^2\\ \le{}& (1-\alpha_n-\gamma_n)^2\left\|z_n-z'_{n-1}\right\|^2+2\langle\alpha_n(T^nz_n-T^nz'_n)+\gamma_n(w_n-w'_n),j(z_{n+1}-z'_n)\rangle\\ \le{}& (1-\alpha_n)^2\left\|z_n-z'_{n-1}\right\|^2+2\alpha_n\langle T^nz_n-T^nz'_n,j(z_n-z'_n)\rangle\\&+2\alpha_n\langle T^nz_n-T^nz'_n,j(z_{n+1}-z'_n)-j(z_n-z'_n)\rangle+2\gamma_n\left\|w_n-w'_n\right\|\left\|z_{n+1}-z'_n\right\|\\ \le{}& (1-\alpha_n)^2\left\|z_n-z'_{n-1}\right\|^2+2\alpha_nk_n\left\|z_n-z'_n\right\|^2\\&-2\alpha_n\phi(\left\|z_n-z'_n\right\|)+2\alpha_nM\sigma_n+2\gamma_nM^2,\end{aligned}\tag{3.21}$$

where $\sigma_n = \|j(z_{n+1} - z'_n) - j(z_n - z'_n)\|$. By (1.15), we get

$$\left\|(z_{n+1} - z'_n) - (z_n - z'_n)\right\| = \|z_{n+1} - z_n\| = \left\|\alpha_n(T^n z_n - z_n) + \gamma_n(w_n - z_n)\right\| \le (\alpha_n + \gamma_n)M. \tag{3.22}$$

It follows from (H1) that $\|(z_{n+1} - z'_n) - (z_n - z'_n)\| \to 0$ as $n \to \infty$, which implies that $\sigma_n \to 0$ as $n \to \infty$. Moreover, for all $n \ge 0$,

$$\begin{aligned}
\left\|z_n - z'_n\right\|^2 &\le \left(\left\|z_n - z'_{n-1}\right\| + \left\|z'_{n-1} - z'_n\right\|\right)^2 \\
&\le \left(\left\|z_n - z'_{n-1}\right\| + \alpha_n\left\|T^n z'_n - z'_{n-1}\right\| + \gamma_n\left\|w'_n - z'_{n-1}\right\|\right)^2 \\
&\le \left[\left\|z_n - z'_{n-1}\right\| + (\alpha_n + \gamma_n)M\right]^2 \\
&= \left\|z_n - z'_{n-1}\right\|^2 + (\alpha_n + \gamma_n)\left[2M\left\|z_n - z'_{n-1}\right\| + (\alpha_n + \gamma_n)M^2\right] \\
&\le \left\|z_n - z'_{n-1}\right\|^2 + 3(\alpha_n + \gamma_n)M^2.
\end{aligned} \tag{3.23}$$

Again by the boundedness of all components, we have that

$$\left\|z'_n - z'_{n-1}\right\| = \left\|\alpha_n(T^n z'_n - z'_{n-1}) + \gamma_n(w'_n - z'_{n-1})\right\| \le (\alpha_n + \gamma_n)M, \tag{3.24}$$

and so

$$\begin{aligned}
\left\|z_{n+1} - z'_n\right\| &= \left\|(z_n - z'_n) + (z'_n - z'_{n-1}) - (\alpha_n + \gamma_n)(z_n - z'_{n-1}) + \alpha_n(T^n z_n - T^n z'_n) + \gamma_n(w_n - w'_n)\right\| \\
&\le \left\|z_n - z'_n\right\| + \left\|z'_n - z'_{n-1}\right\| + (\alpha_n + \gamma_n)\left\|z_n - z'_{n-1}\right\| + \alpha_n\left\|T^n z_n - T^n z'_n\right\| + \gamma_n\left\|w_n - w'_n\right\| \\
&\le \left\|z_n - z'_n\right\| + 3(\alpha_n + \gamma_n)M.
\end{aligned} \tag{3.25}$$

Hence, we have that $\|z_n - z'_n\| \ge \|z_{n+1} - z'_n\| - e_n$, where $e_n = 3(\alpha_n + \gamma_n)M$. Note that $e_n \to 0$ as $n \to \infty$. As in proof of Theorem 3.1, we distinguish two cases:

(i) the set of indices for which $\|z_{n+1} - z'_n\| - e_n \le 0$ contains infinite terms;

(ii) the set of indices for which $\|z_{n+1} - z'_n\| - e_n \le 0$ contains finite terms.

In the first case, (i) we can extract a subsequence such that $\|z_{n_k} - z'_{n_k-1}\| \to 0$, as $k \to \infty$. Substituting (3.23) in (3.21), we have that

$$\begin{aligned}
\|z_{n+1} - z'_n\|^2 \le{}& \left(1 + \alpha_n^2 - 2\alpha_n\right)\|z_n - z'_{n-1}\|^2 + 2\alpha_n k_n \|z_n - z'_{n-1}\|^2 + 6(\alpha_n + \gamma_n)M^2\alpha_n k_n \\
&- 2\alpha_n\phi(\|z_n - z'_n\|) + 2\alpha_n M\sigma_n + 2\gamma_n M^2 \\
\le{}& \|z_n - z'_{n-1}\|^2 + \left(\alpha_n^2 + 2\alpha_n(k_n - 1)\right)\|z_n - z'_{n-1}\|^2 + 6(\alpha_n + \gamma_n)M^2\alpha_n\overline{k} \\
&- 2\alpha_n\phi(\|z_n - z'_n\|) + 2\alpha_n M\sigma_n + 2\gamma_n M^2 \\
\le{}& \|z_n - z'_{n-1}\|^2 - \alpha_n\phi(\|z_n - z'_n\|) + 2\gamma_n M^2 \\
&- \alpha_n\left[\phi(\|z_n - z'_n\|) - \alpha_n M^2 - 2M^2(k_n - 1) - 6(\alpha_n + \gamma_n)M^2\overline{k} - 2M\sigma_n\right] \\
={}& \|z_n - z'_{n-1}\|^2 - \alpha_n\phi(\|z_n - z'_n\|) + 2\gamma_n M^2 \\
&- \alpha_n\left[\phi(\|z_n - z'_n\|) - 7\overline{k}\alpha_n M^2 - 2M^2(k_n - 1) - 6\gamma_n M^2\overline{k} - 2M\sigma_n\right],
\end{aligned} \tag{3.26}$$

where $\overline{k} = \sup_n k_n$. Again by (3.23), for every $\epsilon > 0$, there exists an index l such that if $j > l$,

$$\left\|z_{n_j} - z'_{n_j-1}\right\| < \epsilon, \qquad \left\|z_{n_j} - z'_{n_j}\right\| < 2\epsilon. \tag{3.27}$$

By hypotheses on the control sequence, with the same $\epsilon > 0$, there exists an index N such that definitively

$$\begin{gathered}
|\alpha_n| < \min\left\{\frac{\epsilon}{12M}, \frac{\phi(\epsilon/2)}{56M^2\overline{k}}\right\}, \\
|\gamma_n| < \min\left\{\frac{\epsilon}{12M}, \frac{\phi(\epsilon/2)}{48M^2\overline{k}}\right\}, \qquad \left|\frac{\gamma_n}{\alpha_n}\right| < \frac{\phi(\epsilon/2)}{4M^2}, \\
|k_n - 1| < \frac{\phi(\epsilon/2)}{16M^2}, \\
|\sigma_n| < \frac{\phi(\epsilon/2)}{16M}.
\end{gathered} \tag{3.28}$$

So take $n^* > \max\{n_l, N\}$ with $n^* = n_j$ for a certain j.

We can prove that $\|z_{n+1} - z'_n\| \to 0$ as $n \to \infty$ proving that, for every $i \ge 0$, the result is $\|z_{n^*+i} - z'_{n^*+i-1}\| < \epsilon$.

Let $i = 1$. If we suppose that $\|z_{n^*+1} - z'_{n^*}\| \ge \epsilon$, it results that

$$\epsilon \le \|z_{n^*+1} - z'_{n^*}\| \le \|z_{n^*} - z'_{n^*}\| + 3(\alpha_{n^*} + \gamma_{n^*})M < \|z_{n^*} - z'_{n^*}\| + \frac{\epsilon}{2}, \tag{3.29}$$

so $\|z_{n^*} - z'_{n^*}\| > \epsilon/2$. In consequence of this, $\phi(\|z_{n^*} - z'_{n^*}\|) > \phi(\epsilon/2)$.

In (3.26), we note that

$$\begin{aligned}
&7\overline{k}\alpha_{n^*}M^2 + 2M^2(k_{n^*}-1) + 6\gamma_{n^*}M^2\overline{k} + 2M\sigma_{n^*} \\
&\quad \le 7\overline{k}M^2\frac{\phi(\epsilon/2)}{56M^2\overline{k}} + 2M^2\frac{\phi(\epsilon/2)}{16M^2} + 6M^2\overline{k}\frac{\phi(\epsilon/2)}{48M^2\overline{k}} + 2M\frac{\phi(\epsilon/2)}{16M} \\
&\quad = \frac{\phi(\epsilon/2)}{8}4 = \frac{\phi(\epsilon/2)}{2},
\end{aligned} \tag{3.30}$$

so

$$\phi(\|z_{n^*} - z'_{n^*}\|) - 7\overline{k}\alpha_{n^*}M^2 - 2M^2(k_{n^*}-1) - 6\gamma_{n^*}M^2\overline{k} - 2M\sigma_{n^*} \ge \frac{\phi(\epsilon/2)}{2}. \tag{3.31}$$

hence in (3.26) remains

$$\|z_{n^*+1} - z'_{n^*}\|^2 \le \epsilon^2 - \alpha_n\phi(\|z_n - z'_n\|) + 2\gamma_n M^2 < \epsilon^2, \tag{3.32}$$

as in Theorem 3.1. This is a contradiction. By the same idea, and using the inductive hypothesis, we obtain that $\|z_{n^*+i} - z'_{n^*+i-1}\| < \epsilon$, for every $i \ge 0$. This ensures that $\|z_{n+1} - z'_n\| \to 0$. In the second case (ii), definitively, $\|z_{n+1} - z'_n\| - e_n \ge 0$, then from the strictly increasing function ϕ, we have

$$\phi(\|z_n - z'_n\|) \ge \phi(\|z_{n+1} - z'_n\| - e_n). \tag{3.33}$$

Substituting (3.33) and (3.23) into (3.21) and simplifying, we have

$$\begin{aligned}
\|z_{n+1} - z'_n\|^2 &\le \|z_n - z'_{n-1}\|^2 + \alpha_n^2\|z_n - z'_{n-1}\|^2 + 2\alpha_n(k_n-1)\|z_n - z'_{n-1}\|^2 \\
&\quad + 6\alpha_n k_n(\alpha_n + \gamma_n)M^2 - 2\alpha_n\phi(\|z_{n+1} - z'_n\| - e_n) + 2\alpha_n\sigma_n M + 2\gamma_n M^2 \\
&\le \|z_n - z'_{n-1}\|^2 - 2\alpha_n\phi(\|z_{n+1} - z'_n\| - e_n) + \alpha_n^2 M^2 \\
&\quad + 2\alpha_n(k_n-1)M^2 + 6\alpha_n k_n(\alpha_n + \gamma_n)M^2 + 2\alpha_n\sigma_n M + 2\gamma_n M^2.
\end{aligned} \tag{3.34}$$

By virtue of Lemma 2.3, we obtain that $\lim_{n\to\infty}\|z_n - z'_{n-1}\| = 0$. □

Theorem 3.3. *Let X be a uniformly smooth Banach space, and let $T : X \to X$ be generalized strongly asymptotically ϕ-pseudocontractive mapping with fixed point x^* and bounded range.*

Let $\{z_n\}_n$ be the sequences defined by (1.15) where $\{\alpha_n\}_n, \{\gamma_n\}_n \subset [0,1]$ satisfy

(i) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\gamma_n = 0$,

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$.

and the sequence $\{w_n\}_n$ is bounded on X, then for any initial point $z_0 \in X$, the sequence $\{z_n\}_n$ strongly converges to x^*.

Proof. Firstly, we observe that, by the boundedness of the range of T, of the sequence $\{w_n\}_n$, and by Lemma 2.4, we have that $\{z_n\}_n$ is bounded.

By Lemma 2.2, we observe that

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 &\le (1 - \alpha_n - \gamma_n)^2 \|z_n - x^*\|^2 + 2\alpha_n \langle T^n z_n - x^*, j(z_{n+1} - x^*\rangle \\
&\quad + 2\gamma_n \langle w_n - x^*, j(z_{n+1} - x^*)\rangle \\
&\le (1 - \alpha_n)^2 \|z_n - x^*\|^2 + 2\alpha_n \langle T^n z_n - x^*, j(z_{n+1} - x^*) - j(z_n - x^*)\rangle \\
&\quad + 2\alpha_n \langle T^n z_n - x^*, j(z_n - x^*)\rangle + 2\gamma_n \langle w_n - x^*, j(z_{n+1} - x^*)\rangle \\
&\le (1 - \alpha_n)^2 \|z_n - x^*\|^2 + 2\alpha_n \langle T^n n z_n - x^*, j(z_{n+1} - x^*) - j(z_n - x^*)\rangle \\
&\quad + 2\alpha_n k_n \|z_n - x^*\|^2 - 2\alpha_n \phi(\|z_n - x^*\|) + 2\gamma_n \langle w_n - x^*, j(z_{n+1} - x^*)\rangle \\
&= \left(1 + \alpha_n^2 - 2\alpha_n\right) \|z_n - x^*\|^2 + 2\alpha_n \langle T^n z_n - x^*, j(z_{n+1} - x^*) - j(z_n - x^*)\rangle \\
&\quad + 2\alpha_n k_n \|z_n - x^*\|^2 - 2\alpha_n \phi(\|z_n - x^*\|) + 2\gamma_n \langle w_n - x^*, j(z_{n+1} - x^*)\rangle \\
&= \|z_n - x^*\|^2 + \left(\alpha_n^2 - 2\alpha_n + 2\alpha_n k_n\right) \|z_n - x^*\|^2 - 2\alpha_n \phi(\|z_n - x^*\|) + 2\alpha_n \mu_n \\
&\quad + 2\gamma_n \langle w_n - x^*, j(z_{n+1} - x^*)\rangle,
\end{aligned} \tag{3.35}$$

where $\mu_n := \langle T^n z_n - x^*, j(z_{n+1} - x^*) - j(z_n - x^*)\rangle$. Let

$$M := \max\left\{\sup_n \|z_n - x^*\|,\ \sup_n \|T^n z_n - x^*\|,\ \sup_n \|w_n - x^*\|,\ \sup_n \langle w_n - x^*, j(z_{n+1} - x^*)\rangle\right\}. \tag{3.36}$$

We have

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 &\le \|z_n - x^*\|^2 + \left(\alpha_n^2 + 2\alpha_n (k_n - 1)\right) M - 2\alpha_n \phi(\|z_n - x^*\|) + 2\alpha_n \mu_n + 2\gamma_n M \\
&= \|z_n - x^*\|^2 - \alpha_n \phi(\|z_n - x^*\|) + 2\gamma_n M \\
&\quad - \alpha_n \left[\phi(\|z_n - x^*\|) - 2\mu_n - (\alpha_n + 2(k_n - 1))M\right],
\end{aligned} \tag{3.37}$$

so we can observe that

(1) $\mu_n \to 0$ as $n \to \infty$. Indeed from the inequality

$$\left|\|z_{n+1} - x^*\| - \|z_n - x^*\|\right| \le \|z_{n+1} - z_n\| \le (\alpha_n + \gamma_n) M \longrightarrow 0, \quad \text{as } n \longrightarrow \infty\ , \tag{3.38}$$

and since j is norm to norm uniformly continuous, then $j(\|z_{n+1} - x^*\|) - j(\|z_n - x^*\|) \to 0$, as $n \to \infty$,

(2) $\inf_n(\|z_n - x^*\|) = 0$. Indeed, if we supposed that $\sigma := \inf_n(\|z_n - x^*\|) > 0$, by the monotonicity of ϕ,

$$\phi(\|z_n - x^*\|) \geq \phi(\sigma) > 0. \tag{3.39}$$

Thus, by (1) and by the hypotheses on α_n and k_n, the value $-\alpha_n[\phi(\|z_n - x^*\|) - 2\mu_n - (\alpha_n + 2(k_n - 1))M]$ is definitively negative. In this case, we conclude that there exists $N > 0$ such that for every $n > N$,

$$\begin{aligned} \|z_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 - \alpha_n \phi(\|z_n - x^*\|) + 2\gamma_n M \\ &\leq \|z_n - x^*\|^2 - \alpha_n \phi(\sigma) + 2\gamma_n M, \end{aligned} \tag{3.40}$$

and so

$$\alpha_n \phi(\sigma) \leq \|z_n - x^*\|^2 - \|z_{n+1} - x^*\|^2 + 2\gamma_n M \quad \forall n > N. \tag{3.41}$$

In the same way we obtain that

$$\begin{aligned} \phi(\sigma)\sum_{i=N}^{m}\alpha_i &\leq \sum_{i=N}^{m}\left[\|z_i - x^*\|^2 - \|z_{i+1} - x^*\|^2\right] + 2\sum_{i=N}^{m}\gamma_i M \\ &= \|z_N - x^*\|^2 - \|z_m - x^*\|^2 + 2M\sum_{i=N}^{m}\gamma_i. \end{aligned} \tag{3.42}$$

By the hypotheses $\sum_n \gamma_n < \infty$ and $\sum_n \alpha_n = \infty$, the previous is a contradiction, and it follows that $\inf_n(\|z_n - x^*\|) = 0$.

Then, there exists a subsequence $\{z_{n_k}\}_k$ of $\{z_n\}_n$ that strongly converges to x^*. This implies that for every $\epsilon > 0$, there exists an index $n_{k(\epsilon)}$ such that, for all $j \geq n_{k(\epsilon)}$, $\|z_{n_j} - x^*\| < \epsilon$.

Now, we will prove that the sequence $\{z_n\}_n$ converges to x^*. Since the sequences in (3.37) are null sequences and $\sum_n \gamma_n < \infty$, but $\sum_n \alpha_n = \infty$, then, for every $\epsilon > 0$, there exists an index $\overline{n}(\epsilon)$ such that for all $n \geq \overline{n}(\epsilon)$, it results that

$$\begin{gathered} |\alpha_n| < \frac{1}{4M}\min\left\{\epsilon, \phi\left(\frac{\epsilon}{2}\right)\right\}, \\ |\gamma_n| < \frac{\epsilon}{4M}, \qquad \left|\frac{\gamma_n}{\alpha_n}\right| < \frac{\phi(\epsilon/2)}{4M}, \\ |k_n - 1| < \frac{\phi(\epsilon/2)}{8M}, \\ |\mu_n| < \frac{\phi(\epsilon/2)}{8}. \end{gathered} \tag{3.43}$$

So, fixing $\epsilon > 0$, let $n^* > \max(n_{k(\epsilon)}, \overline{n}(\epsilon))$ with $n^* = n_j$ for a certain n_j. We will prove, by induction, that $\|z_{n^*+i} - x^*\| < \epsilon$ for every $i \in \mathbb{N}$. Let $i = 1$. If not, it results that $\|z_{n^*+1} - x^*\| \geq \epsilon$. Thus,

$$\begin{aligned}\epsilon \leq \|z_{n^*+1} - x^*\| &\leq \|z_{n^*} - x^*\| + \alpha_{n^*}M + \gamma_{n^*}M, \\ &< \|z_{n^*} - x^*\| + \frac{\epsilon}{4M}M + \frac{\epsilon}{4M}M = \|z_{n^*} - x^*\| + \frac{\epsilon}{2},\end{aligned} \tag{3.44}$$

that is, $\|z_{n^*} - x^*\| > \epsilon/2$. By the strict increasing of ϕ, $\phi(\|z_{n^*} - x^*\|) > \phi(\epsilon/2)$.

By (3.37), it results that

$$\begin{aligned}\|z_{n^*+1} - x^*\|^2 < \epsilon^2 - \alpha_{n^*}\left(\phi(\|z_{n^*} - x^*\|) - 2M\frac{\gamma_{n^*}}{\alpha_{n^*}}\right) \\ - \alpha_{n^*}\left[\phi(\|z_{n^*} - x^*\|) - 2\mu_{n^*} - (\alpha_{n^*} + 2(k_{n^*} - 1))M\right].\end{aligned} \tag{3.45}$$

We can note that

$$2\mu_{n^*} + (\alpha_{n^*} + 2(k_{n^*} - 1))M \leq \frac{\phi(\epsilon/2)}{4} + \left(\frac{\phi(\epsilon/2)}{4M} + \frac{\phi(\epsilon/2)}{4M}\right)M, \tag{3.46}$$

so

$$\phi(\|z_{n^*} - x^*\|) - 2\mu_{n^*} - (\alpha_{n^*} + 2(k_{n^*} - 1))M > \phi(\epsilon/2) - \frac{3\phi(\epsilon/2)}{4} > 0. \tag{3.47}$$

Moreover, $\phi(\|z_{n^*} - x^*\|) - 2M\gamma_{n^*}/\alpha_{n^*} > \phi(\epsilon/2)/2 > 0$, so it results that

$$\|z_{n^*+1} - x^*\|^2 < \epsilon^2. \tag{3.48}$$

This is a contradiction. Thus, $\|z_{n^*+1} - x^*\| < \epsilon$.

In the same manner, by induction, one obtains that, for every $i \geq 1$, $\|z_{n^*+i} - x^*\| < \epsilon$. So $\|z_n - x^*\| \to 0$. □

Corollary 3.4. *Let X be a uniformly smooth Banach space, and let $T : X \to X$ be generalized strongly asymptotically ϕ-pseudocontractive mapping with bounded range and fixed point x^*. The sequences $\{x_n\}_n$, $\{z_n\}_n$, and $\{z'_n\}_n$ are defined by* (1.14), (1.15), *and* (1.16), *respectively, where the sequences $\{\alpha_n\}_n$, $\{\beta_n\}_n$, $\{\gamma_n\}_n$, $\{\delta_n\}_n \subset [0,1]$ satisfy*

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$,

and the sequences $\{u_n\}_n$, $\{v_n\}_n$, $\{w_n\}_n$, and $\{w'_n\}_n$ are bounded in X. Then for any initial point $x_0, z_0, z'_0 \in X$, the following two assertions are equivalent and true:

(i) *the modified Ishikawa iteration sequence with errors* (1.14) *converges to the fixed point x^*;*

(ii) *the modified Mann iteration sequence with errors* (1.15) *converges to the fixed point x^*;*

(iii) *the implicit iteration sequence with errors* (1.16) *converges to the fixed point x^*.*

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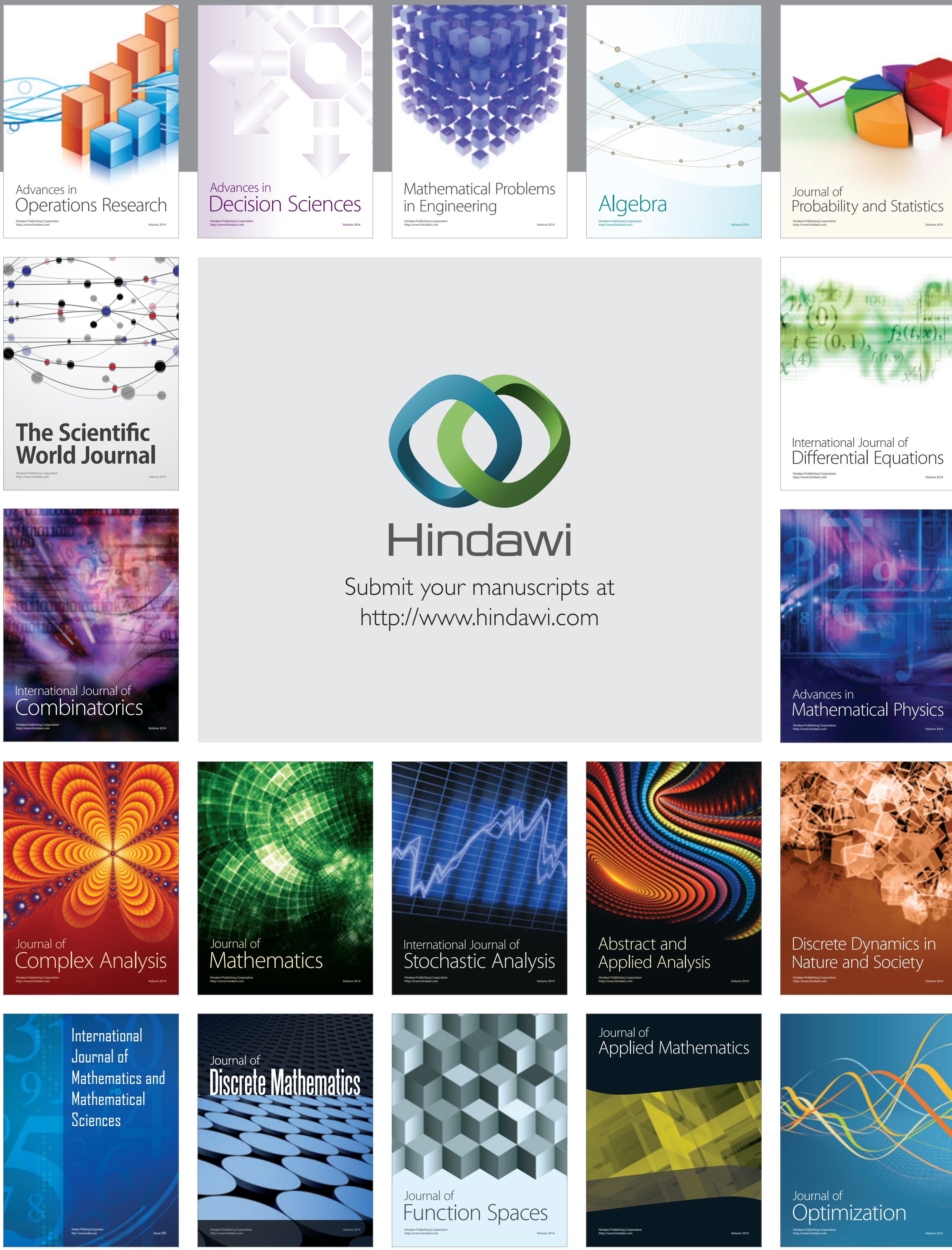
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