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Research Article

The Group Involutory Matrix of the Combinations of Two Idempotent Matrices

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We discuss the following problem: when $aP + bQ + cPQ + dQP + ePQP + fQPQ + gPQPQ$ of idempotent matrices P and Q, where $a, b, c, d, e, f, g \in \mathbb{C}$ and $a \neq 0, b \neq 0$, is group involutory.

1. Introduction

Throughout this paper $\mathbb{C}^{n\times n}$ stands for the set of $n \times n$ complex matrices. Let $A \in \mathbb{C}^{n\times n}$. A is said to be idempotent if $A^2 = A$. A is said to be group invertible if there exists an $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A, \quad XAX = X, \quad AX = XA \tag{1.1}$$

hold. If such an X exists, then it is unique, denoted by A_g, and called the group inverse of A. It is well known that the group inverse of a square matrix A exists if and only if $\operatorname{rank}(A^2) = \operatorname{rank}(A)$ (see, e.g., [1] for details). Clearly, not every matrix is group invertible. But the group inverse of every idempotent matrix exists and is this matrix itself.

Recall that a matrix A with the group inverse is said to be group involutory if $A_g = A$. A is the group involutory matrix if and only if it is tripotent, that is, satisfies $A^3 = A$ (see [2]). Thus, for a nonzero idempotent matrix P and a nonzero scalar a, aP is a group involutory matrix if and only if either $a = 1$ or $a = -1$.

Recently, some properties of linear combinations of idempotents or projections are widely discussed (see, e.g., [3–12] and the literature mentioned below). In [13], authors established a complete solution to the problem of when a linear combination of two different projectors is also a projector. In [14], authors considered the following problem: when a linear combination of nonzero different idempotent matrices is the group involutory matrix. In [15], authors provided the complete list of situations in which a linear combination of two idempotent matrices is the group involutory matrix. In [16], authors discussed the group inverse of $aP+bQ+cPQ+dQP+ePQP+fQPQ+gPQPQ$ of idempotent matrices P and Q, where $a,b,c,d,e,f,g \in \mathbb{C}$ with $a,b \neq 0$, deduced its explicit expressions, and some necessary and sufficient conditions for the existence of the group inverse of $aP+bQ+cPQ$.

In this paper, we will investigate the following problem: when $aP+bQ+cPQ+dQP+ePQP+fQPQ+gPQPQ$ is group involutory. To this end, we need the results below.

Lemma 1.1 (see [16, Theorems 2.1 and 2.4]). *Let $P,Q \in \mathbb{C}^{n\times n}$ be two different nonzero idempotent matrices. Suppose $(PQ)^2 = (QP)^2$. Then for any scalars a, b, c, d, e, f, g, where $a,b \neq 0$ and $\theta = a+b+c+d+e+f+g$, $aP+bQ+cPQ+dQP+ePQP+fQPQ+g(PQ)^2$ is group invertible, and*

(i) *if $\theta \neq 0$, then*

$$\begin{aligned}
&\left(aP+bQ+cPQ+dQP+ePQP+fQPQ+g(PQ)^2\right)_g \\
&\quad= \frac{1}{a}P+\frac{1}{b}Q-\left(\frac{1}{a}+\frac{1}{b}+\frac{c}{ab}\right)PQ-\left(\frac{1}{a}+\frac{1}{b}+\frac{d}{ab}\right)QP \\
&\qquad+\left(\frac{2}{a}+\frac{1}{b}+\frac{c+d}{ab}+\frac{cd-be}{a^2b}\right)PQP+\left(\frac{1}{a}+\frac{2}{b}+\frac{c+d}{ab}+\frac{cd-af}{ab^2}\right)QPQ \\
&\qquad-\left(\frac{2}{a}+\frac{2}{b}+\frac{c+d}{ab}+\frac{cd-be}{a^2b}+\frac{cd-af}{ab^2}-\frac{1}{\theta}\right)PQPQ;
\end{aligned} \tag{1.2}$$

(ii) *if $\theta = 0$, then*

$$\begin{aligned}
&\left(aP+bQ+cPQ+dQP+ePQP+fQPQ+g(PQ)^2\right)_g \\
&\quad= \frac{1}{a}P+\frac{1}{b}Q-\left(\frac{1}{a}+\frac{1}{b}+\frac{c}{ab}\right)PQ-\left(\frac{1}{a}+\frac{1}{b}+\frac{d}{ab}\right)QP \\
&\qquad+\left(\frac{2}{a}+\frac{1}{b}+\frac{c+d}{ab}+\frac{cd-be}{a^2b}\right)PQP+\left(\frac{1}{a}+\frac{2}{b}+\frac{c+d}{ab}+\frac{cd-af}{ab^2}\right)QPQ \\
&\qquad-\left(\frac{2}{a}+\frac{2}{b}+\frac{c+d}{ab}+\frac{cd-be}{a^2b}+\frac{cd-af}{ab^2}\right)(PQ)^2.
\end{aligned} \tag{1.3}$$

Lemma 1.2 (see [16, Theorem 3.1]). *Let $P,Q \in \mathbb{C}^{n\times n}$ be two different nonzero idempotent matrices. Suppose $(QP)^2 = 0$. Then for any scalars a, b, c, d, e, f, and g, where $a,b \neq 0$, $aP+bQ+cPQ+dQP+ePQP+fQPQ+g(PQ)^2$ is group invertible, and*

$$\begin{aligned}
&\left(aP+bQ+cPQ+dQP+ePQP+fQPQ+g(PQ)^2\right)_g \\
&\quad= \frac{1}{a}P+\frac{1}{b}Q-\left(\frac{1}{a}+\frac{1}{b}+\frac{c}{ab}\right)PQ-\left(\frac{1}{a}+\frac{1}{b}+\frac{d}{ab}\right)QP
\end{aligned}$$

$$+\left(\frac{2}{a}+\frac{1}{b}+\frac{c+d}{ab}+\frac{cd-be}{a^2b}\right)PQP+\left(\frac{1}{a}+\frac{2}{b}+\frac{c+d}{ab}+\frac{cd-af}{ab^2}\right)QPQ$$
$$-\left(\frac{2}{a}+\frac{2}{b}+\frac{2c+d+g}{ab}+\frac{cd-be-ce}{a^2b}+\frac{cd-af-cf}{ab^2}+\frac{c^2d}{a^2b^2}\right)(PQ)^2. \tag{1.4}$$

2. Main Results

In this section, we will research when some combination of two nonzero idempotent matrices is a group involutory matrix.

First, we will discuss some situations lying in the category of $(PQ)^2 = (QP)^2$.

Theorem 2.1. *Let $P, Q \in \mathbb{C}^{n\times n}$ be two different nonzero idempotent matrices with $(PQ)^2 = (QP)^2$, and let A be a combination of the form*

$$A = aP + bQ + cPQ + dQP + ePQP + fQPQ + gPQPQ, \tag{2.1}$$

where $a, b, c, d, e, f, g \in \mathbb{C}$ with $a, b \neq 0$. Denote $\theta = a + b + c + d + e + f + g$. Then the following list comprises characteristics of all cases where A is the group involutory matrix:

(a) the cases denoted by $(a_1) \sim (a_3)$, in which

$$PQ = QP, \tag{2.2}$$

and any of the following sets of additional conditions hold:

(a_1) *either $a = 1$ or $a = -1$, either $\theta = 1$ or $\theta = -1$ or $\theta = 0$, and $Q = PQ$;*

(a_2) *either $b = 1$ or $b = -1$, either $\theta = 1$ or $\theta = -1$ or $\theta = 0$, and $P = PQ$;*

(a_3) *either $a = 1$ or $a = -1$, either $b = 1$ or $b = -1$, either $\theta = 1$ or $\theta = -1$ or $\theta = 0$ or $PQ = 0$.*

(b) the cases denoted by $(b_1) \sim (b_6)$, in which

$$PQ \neq QP, \quad PQP = QPQ, \tag{2.3}$$

and any of the following sets of additional conditions hold:

(b_1) $a = \pm 1, b = \mp 1$, *either $\theta = 1$ or $\theta = -1$ or $\theta = 0$ or $PQP = 0$;*

(b_2) $a = b = \pm 1, c = d = \mp 1$, *either $\theta = 1$ or $\theta = -1$ or $\theta = 0$ or $PQP = 0$;*

(b_3) $a = b = \pm 1, c = \mp 1$, *either $\theta = 1$ or $\theta = -1$ or $\theta = 0$, and $QP = PQP$;*

(b_4) $a = b = \pm 1, d = \mp 1$, *either $\theta = 1$ or $\theta = -1$ or $\theta = 0$, and $PQ = PQP$;*

(b_5) $a = b = \pm 1$, $c = \mp 1$, *and $QP = 0$;*

(b_6) $a = b = \pm 1$, $d = \mp 1$, *and $PQ = 0$,*

(c) the cases denoted by $(c_1) \sim (c_{18})$, in which

$$PQP \neq QPQ, \qquad PQPQ = QPQP, \tag{2.4}$$

and any of the following sets of additional conditions hold:

(c_1) $a = \pm 1, b = \mp 1, c + d + 2e \pm cd = \pm 1$, *either* $\theta = 1$ *or* $\theta = -1$, *and* $QPQ = PQPQ$;

(c_2) $a = b = e = \pm 1, c = d = \mp 1$, *either* $\theta = 1$ *or* $\theta = -1$, *and* $QPQ = PQPQ$;

(c_3) $a = \pm 1, b = \mp 1, c + d + 2f \mp cd = \mp 1$, *either* $\theta = 1$ *or* $\theta = -1$, *and* $PQP = PQPQ$;

(c_4) $a = b = f = \pm 1, c = d = \mp 1$, *either* $\theta = 1$ *or* $\theta = -1$, *and* $PQP = PQPQ$;

(c_5) $a = \pm 1, b = \mp 1, c + d + 2e \pm cd = \pm 1, c + d + 2f \mp cd = \mp 1$, *either* $g = 1$ *or* $g = -1$;

(c_6) $a = b = e = f = \pm 1, c = d = \mp 1$, *either* $g = \mp 1$ *or* $g = \mp 3$;

(c_7) $a = \pm 1, b = \mp 1, c + d + 2e \pm cd = \pm 1$, *and* $QPQ = 0$;

(c_8) $a = b = e = \pm 1, c = d = \mp 1$, *and* $QPQ = 0$;

(c_9) $a = \pm 1, b = \mp 1, c + d + 2f \mp cd = \mp 1$, *and* $PQP = 0$;

(c_{10}) $a = b = f = \pm 1, c = d = \mp 1$, *and* $PQP = 0$;

(c_{11}) $a = \pm 1, b = \mp 1, c + d + 2e \pm cd = \pm 1, c + d + 2f \mp cd = \mp 1$, *and* $PQPQ = 0$;

(c_{12}) $a = b = e = f = \pm 1, c = d = \mp 1$, *and* $PQPQ = 0$;

(c_{13}) $a = \pm 1, b = \mp 1, 2e + c + d \pm cd = \pm 1, \theta = 0$, *and* $QPQ = PQPQ$;

(c_{14}) $a = b = e = \pm 1, c = d = \mp 1, \theta = 0$, *and* $QPQ = PQPQ$;

(c_{15}) $a = \pm 1, b = \mp 1, 2f + c + d \mp cd = \mp 1, \theta = 0$, *and* $PQP = PQPQ$;

(c_{16}) $a = b = f = \pm 1, c = d = \mp 1, \theta = 0$, *and* $PQP = PQPQ$;

(c_{17}) $a = \pm 1, b = \mp 1, 2e + c + d \pm cd = \pm 1, 2f + c + d \mp cd = \mp 1, g = 0$;

(c_{18}) $a = b = e = f = \pm 1, c = d = \mp 1, g = \mp 2$.

Proof. Obviously, the condition (2.2) implies that the group inverse of A exists and is of the form (1.2) when $\theta \neq 0$ or the form (1.3) when $\theta = 0$ by Lemma 1.1. So do the conditions (2.2), (2.3), and (2.4). We will straightforwardly show that a matrix A of the form (2.1) is the group involutory matrix if and only if $A - A_g = 0$.

(a) Under the condition (2.2), $A = aP + bQ + \mu PQ$, where $\mu = c + d + e + f + g$.

(1) If $\theta \neq 0$, then

$$A_g = \frac{1}{a}P + \frac{1}{b}Q + \left(\frac{1}{\theta} - \frac{1}{a} - \frac{1}{b}\right)PQ, \tag{2.5}$$

and so

$$A - A_g = \left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)Q + \left(\mu - \frac{1}{\theta} + \frac{1}{a} + \frac{1}{b}\right)PQ = 0. \tag{2.6}$$

Multiplying (2.6) by P and Q, respectively, leads to

$$\begin{aligned} &\left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)PQ + \left(\mu - \frac{1}{\theta} + \frac{1}{a} + \frac{1}{b}\right)PQ = 0, \\ &\left(a - \frac{1}{a}\right)PQ + \left(b - \frac{1}{b}\right)Q + \left(\mu - \frac{1}{\theta} + \frac{1}{a} + \frac{1}{b}\right)PQ = 0, \end{aligned} \tag{2.7}$$

and then

$$\left(a-\frac{1}{a}\right)P+\left(b-\frac{1}{b}\right)PQ=\left(a-\frac{1}{a}\right)PQ+\left(b-\frac{1}{b}\right)Q. \tag{2.8}$$

Multiplying the above equation, respectively, by P and by Q, we get

$$\left(a-\frac{1}{a}\right)(P-PQ)=0, \qquad \left(b-\frac{1}{b}\right)(Q-PQ)=0. \tag{2.9}$$

Thus, since $P \neq Q$, we have three situations: $P=PQ$ and $b=b^{-1}$; $a=a^{-1}$ and $Q=PQ$; $a=a^{-1}$ and $b=b^{-1}$.

When $Q=PQ$ and $a=a^{-1}$, (2.6) becomes $(\theta-\theta^{-1})Q=0$ and then $\theta=\pm 1$. Therefore, we obtain (a_1) except the situation $\theta=0$. Similarly, when $b=b^{-1}$ and $P=PQ$, we have (a_2) except the situation $\theta=0$. When $a=a^{-1}$ and $b=b^{-1}$, (2.6) becomes $(\theta-\theta^{-1})PQ=0$ and then $\theta=\pm 1$ or $PQ=0$. Therefore, we obtain (a_3) except the situation $\theta=0$.

(2) If $\theta=0$, then

$$A_g=\frac{1}{a}P+\frac{1}{b}Q-\left(\frac{1}{a}+\frac{1}{b}\right)PQ, \tag{2.10}$$

and then

$$A-A_g=\left(a-\frac{1}{a}\right)P+\left(b-\frac{1}{b}\right)Q+\left(\mu+\frac{1}{a}+\frac{1}{b}\right)PQ=0. \tag{2.11}$$

Analogous to the process of reaching (2.9) in (a)(1), we have

$$\left(b-\frac{1}{b}\right)(Q-PQ)=0, \qquad \left(a-\frac{1}{a}\right)(P-PQ)=0. \tag{2.12}$$

Thus, we have three situations: $P=PQ$ and $b=b^{-1}$; $a=a^{-1}$ and $Q=PQ$; $a=a^{-1}$ and $b=b^{-1}$, since $P \neq Q$. Similar to the argument in $(a)(1)$, substituting them, respectively, into (2.11), we can obtain the situation $\theta=0$, respectively, in (a_1), (a_2), and (a_3).

(b) Under the condition (2.3), $A=aP+bQ+cPQ+dQP+\nu PQP$, where $\nu=e+f+g$. (1) If $\theta \neq 0$, then

$$\begin{aligned} A_g=&\frac{1}{a}P+\frac{1}{b}Q-\left(\frac{1}{a}+\frac{1}{b}+\frac{c}{ab}\right)PQ-\left(\frac{1}{a}+\frac{1}{b}+\frac{d}{ab}\right)QP \\ &+\left(\frac{1}{a}+\frac{1}{b}+\frac{c+d}{ab}+\frac{1}{\theta}\right)PQP, \end{aligned} \tag{2.13}$$

and so

$$A - A_g = \left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)Q + \left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ$$
$$+ \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP + \left(\nu - \frac{1}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{1}{\theta}\right)PQP = 0. \tag{2.14}$$

Multiplying the above equation, respectively, on the two sides by P yields

$$0 = \left(a - \frac{1}{a}\right)P + \left(c + b + \frac{1}{a} + \frac{c}{ab}\right)PQ + \left(\nu + d - \frac{c}{ab} - \frac{1}{\theta}\right)PQP, \tag{2.15}$$

$$0 = \left(a - \frac{1}{a}\right)P + \left(b + d + \frac{1}{a} + \frac{d}{ab}\right)QP + \left(\nu + c - \frac{d}{ab} - \frac{1}{\theta}\right)PQP. \tag{2.16}$$

Multiplying (2.15) on the left sides by Q and (2.16) on the right sides by Q, by (2.3), we have

$$\left(a - \frac{1}{a}\right)QP + \left(b + c + d + \nu + \frac{1}{a} - \frac{1}{\theta}\right)QPQ = 0,$$
$$\left(a - \frac{1}{a}\right)PQ + \left(b + c + d + \nu + \frac{1}{a} - \frac{1}{\theta}\right)QPQ = 0, \tag{2.17}$$

and then $(a - a^{-1})(QP - PQ) = 0$. Since $QP \neq PQ$, $a = a^{-1}$. Similarly, $b = b^{-1}$.

Substituting $a = a^{-1}$ inside (2.17) yields $(\theta - \theta^{-1})QPQ = 0$ and then $\theta = \theta^{-1}$ or $QPQ = 0$. We will discuss the remainder for detail as follows:

When $a = a^{-1}$, $b = b^{-1}$, (2.14) becomes

$$0 = \left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ + \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP$$
$$+ \left(\nu - \frac{1}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{1}{\theta}\right)PQP, \tag{2.18}$$

(i) if $a + b = 0$, then

$$c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} = 0, \qquad d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} = 0, \tag{2.19}$$

and so it follows from (2.18) that

$$\left(\theta - \frac{1}{\theta}\right)PQP = \left(\nu + c + d - \frac{1}{\theta}\right)PQP = 0. \tag{2.20}$$

Therefore, either $\theta = \theta^{-1}$ or $PQP = 0$ implies that (2.18) holds, namely, (2.14) holds. Thus, we have (b_1) except the situation $\theta = 0$.

(ii) if $a = b$, then (2.18) becomes

$$0 = (2c + 2a)PQ + (2d + 2a)QP + \left(2\nu - \theta - \frac{1}{\theta}\right)PQP. \tag{2.21}$$

Multiplying the above equation, respectively, on the right side by P and on the left side by Q, we have

$$0 = (2c + 2a)PQ + \left(\nu + d - c - \frac{1}{\theta}\right)PQP, \tag{2.22}$$

$$0 = (2d + 2a)QP + \left(\nu + c - d - \frac{1}{\theta}\right)PQP. \tag{2.23}$$

So if $\theta = \theta^{-1}$, then the two equations above (2.22) and (2.23) become, respectively,

$$(c + a)(PQ - PQP) = 0, \qquad (d + a)(QP - PQP) = 0. \tag{2.24}$$

Or if $PQP = 0$, then (2.22) and (2.23) become, respectively,

$$(c + a)PQ = 0, \qquad (d + a)QP = 0. \tag{2.25}$$

Since $PQ \neq QP$, it follows from (2.24) and (2.25) that we have the six situations: $\theta = \theta^{-1}$ and $c = d = -a$; $\theta = \theta^{-1}$, $c = -a$ and $QP = PQP$; $\theta = \theta^{-1}$, $d = -a$, and $PQ = PQP$; $c = -a$ and $QP = 0$; $d = -a$ and $PQ = 0$; $c = d = -a$ and $PQP = 0$. Thus, we have $(b_2) \sim (b_4)$ except the situation $\theta = 0$, and (b_5) and (b_6).

(2) If $\theta = 0$, then

$$A_g = \frac{1}{a}P + \frac{1}{b}Q - \left(\frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ - \left(\frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP + \left(\frac{1}{a} + \frac{1}{b} + \frac{c + d}{ab}\right)PQP, \tag{2.26}$$

and then

$$\begin{aligned} A - A_g = {} & \left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)Q + \left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ \\ & + \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP + \left(\nu - \frac{1}{a} - \frac{1}{b} - \frac{c + d}{ab}\right)PQP = 0. \end{aligned} \tag{2.27}$$

Analogous to the process in (b)(1), using (2.27) we can obtain

$$\begin{aligned} & \left(a - \frac{1}{a}\right)QP - \left(a - \frac{1}{a}\right)PQP = 0, \\ & \left(a - \frac{1}{a}\right)PQ - \left(a - \frac{1}{a}\right)PQP = 0. \end{aligned} \tag{2.28}$$

Thus, since $PQ \neq QP$, $PQ \neq PQP$ and/or $QP \neq PQP$ and then $a = a^{-1}$. Similarly, $b = b^{-1}$. Hence, $a = \pm b$.

(i) If $a = -b$, then

$$c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} = 0,$$
$$d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} = 0, \tag{2.29}$$
$$\nu - \frac{1}{a} - \frac{1}{b} - \frac{c+d}{ab} = -2(a+b) = 0.$$

Thus, (2.27) holds. Hence we have the situation $\theta = 0$ in (b_1).

(ii) If $a = b$, then (2.27) becomes

$$(c+a)PQ + (d+a)QP + \nu PQP = 0. \tag{2.30}$$

Multiplying the above equation on the left side, respectively, by P and by Q, we have

$$(c+a)(PQ - PQP) = 0, \qquad (d+a)(QP - PQP) = 0. \tag{2.31}$$

Thus, $c = d = -a$; $c = -a$ and $QP = PQP$; $d = -a$ and $PQ = PQP$. Hence, we have the situation $\theta = 0$, respectively, in (b_2), (b_3), and (b_4).

(c) Under the condition (2.4),

$$A = aP + bQ + cPQ + dQP + ePQP + fQPQP + gPQPQ. \tag{2.32}$$

(1) If $\theta \neq 0$, then

$$\begin{aligned} A_g = {} & \frac{1}{a}P + \frac{1}{b}Q - \left(\frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ - \left(\frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP \\ & + \left(\frac{2}{a} + \frac{1}{b} + \frac{c+d}{ab} + \frac{cd - be}{a^2 b}\right)PQP + \left(\frac{1}{a} + \frac{2}{b} + \frac{c+d}{ab} + \frac{\mathrm{cd} - af}{ab^2}\right)QPQ \\ & - \left(\frac{2}{a} + \frac{2}{b} + \frac{c+d}{ab} + \frac{cd - be}{a^2 b} + \frac{cd - af}{ab^2} - \frac{1}{\theta}\right)PQPQ, \end{aligned} \tag{2.33}$$

and so

$$\begin{aligned} A - A_g = {} & \left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)Q + \left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ + \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP \\ & + \left(e - \frac{2}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{cd - be}{a^2 b}\right)PQP \end{aligned}$$

$$+\left(f-\frac{1}{a}-\frac{2}{b}-\frac{c+d}{ab}-\frac{cd-af}{ab^2}\right)QPQ$$
$$+\left(g+\frac{2}{a}+\frac{2}{b}+\frac{c+d}{ab}+\frac{cd-be}{a^2b}+\frac{cd-af}{ab^2}-\frac{1}{\theta}\right)PQPQ\ =0. \tag{2.34}$$

If $PQ=0$, then $QPQ=0=PQP$ and so it contradicts (2.4). Thus $PQ\neq 0$. Similarly, $QP\neq 0$.

Multiplying (2.34) on the left side by QP yields

$$\left(a-\frac{1}{a}\right)QP+\left(b+c+\frac{1}{a}+\frac{c}{ab}\right)QPQ+\left(d+e+f+g-\frac{c}{ab}-\frac{1}{\theta}\right)PQPQ=0. \tag{2.35}$$

Multiplying the above equation, respectively, on the left side by P and on the right side by PQ yields, by (2.4),

$$0=\left(a-\frac{1}{a}\right)PQP+\left(\frac{1}{a}-a+\theta-\frac{1}{\theta}\right)PQPQ, \tag{2.36}$$
$$0=\left(a-\frac{1}{a}\right)QPQ+\left(\frac{1}{a}-a+\theta-\frac{1}{\theta}\right)PQPQ. \tag{2.37}$$

Since $PQP\neq QPQ$, $a=a^{-1}$ by (2.36) and (2.37). Similarly, we can gain $b=b^{-1}$. Substituting $a=a^{-1}$ inside (2.36) yields $\theta=\theta^{-1}$ or $PQPQ=0$.

(i) Consider the case of $a=a^{-1}$, $b=b^{-1}$ and $\theta=\theta^{-1}$.

Substituting $a=a^{-1}$, $b=b^{-1}$, and $\theta=\theta^{-1}$ inside (2.35) yields

$$\left(a+b+c+\frac{c}{ab}\right)(QPQ-PQPQ)=0. \tag{2.38}$$

Similarly, we have

$$\left(a+b+d+\frac{d}{ab}\right)(PQP-PQPQ)=0. \tag{2.39}$$

If $PQP=PQPQ$, then $QPQ\neq PQPQ$ by the hypothesis $PQP\neq QPQ$ and so $a+b+c+c/ab=0$ by (2.38). Multiplying (2.34) on the right side by Q yields

$$\left(a+c+d+2f-\frac{cd}{a}\right)(QPQ-PQPQ)=0. \tag{2.40}$$

Thus, $a+c+d+2f-cd/a=0$ and then (2.14) becomes

$$\left(a+b+d+\frac{d}{ab}\right)QP+\left(f-a-2b-\frac{c+d}{ab}-\frac{cd-af}{a}\right)QPQ$$
$$+\left(b+e+g+\frac{cd-af}{a}-\theta\right)PQP=0. \tag{2.41}$$

Multiplying the above equation on the right side by P yields

$$\left(a+b+d+\frac{d}{ab}\right)(QP-PQP)=0. \tag{2.42}$$

Assume $PQ = PQP$. Then $QPQ = QPQP = PQPQ = PQ = PQP$ and it contradicts the hypothesis $PQP \neq QPQ$. Thus, $a+b+d+d/ab=0$.

Similarly, if $QPQ = PQPQ$, then we can obtain $a+b+d+\dfrac{d}{ab}=0$, $b+c+d+2e-cd/b=0$, and $a+b+c+c/ab=0$.

Obviously, if $QPQ \neq QPQP$ and $QPQ \neq PQPQ$, we have $a+b+d+d/ab=0$, $a+b+c+c/ab=0$, $b+c+d+2e-cd/b=0$, and $a+c+d+2f-cd/a=0$.

Next, we calculate these scalars. If $a+b=0$, then $a+b+c+c/ab=0$ for any c and $a+b+d+d/ab=0$ for any d, and so c,d,e are chosen to satisfy $b+c+d+2e-cd/b=0$. Similarly c,d,f are chosen to satisfy $a+c+d+2f-cd/a=0$.

If $a=b$, then $c=d=-a$, and $e=a$ by solving $b+c+d+2e-cd/b=0$, and $f=a$ by solving $a+c+d+2f-cd/a=0$.

Note that $b+c+d+2e-cd/b=0$ and $a+c+d+2f-cd/a=0$ imply $g=\theta-(a+b)$. Hence, we have $(c_1) \sim (c_6)$.

(ii) Consider the case of $a=a^{-1}$, $b=b^{-1}$, and $PQPQ=0$.

Multiplying (2.34), respectively, on the right side by QP and on the left side by PQ yields

$$\begin{aligned}\left(c+\frac{1}{a}+\frac{1}{b}+\frac{c}{ab}\right)QPQ=0,\\ \left(d+\frac{1}{a}+\frac{1}{b}+\frac{d}{ab}\right)PQP=0.\end{aligned} \tag{2.43}$$

If $QPQ=0$, then $PQP \neq 0$ and so $a+b+d+d/ab=0$ and (2.34) becomes

$$0=\left(c+\frac{1}{a}+\frac{1}{b}+\frac{c}{ab}\right)PQ+\left(e-\frac{2}{a}-\frac{1}{b}-\frac{c+d}{ab}-\frac{cd-be}{a^2b}\right)PQP. \tag{2.44}$$

Multiplying (2.44) on right side by Q yields

$$\left(c+\frac{1}{a}+\frac{1}{b}+\frac{c}{ab}\right)PQ=0. \tag{2.45}$$

Since $PQ \neq 0$, $a+b+c+c/ab=0$ and then (2.44) becomes

$$\left(2e+b+c+d-\frac{cd}{b}\right)PQP. \tag{2.46}$$

Thus, $2e+b+c+d-cd/b=0$.

If $PQP=0$, then we, similarly, have $a+b+c+c/ab=0$, $a+b+d+d/ab=0$, and $2f+a+c+d-cd/a=0$.

If $PQP \neq 0$ and $QPQ \neq 0$, then, multiplying (2.34), on the right side by Q and on the left side by P yields $a + b + c + c/ab = 0$, and multiplying (2.34) on the right side by P and on the left side by Q yields $a + b + d + d/ab = 0$. Thus, (2.34) becomes

$$\left(e - \frac{2}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{cd-be}{a^2b}\right)PQP + \left(f - \frac{1}{a} - \frac{2}{b} - \frac{c+d}{ab} - \frac{cd-af}{ab^2}\right)QPQ = 0. \tag{2.47}$$

Multiplying the equation above on the right side, respectively, by P and by Q yields

$$2e + b + c + d - \frac{cd}{b} = 0, \qquad 2f + a + c + d - \frac{cd}{a} = 0. \tag{2.48}$$

As the argument above in (i), we have $(c_7) \sim (c_{12})$.

(2) If $\theta = 0$, then

$$\begin{aligned} A_g = {} & \frac{1}{a}P + \frac{1}{b}Q - \left(\frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ - \left(\frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP \\ & + \left(\frac{2}{a} + \frac{1}{b} + \frac{c+d}{ab} + \frac{cd-be}{a^2b}\right)PQP + \left(\frac{1}{a} + \frac{2}{b} + \frac{c+d}{ab} + \frac{cd-af}{ab^2}\right)QPQ \\ & - \left(\frac{2}{a} + \frac{2}{b} + \frac{c+d}{ab} + \frac{cd-be}{a^2b} + \frac{cd-af}{ab^2}\right)PQPQ, \end{aligned} \tag{2.49}$$

and so

$$\begin{aligned} A - A_g = {} & \left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)Q + \left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ \\ & + \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP + \left(e - \frac{2}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{cd-be}{a^2b}\right)PQP \\ & + \left(f - \frac{1}{a} - \frac{2}{b} - \frac{c+d}{ab} - \frac{cd-af}{ab^2}\right)QPQ \\ & + \left(g + \frac{2}{a} + \frac{2}{b} + \frac{c+d}{ab} + \frac{cd-be}{a^2b} + \frac{cd-af}{ab^2}\right)PQPQ = 0. \end{aligned} \tag{2.50}$$

Analogous to the process in (c)(1), using (2.50), we can get

$$\begin{aligned} & \left(a - \frac{1}{a}\right)(PQP - PQPQ) = 0, \\ & \left(a - \frac{1}{a}\right)(QPQ - PQPQ) = 0. \end{aligned} \tag{2.51}$$

Thus, since $PQP \neq QPQ$, $PQP \neq PQPQ$ and/or $QPQ \neq PQPQ$ and then $a = a^{-1}$. Similarly, $b = b^{-1}$. Therefore, multiplying (2.50) on the right side by Q and on the left side by P yields

$$\left(a + b + c + \frac{c}{ab}\right)(PQ - PQPQ) = 0. \tag{2.52}$$

Multiplying (2.50) on the right side by P and on the left side by Q yields

$$\left(a + b + d + \frac{d}{ab}\right)(QP - PQPQ) = 0. \tag{2.53}$$

Since $PQ \neq PQPQ$ and $QP \neq PQPQ$, $a + b + c + c/ab = 0$ and $a + b + d + d/ab = 0$. Multiplying (2.50) on the left side, respectively, by P and by Q yields

$$\begin{aligned} &\left(2e + b + c + d - \frac{cd}{b}\right)(PQP - PQPQ) = 0, \\ &\left(2f + a + c + d - \frac{cd}{a}\right)(QPQ - PQPQ) = 0. \end{aligned} \tag{2.54}$$

Thus, we have $2e + b + c + d - cd/b = 0$ and $QPQ = PQPQ$; $2f + a + c + d - cd/a = 0$ and $PQP = PQPQ$; $2e + b + c + d - cd/b = 0$ and $2f + a + c + d - cd/a = 0$.

Note that $2e + b + c + d - cd/b = 0$ and $2f + a + c + d - cd/a = 0$ imply $g = -(a + b)$ by $\theta = 0$. As the argument above in (c)(1), we have $(c_{13}) \sim (c_{18})$. □

Remark 2.2. Clearly, [15, (a) and (b) in Theorem] are the special cases in Theorem 2.1.

Example 2.3. Let

$$P = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&-1&1&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}. \tag{2.55}$$

Then they, obviously, are idempotent, and $(PQ)^2 = (QP)^2$ but $PQP \neq QPQ$. By Theorem 2.1(c_5),

$$A = P - Q + 2PQ + 2QP - \frac{7}{2}PQP - \frac{1}{2}QPQ + PQPQ \tag{2.56}$$

is the group involutory matrix, namely, $A = A_g$, since $2 + 2 + 2 * (-7/2) + 2 * 2 = 1$ and $2 + 2 + 2 * (-1/2) - 2 * 2 = -1$. By Theorem 2.1($c_{17}$),

$$P - Q + PQ - 2QP + 2PQP - QPQ \tag{2.57}$$

is group involutory since $1 - 2 + 2 * 2 + 1 * (-2) = 1$ and $1 - 2 + 2 * (-1) - 1 * (-2) = -1$.

Next, we will study the situation $(PQ)^2 = 0$ or $(QP)^2 = 0$.

Theorem 2.4. *Let $P, Q \in \mathbb{C}^{n\times n}$ be two different nonzero idempotent matrices, and let A be a combination of the form*

$$A = aP + bQ + cPQ + dQP + ePQP + fQPQ + gPQPQ, \tag{2.58}$$

where $a, b, c, d, e, f, g \in \mathbb{C}$ with $a, b \neq 0$. Suppose that

$$PQPQ \neq 0, \qquad QPQP = 0, \tag{2.59}$$

and any of the following sets of additional conditions hold:

(d_1) $a = b = \pm 1$, $c = d = \mp 1$, $e = f = \pm 1$, $g = \mp 1$;

(d_2) $a = \pm 1$, $b = \mp 1$, $2e + c + d \pm cd = \pm 1$, $2f + c + d \mp cd = \mp 1$.

Then A is the group involutory matrix.

Proof. By Lemma 1.2,

$$\begin{aligned}
0 &= A - A_g \\
&= \left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)Q + \left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ + \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP \\
&\quad + \left(e - \frac{2}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{cd - be}{a^2 b}\right)PQP \\
&\quad + \left(f - \frac{1}{a} - \frac{2}{b} - \frac{c+d}{ab} - \frac{cd - af}{ab^2}\right)QPQ \\
&\quad + \left(g + \frac{2}{a} + \frac{2}{b} + \frac{2c + d + g}{ab} + \frac{cd - be - ce}{a^2 b} + \frac{cd - af - cf}{ab^2} + \frac{c^2 d}{a^2 b^2}\right)(PQ)^2.
\end{aligned} \tag{2.60}$$

Since $PQPQ \neq 0$, multiplying (2.60), respectively, on the right side and on the right side by $PQPQ$ yields

$$\left(a - \frac{1}{a}\right)PQPQ = 0, \qquad \left(b - \frac{1}{b}\right)PQPQ = 0, \tag{2.61}$$

and so $a = a^{-1}$ and $b = b^{-1}$. Substituting them inside (2.60), we get

$$\begin{aligned} 0 = &\left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ + \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP \\ &+ \left(e - \frac{2}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{cd-be}{a^2b}\right)PQP \\ &+ \left(f - \frac{1}{a} - \frac{2}{b} - \frac{c+d}{ab} - \frac{cd-af}{ab^2}\right)QPQ \\ &+ \left(g + \frac{2}{a} + \frac{2}{b} + \frac{2c+d+g}{ab} + \frac{cd-be-ce}{a^2b} + \frac{cd-af-cf}{ab^2} + \frac{c^2d}{a^2b^2}\right)PQPQ. \end{aligned} \tag{2.62}$$

Multiplying (2.62) on the left side by PQP yields

$$\left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQPQ = 0, \tag{2.63}$$

and then

$$c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} = 0. \tag{2.64}$$

So (2.62) becomes

$$\begin{aligned} 0 = &\left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP + \left(e - \frac{2}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{cd-be}{a^2b}\right)PQP \\ &+ \left(f - \frac{1}{a} - \frac{2}{b} - \frac{c+d}{ab} - \frac{cd-af}{ab^2}\right)QPQ \\ &+ \left(g + \frac{2}{a} + \frac{2}{b} + \frac{2c+d+g}{ab} + \frac{cd-be-ce}{a^2b} + \frac{cd-af-cf}{ab^2} + \frac{c^2d}{a^2b^2}\right)PQPQ. \end{aligned} \tag{2.65}$$

Multiplying (2.65) on the left side by PQ and on the right side by P yields

$$\left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)PQPQ = 0. \tag{2.66}$$

Therefore,

$$d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} = 0. \tag{2.67}$$

Similarly, we can obtain

$$\begin{aligned} 0 &= e - \frac{2}{a} - \frac{1}{b} - \frac{c+d}{ab} - \frac{cd-be}{a^2b}, \\ 0 &= f - \frac{1}{a} - \frac{2}{b} - \frac{c+d}{ab} - \frac{cd-af}{ab^2}, \\ 0 &= g + \frac{2}{a} + \frac{2}{b} + \frac{2c+d+g}{ab} + \frac{cd-be-ce}{a^2b} + \frac{cd-af-cf}{ab^2} + \frac{c^2d}{a^2b^2}. \end{aligned} \tag{2.68}$$

By (2.64) and (2.67), we can obtain

$$\frac{1}{b} + c + d + 2e - \frac{cd}{b} = 0, \qquad \frac{1}{a} + c + d + 2f - \frac{cd}{a} = 0. \tag{2.69}$$

Since $a = a^{-1}$ and $b = b^{-1}$, $a = \pm b$. If $a = -b$, then (2.64) holds for any c, (2.67) holds for any d, and, for any $c,\ d,\ e,\ f$ satisfying (2.69) and any g,

$$\begin{aligned} g &+ \frac{2}{a} + \frac{2}{b} + \frac{2c+d+g}{ab} + \frac{cd-be-ce}{a^2b} + \frac{cd-af-cf}{ab^2} + \frac{c^2d}{a^2b^2} \\ &= c^2d - 2c - d - (e+f) + \frac{c}{a}(e-f) \\ &= c^2d - 2c - d + (c+d) + \frac{c}{a}\left(\frac{1}{a} - \frac{cd}{a}\right) = 0. \end{aligned} \tag{2.70}$$

If $a = b$, then, by (2.64) ~ (2.69), $c = d = -a$ and $e = f = a$ and so $g = -a$ from (2.68).

Hence, we have (d_1) and (d_2). □

Example 2.5. Let

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix}. \tag{2.71}$$

Obviously they are idempotent, and $(QP)^2 = 0$ but $(PQ)^2 \neq 0$. By Theorem 2.4(d_2),

$$P - Q + 2PQ - 2QP + \frac{5}{2}PQP - \frac{5}{2}QPQ - 2PQPQ \tag{2.72}$$

is group involutory since $2 - 2 + 2 * (5/2) + 2 * (-2) = 1$ and $2 - 2 + 2 * (-5/2) - 2 * (-2) = -1$.

Similarly, we have the following result.

Theorem 2.6. *Let $P, Q \in \mathbb{C}^{n\times n}$ be two different nonzero idempotent matrices, and let A be a combination of the form*

$$A = aP + bQ + cPQ + dQP + ePQP + fQPQ + hQPQP, \tag{2.73}$$

where $a, b, c, d, e, f, h \in \mathbb{C}$ with $a, b \neq 0$. Suppose that

$$QPQP \neq 0, \qquad PQPQ = 0, \tag{2.74}$$

and any of the following sets of additional conditions hold:

(e_1) $a = b = \pm 1$, $c = d = \mp 1$, $e = f = \pm 1$, $h = \mp 1$;

(e_2) $a = \pm 1$, $b = \mp 1$, $2e + c + d \pm cd = \pm 1$, $2f + c + d \mp cd = \mp 1$.

Then A is the group involutory matrix.

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