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Research Article

The Generalized Order-k Lucas Sequences in Finite Groups

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We study the generalized order-k Lucas sequences modulo m. Also, we define the ith generalized order-k Lucas orbit $l_A^{i,\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}}(G)$ with respect to the generating set A and the constants α_1, α_2, and α_{k-1} for a finite group $G = \langle A \rangle$. Then, we obtain the lengths of the periods of the ith generalized order-k Lucas orbits of the binary polyhedral groups $\langle n,2,2\rangle$, $\langle 2,n,2\rangle$, $\langle 2,2,n\rangle$ and the polyhedral groups $(n,2,2)$, $(2,n,2)$, $(2,2,n)$ for $1 \le i \le k$.

1. Introduction

The well-known Fibonacci sequence $\{F_n\}$ is defined as

$$F_1 = F_2 = 1, \quad \text{for } n > 2, \quad F_n = F_{n-1} + F_{n-2}. \tag{1.1}$$

We call F_n the nth Fibonacci number. The Fibonacci sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots. \tag{1.2}$$

Definition 1.1. Let $f_n^{(k)}$ denote the nth member of the k-step Fibonacci sequence defined as

$$f_n^{(k)} = \sum_{j=1}^{k} f_{n-j}^{(k)} \quad \text{for } n > k \tag{1.3}$$

with boundary conditions $f_i^{(k)} = 0$ for $1 \leq i < k$ and $f_k^{(k)} = 1$. Reducing this sequence by modulus m, we can get a repeating sequence, which we denote by

$$f(k,m) = \left(f_1^{(k,m)}, f_2^{(k,m)}, \ldots, f_n^{(k,m)} \ldots\right), \tag{1.4}$$

where $f_i^{(k,m)} = f_i^{(k)} (\bmod m)$. We then have that $(f_1^{(k,m)}, f_2^{(k,m)}, \ldots, f_k^{(k,m)}) = (0, 0, \ldots 0, 1)$ and it has the same recurrence relation as in (1.3) [1].

Theorem 1.2. *$f(k,m)$ is a periodic sequence [1].*

Let $h_k(m)$ denote the smallest period of $f(k,m)$, called the period of $f(k,m)$ or the Wall number of the k-step Fibonacci sequence modulo m. For more information see [1].

Definition 1.3. Let $h_{k(a_1,a_2,\ldots,a_k)}(m)$ denote the smallest period of the integer-valued recurrence relation $u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-k}$, $u_1 = a_1$, $u_2 = a_2, \ldots,$ $u_k = a_k$ when each entry is reduced modulo m [2].

Lemma 1.4. *For $a_1, a_2, \ldots, a_k, x_1, x_2, \ldots, x_k \in \mathbb{Z}$ with $m > 0$, $a_1, a_2, \ldots, a_k$ not all congruent to zero modulo m and $x_1, x_2, \ldots, x_k$ not all congruent to zero modulo m,*

$$h_{k(a_1,a_2,\ldots,a_k)}(m) = h_{k(x_1,x_2,\ldots,x_k)}(m), \tag{1.5}$$

see [2].

In [3], Taşçı and Kılıç defined the k sequences of the generalized order-k Lucas numbers as follows:

$$l_n^i = \sum_{j=1}^{k} l_{n-j}^i, \tag{1.6}$$

for $n > 0$ and $1 \leq i \leq k$, with boundary (initial) conditions

$$l_n^i = \begin{cases} 2 & \text{if } i = 2 - n, \\ -1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

for $1 - k \leq n \leq 0$, where l_n^i is the nth term of the ith sequence. When $i = 1$ and $k = 2$, the generalized order-k Lucas sequence reduces to the usual negative Fibonacci sequence, that is, $l_n^1 = -F_{n+1}$ for all $n \in \mathbb{Z}^+$.

In [3], it is obtained that

$$\begin{bmatrix} l_{n+1}^{i} \\ l_{n}^{i} \\ l_{n-1}^{i} \\ \vdots \\ l_{n-k+2}^{i} \end{bmatrix} = A \begin{bmatrix} l_{n}^{i} \\ l_{n-1}^{i} \\ l_{n-2}^{i} \\ \vdots \\ l_{n-k+1}^{i} \end{bmatrix}, \tag{1.8}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.9}$$

The Lucas sequence, the generalized Lucas sequence, and their properties have been studied by several authors; see for example, [4–9]. The study of the Fibonacci sequences in groups began with the earlier work of Wall [10]. Knox examined the k-nacci (k-step Fibonacci) sequences in finite groups [11]. Karaduman and Aydin examined the periods of the 2-step general Fibonacci sequences in dihedral groups D_n [12]. Lü and Wang contributed to the study of the Wall number for the k-step Fibonacci sequence [1]. C. M. Campbell and P. P. Campbell examined the behaviour of the Fibonacci lengths of finite binary polyhedral groups [13]. Also, Deveci et al. obtained the periods of the k-nacci sequences in finite binary polyhedral groups [14]. Now, we extend the concept to k sequences of the generalized order-k Lucas numbers and we examine the periods of the ith generalized order-k Lucas orbits of the binary polyhedral groups $\langle n,2,2\rangle$, $\langle 2,n,2\rangle$, $\langle 2,2,n\rangle$ and the polyhedral groups $(n,2,2)$, $(2,n,2)$, $(2,2,n)$ for $1 \le i \le k$.

In this paper, the usual notation p is used for a prime number.

2. Main Results and Proofs

Reducing the k sequences of the generalized order-k Lucas numbers by modulus m, we can get a repeating sequence denoted by

$$l(i,m) = \left(\ldots, l_1^{(i,m)}, l_2^{(i,m)}, \ldots, l_n^{(i,m)}, \ldots\right) \quad \text{for } n > 0,\ 1 \le i \le k, \tag{2.1}$$

where $l_n^{(i,m)} = l_n^i (\text{mod } m)$. It has the same recurrence relation as that in (1.6).

Let the notation $hl_k^i(m)$ denote the smallest period of $l(i,m)$. It is easy to see from Lemma 1.4 that $h_k(m) = hl_k^i(m)$.

For a given matrix $M = [b_{ij}]$ with b_{ij}'s being integers, $M(\text{mod}\, m)$ means that every entry of M is reduced modulo m, that is, $M(\text{mod}\, m) = (b_{ij}(\text{mod}\, m))$.

Let $\langle A\rangle_{p^\alpha} = \{A^i (\text{mod } p^\alpha) \mid i \geq 0\}$ be a cyclic group, and let $|\langle A\rangle_{p^\alpha}|$ denote the order of $\langle A\rangle_{p^\alpha}$. Then, we have the following.

Theorem 2.1. $h_k(p^\alpha) = |\langle A\rangle_{p^\alpha}|$.

Proof. It is clear that $|\langle A\rangle_{p^\alpha}|$ is divisible by $h_k(p^\alpha)$. Then we need only to prove that $h_k(p^\alpha)$is divisible by $|\langle A\rangle_{p^\alpha}|$. Let $h_k(p^\alpha) = \lambda$. Then we have

$$A^\lambda = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}. \tag{2.2}$$

By mathematical induction it is easy to prove that the elements of the matrix A^λ are in the following forms:

$$\begin{array}{lllll} a_{11} = f^{(k)}_{\lambda+k}, & a_{12} = f^{(k)}_{\lambda+k-1} + \cdots + f^{(k)}_{\lambda+1}, & a_{13} = f^{(k)}_{\lambda+k-1} + \cdots + f^{(k)}_{\lambda+2}, \ldots, & a_{1k-1} = f^{(k)}_{\lambda+k-1} + f^{(k)}_{\lambda+k-2}, & a_{1k} = f^{(k)}_{\lambda+k-1}, \\ a_{21} = f^{(k)}_{\lambda+k-1}, & a_{22} = f^{(k)}_{\lambda+k-2} + \cdots + f^{(k)}_{\lambda}, & a_{23} = f^{(k)}_{\lambda+k-2} + \cdots + f^{(k)}_{\lambda+1}, \ldots, & a_{1k-1} = f^{(k)}_{\lambda+k-2} + f^{(k)}_{\lambda+k-3}, & a_{2k} = f^{(k)}_{\lambda+k-2}, \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} = f^{(k)}_{\lambda+1}, & a_{k2} = f^{(k)}_{\lambda} + \cdots + f^{(k)}_{\lambda-k+2}, & a_{k3} = f^{(k)}_{\lambda} + \cdots + f^{(k)}_{\lambda-k+3}, \ldots, & a_{kk-1} = f^{(k)}_{\lambda} + f^{(k)}_{\lambda-1}, & a_{kk} = f^{(k)}_{\lambda}. \end{array} \tag{2.3}$$

We thus obtain that

$$\begin{gathered} a_{ii} \equiv 1 (\text{mod } p^\alpha) \quad \text{for } 1 \leq i \leq k, \\ a_{ij} \equiv 0 (\text{mod } p^\alpha) \quad \text{for } 1 \leq i, j \leq k \text{ such that } i \neq j. \end{gathered} \tag{2.4}$$

So we get that $A^\lambda \equiv I(\text{mod } p^\alpha)$, which yields that n is divisible by $|\langle A\rangle_{p^\alpha}|$. We are done. □

Definition 2.2. Let G be a finitely generated group $G = \langle A\rangle$, where $A = \{a_1, a_2, \ldots, a_k\}$ and $1 \leq i \leq k$. The sequence

$$x_0 = (a_1)_{\alpha_1}, \qquad x_1 = (a_2)_{\alpha_2}, \ldots, \qquad x_{k-2} = (a_{k-1})_{\alpha_{k-1}}, \tag{2.5}$$

where

$$(a_u)_{\alpha_u} = \begin{cases} a_u a_k^{l^i_{u-k}} & \text{if } \alpha_u = 1, \\ a_k^{l^i_{u-k}} a_u & \text{if } \alpha_u = 2 \end{cases} \tag{2.6}$$

such that $1 \le u \le k-1$ and $1 \le \alpha_u \le 2$, $x_{k-1} = a_k^{l_0^i}$, $x_{k+\beta} = \prod_{j=1}^{k} x_{\beta+j-1}$ for $\beta \ge 0$, is called the ith generalized order-k Lucas orbit of G with respect to the generating set A and the $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ constants, denoted by $l_A^{i,\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}}(G)$.

Example 2.3. The 3rd generalized order-4 Lucas orbits $l_{\{a_1,a_2,a_3,a_4\}}^{3,\{1,1,1\}}(G)$, $l_{\{a_1,a_2,a_3,a_4\}}^{3,\{1,2,1\}}(G)$, $l_{\{a_1,a_2,a_3,a_4\}}^{3,\{1,1,2\}}(G)$, $l_{\{a_1,a_2,a_3,a_4\}}^{3,\{1,2,2\}}(G)$, $l_{\{a_1,a_2,a_3,a_4\}}^{3,\{2,1,1\}}(G)$, $l_{\{a_1,a_2,a_3,a_4\}}^{3,\{2,2,1\}}(G)$, $l_{\{a_1,a_2,a_3,a_4\}}^{3,\{2,1,2\}}(G)$, and $l_{\{a_1,a_2,a_3,a_4\}}^{3,\{2,2,2\}}(G)$ of the finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, a_3, a_4\}$, respectively, are as follows:

$$\begin{gathered}
x_0 = a_1 a_4^{l_{-3}^3} = a_1,\ x_1 = a_2 a_4^{l_{-2}^3} = a_2 a_4^{-1},\ x_2 = a_3 a_4^{l_{-1}^3} = a_3 a_4^2,\ x_3 = a_4^{l_0^3} = e,\ x_{4+\beta} = \prod_{j=1}^{4} x_{\beta+j-1} \\
(\beta \ge 0), \\
x_0 = a_1 a_4^{l_{-3}^3} = a_1,\ x_1 = a_4^{l_{-2}^3} a_2 = a_4^{-1} a_2,\ x_2 = a_3 a_4^{l_{-1}^3} = a_3 a_4^2,\ x_3 = a_4^{l_0^3} = e,\ x_{4+\beta} = \prod_{j=1}^{4} x_{\beta+j-1} \\
(\beta \ge 0), \\
x_0 = a_1 a_4^{l_{-3}^3} = a_1,\ x_1 = a_2 a_4^{l_{-2}^3} = a_2 a_4^{-1},\ x_2 = a_4^{l_{-1}^3} a_3 = a_4^2 a_3,\ x_3 = a_4^{l_0^3} = e,\ x_{4+\beta} = \prod_{j=1}^{4} x_{\beta+j-1} \\
(\beta \ge 0), \\
x_0 = a_1 a_4^{l_{-3}^3} = a_1,\ x_1 = a_4^{l_{-2}^3} a_2 = a_4^{-1} a_2,\ x_2 = a_4^{l_{-1}^3} a_3 = a_4^2 a_3,\ x_3 = a_4^{l_0^3} = e,\ x_{4+\beta} = \prod_{j=1}^{4} x_{\beta+j-1} \\
(\beta \ge 0), \\
x_0 = a_4^{l_{-3}^3} a_1 = a_1,\ x_1 = a_2 a_4^{l_{-2}^3} = a_2 a_4^{-1},\ x_2 = a_3 a_4^{l_{-1}^3} = a_3 a_4^2,\ x_3 = a_4^{l_0^3} = e,\ x_{4+\beta} = \prod_{j=1}^{4} x_{\beta+j-1} \\
(\beta \ge 0), \\
x_0 = a_4^{l_{-3}^3} a_1 = a_1,\ x_1 = a_4^{l_{-2}^3} a_2 = a_4^{-1} a_2,\ x_2 = a_3 a_4^{l_{-1}^3} = a_3 a_4^2,\ x_3 = a_4^{l_0^3} = e,\ x_{4+\beta} = \prod_{j=1}^{4} x_{\beta+j-1} \\
(\beta \ge 0), \\
x_0 = a_4^{l_{-3}^3} a_1 = a_1,\ x_1 = a_2 a_4^{l_{-2}^3} = a_2 a_4^{-1},\ x_2 = a_4^{l_{-1}^3} a_3 = a_4^2 a_3,\ x_3 = a_4^{l_0^3} = e,\ x_{4+\beta} = \prod_{j=1}^{4} x_{\beta+j-1} \\
(\beta \ge 0), \\
x_0 = a_4^{l_{-3}^3} a_1 = a_1,\ x_1 = a_4^{l_{-2}^3} a_2 = a_4^{-1} a_2,\ x_2 = a_4^{l_{-1}^3} a_3 = a_4^2 a_3,\ x_3 = a_4^{l_0^3} = e,\ x_{4+\beta} = \prod_{j=1}^{4} x_{\beta+j-1} \\
(\beta \ge 0).
\end{gathered} \tag{2.7}$$

It is well known that a sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence.

Theorem 2.4. *The ith generalized order-k Lucas orbits in a finite group are periodic.*

Proof. The proof is similar to the proof of Theorem 1 in [10] and is omitted.

We denote the length of the period of the sequence $l_A^{i,\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}}(G)$ by $\mathrm{LEN}_A l^{i,\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}}(G)$ and call it the ith generalized order-k Lucas length of G with respect to the generating set A and the constants $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$.

From the definition it is clear that the ith generalized order-k Lucas length of a group depends on the chosen generating set and the order in which the assignments of $x_0, x_1, \ldots x_{n-1}$ are made.

We will now address the ith generalized order-k Lucas lengths in specific classes of groups.

The binary polyhedral group $\langle l, m, n\rangle$, for $l, m, n > 1$, is defined by the presentation

$$\left\langle x, y, z \mid x^l = y^m = z^n = xyz \right\rangle \tag{2.8}$$

or

$$\left\langle x, y \mid x^l = y^m = (xy)^n \right\rangle. \tag{2.9}$$

The binary polyhedral group $\langle l, m, n\rangle$ is finite if, and only if, the number $k = lmn(1/l + 1/m + 1/n - 1) = mn + nl + lm - lmn$ is positive. Its order is $4lmn/k$.

For more information on these groups, see [15, pages 68–71].

The polyhedral group (l, m, n), for $l, m, n > 1$, is defined by the presentation

$$\left\langle x, y, z \mid x^l = y^m = z^n = xyz = e \right\rangle \tag{2.10}$$

or

$$\left\langle x, y \mid x^l = y^m = (xy)^n = e \right\rangle. \tag{2.11}$$

The polyhedral group (l, m, n) is finite if, and only if, the number $k = lmn(1/l + 1/m + 1/n - 1) = mn + nl + lm - lmn$ is positive. Its order is $2lmn/k$.

For more information on these groups, see [15, pages 67-68]. □

Theorem 2.5. *The ith generalized order-3 Lucas lengths of the binary polyhedral group $\langle n, 2, 2\rangle$ for every i integer such that $1 \le i \le 3$ and the generating triple $\{x, y, z\}$ are as follows:*

(i) $\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}(\langle n, 2, 2\rangle) = 8$ *for* $1 \le \alpha_1, \alpha_2 \le 2$,

(ii) $\mathrm{LEN}_{\{x,y,z\}} l^{2,\{\alpha_1,\alpha_2\}}(\langle n, 2, 2\rangle) = h_3(2n)$ *for* $1 \le \alpha_1, \alpha_2 \le 2$,

(iii) (1) $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{1,1\}}(\langle n,2,2\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{3,\{1,2\}}(\langle n,2,2\rangle) = 8$,
(2) $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,1\}}(\langle n,2,2\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,2\}}(\langle n,2,2\rangle) = 4n$ *if* n *is even*, $8n$ *if* n *is odd.*

Proof. We prove the result by direct calculation. We first note that in the group defined by $\langle x,y,z \mid x^n = y^2 = z^2 = xyz\rangle, |x| = 2n$ (where $|x|$ means the order of x), $|y| = 4$, $|z| = 4$, $x = zy^3$, $y = x^{-1}z$, and $z = xy$.

(i) The 1st generalized order-3 Lucas orbits of the group $\langle n,2,2\rangle$ for generating triple $\{x,y,z\}$ and every constant α_1,α_2 such that $1 \le \alpha_1,\alpha_2 \le 2$ are the same and are as follows:

$$\begin{gathered} x_0 = x, x_1 = y, x_2 = z^{-1}, x_3 = e, x_4 = yz^{-1}, x_5 = xz^{-1}, x_6 = z^{-1}, \\ x_7 = x^n, x_8 = x, x_9 = y, x_{10} = z^{-1}, \dots. \end{gathered} \tag{2.12}$$

Since the elements succeeding x_8, x_9, and x_{10} depend on x, y, and z^{-1} for their values, the cycle is again the 8th element; that is, $x_0 = x_8$, $x_1 = x_9$, $x_2 = x_{10}, \dots$. Thus, $\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}(\langle n,2,2\rangle) = 8$ for $1 \le \alpha_1,\alpha_2 \le 2$.

(ii) Firstly, let us consider the orbits $l^{2,\{1,1\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ and $l^{2,\{2,1\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$. The orbits $l^{2,\{1,1\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ and $l^{2,\{2,1\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ are the same and are as follows:

$$\begin{gathered} x_0 = x, x_1 = x^{-1}, x_2 = z^2, x_3 = z^2, x_4 = x^{-1}, x_5 = x^{-1}, x_6 = x^{-2}z^2, \\ x_7 = x^{-4}z^2, x_8 = x^{-7}, x_9 = x^{-13}, x_{10} = x^{-24}z^2, x_{11} = x^{-44}z^2, \dots. \end{gathered} \tag{2.13}$$

For $m > 3$ we can see that the sequence will separate into some natural layers and each layer will be of the form

$$x_m = \begin{cases} x^{u_m} & \text{if } m \equiv 0 (\mathrm{mod}\ 4), \\ x^{u_m} & \text{if } m \equiv 1 (\mathrm{mod}\ 4), \\ x^{u_m} z^2 & \text{if } m \equiv 2 (\mathrm{mod}\ 4), \\ x^{u_m} z^2 & \text{if } m \equiv 3 (\mathrm{mod}\ 4), \end{cases} \tag{2.14}$$

where

$$u_m = u_{m-3} + u_{m-2} + u_{m-1}, \qquad u_0 = 1, \qquad u_1 = -1, \qquad u_2 = 0. \tag{2.15}$$

Now the proof is finished when we note that the sequence will repeat when $x_{h_3(2n)} = x$, $x_{h_3(2n)+1} = x^{-1}$, and $x_{h_3(2n)+2} = z^2$, where $h_3(2n)$ is the 3-step Wall number of the positive integer $2n$ and $h_3(2n) = 4\mu$ ($\mu \in N$). Letting $L = \mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,1\}}(\langle n,2,2\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{2,\{2,1\}}(\langle n,2,2\rangle)$, we have

$$x_L = x^{u_L}, x_{L+1} = x^{u_{L+1}}, x_{L+2} = x^{u_{L+2}} z^2. \tag{2.16}$$

Using Lemma 1.4, we obtain $u_L = u_0 = 1,\ u_{L+1} = u_1 = -1$, and $u_{L+2} = u_2 = 0$. In this case the above equalities give

$$x_L = x^{u_L} = x, x_{L+1} = x^{u_{L+1}} = x^{-1}, x_{L+2} = x^{u_{L+2}}z^2 = x^0z^2 = z^2. \tag{2.17}$$

The smallest nontrivial integer satisfying the above conditions occurs when the period is $h_3(2n)$.

Secondly, let us consider the orbits $l^{2,\{1,2\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ and $l^{2,\{2,2\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$. The orbits $l^{2,\{1,2\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ and $l^{2,\{2,2\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ are the same and are as follows:

$$\begin{gathered} x_0 = x, x_1 = x, x_2 = z^2, x_3 = x^2z^2, x_4 = x^3, x_5 = x^5, x_6 = x^{10}z^2, \\ x_7 = x^{18}z^2, x_8 = x^{33}, x_9 = x^{61}, x_{10} = x^{112}z^2,\ x_{11} = x^{206}z^2,\ \ldots. \end{gathered} \tag{2.18}$$

For $m > 3$ we can see that the sequence will separate into some natural layers and each layer will be of the form

$$x_m = \begin{cases} x^{v_m} & \text{if } m \equiv 0 (\text{mod } 4), \\ x^{v_m} & \text{if } m \equiv 1 (\text{mod } 4), \\ x^{v_m}z^2 & \text{if } m \equiv 2 (\text{mod } 4), \\ x^{v_m}z^2 & \text{if } m \equiv 3 (\text{mod } 4), \end{cases} \tag{2.19}$$

where

$$v_m = v_{m-3} + v_{m-2} + v_{m-1}, \qquad v_0 = 1, \qquad v_1 = 1, \qquad v_2 = 0. \tag{2.20}$$

Now the proof is finished when we note that the sequence will repeat when $x_{h_3(2n)} = x$, $x_{h_3(2n)+1} = x$ and $x_{h_3(2n)+2} = z^2$. Letting $L = \text{LEN}_{\{x,y,z\}} l^{2,\{1,2\}}(\langle n,2,2\rangle) = \text{LEN}_{\{x,y,z\}} l^{2,\{2,2\}}(\langle n,2,2\rangle)$, we have

$$x_L = x^{v_L}, x_{L+1} = x^{v_{L+1}}, x_{L+2} = x^{v_{L+2}}z^2. \tag{2.21}$$

Using Lemma 1.4, we obtain $v_L = v_0 = 1,\ v_{L+1} = v_1 = 1$, and $v_{L+2} = v_2 = 0$. In this case the above equalities give

$$x_L = x^{v_L} = x, x_{L+1} = x^{v_{L+1}} = x, x_{L+2} = x^{v_{L+2}}z^2 = x^0z^2 = z^2. \tag{2.22}$$

The smallest nontrivial integer satisfying the above conditions occurs when the period is $h_3(2n)$.

(iii) (1) The orbits $l^{3,\{1,1\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ and $l^{3,\{1,2\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ are the same and are as follows:

$$\begin{gathered} x_0 = xz^{-1}, x_1 = y^3, x_2 = e, x_3 = x^{n+2},\ x_4 = yx^2, \\ x_5 = y^3, x_6 = x^n, x_7 = x^{n-2}, x_8 = xz^{-1}, x_9 = y^3, x_{10} = e, \ldots. \end{gathered} \tag{2.23}$$

So, we get $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{1,1\}}(\langle n,2,2\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{3,\{1,2\}}(\langle n,2,2\rangle) = 8$.

(2) The orbits $l^{3,\{2,1\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ and $l^{3,\{2,2\}}_{\{x,y,z\}}(\langle n,2,2\rangle)$ are the same and are as follows:

$$\begin{gathered} x_0 = y^3, x_1 = x^{n+1}, x_2 = e, x_3 = yx, x_4 = x^{-1}, x_5 = x^{-2}, x_6 = y^3x^{-3}, \\ x_7 = x^{-3}yx^{-3}, x_8 = y^3, x_9 = x^{n+1}, x_{10} = x^4, x_{11} = yx^5, x_{12} = y^3, \\ x_{13} = x^{-1}, x_{14} = x^{-6}, x_{15} = y^3x^{-7}, x_{16} = y^3, x_{17} = x^{n+1}, x_{18} = x^8, \ldots. \end{gathered} \tag{2.24}$$

The sequence can be said to form layers of length eight. Using the above, the sequence becomes

$$\begin{gathered} x_0 = y^3, x_1 = x^{n+1}, x_2 = e, \ldots, \\ x_8 = y^3, x_9 = x^{n+1}, x_{10} = x^4, \ldots, \\ x_{16} = y^3, x_{17} = x^{n+1}, x_{18} = x^8, \ldots, \\ x_{8i} = y^3, x_{8i+1} = x^{n+1}, x_{8i+2} = x^{4i}, \ldots. \end{gathered} \tag{2.25}$$

So we need the smallest $i \in \mathbb{N}$ such that $4i = 2nk$ for $k \in \mathbb{N}$.

If n is even, then $i = n/2$. Thus, $8i = 4n$ and $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,1\}}(\langle n,2,2\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,2\}}(\langle n,2,2\rangle) = 4n$.

If n is odd, then $i = n$. Thus, $8i = 8n$ and $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,1\}}(\langle n,2,2\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,2\}}(\langle n,2,2\rangle) = 8n$. □

Theorem 2.6. *The ith generalized order-2 Lucas lengths of the binary polyhedral group $\langle n,2,2\rangle$ for every i such that $1 \le i \le 2$ and the generating pair $\{x,y\}$ are 6.*

Proof. We prove the result by direct calculation. We first note that in the group defined by

$$\left\langle x, y \mid x^n = y^2 = (xy)^2 \right\rangle, \quad |x| = 2n,\ |y| = 4,\ xy = yx^{-1},\ yx = x^{-1}y. \tag{2.26}$$

Firstly, let us consider the orbits $l^{1,\{1\}}_{\{x,y\}}(\langle n,2,2\rangle)$ and $l^{1,\{2\}}_{\{x,y\}}(\langle n,2,2\rangle)$. The orbits $l^{1,\{1\}}_{\{x,y\}}(\langle n,2,2\rangle)$ and $l^{1,\{2\}}_{\{x,y\}}(\langle n,2,2\rangle)$ are the same and are as follows:

$$x_0 = x, x_1 = y^{-1}, x_2 = xy^{-1}, x_3 = x^{n-1}, x_4 = x^2y, x_5 = y^{-1}x^{-1}, x_6 = x, x_7 = y^{-1}, \ldots. \tag{2.27}$$

So, we get $\mathrm{LEN}_{\{x,y\}} l^{1,\{1\}}(\langle n,2,2\rangle) = \mathrm{LEN}_{\{x,y\}} l^{1,\{2\}}(\langle n,2,2\rangle) = 6$.

Secondly, let us consider the orbit $l_{\{x,y\}}^{2,\{1\}}(\langle n,2,2\rangle)$. The orbit $l_{\{x,y\}}^{2,\{1\}}(\langle n,2,2\rangle)$ is as follows:

$$x_0 = xy^{-1}, x_1 = x^n, x_2 = xy, x_3 = xy^{-1}, x_4 = e, x_5 = xy^{-1}, x_6 = xy^{-1}, x_7 = x^n, \dots. \tag{2.28}$$

So, we get $\mathrm{LEN}_{\{x,y\}} l^{2,\{1\}}(\langle n,2,2\rangle) = 6$.

Thirdly, let us consider the orbit $l_{\{x,y\}}^{2,\{2\}}(\langle n,2,2\rangle)$. The orbit $l_{\{x,y\}}^{2,\{2\}}(\langle n,2,2\rangle)$ is as follows:

$$x_0 = y^{-1}x, x_1 = x^n, x_2 = yx, x_3 = y^{-1}x, x_4 = e, x_5 = y^{-1}x, x_6 = y^{-1}x, x_7 = x^n, \dots. \tag{2.29}$$

So, we get $\mathrm{LEN}_{\{x,y\}} l^{2,\{2\}}(\langle n,2,2\rangle) = 6$. □

Theorem 2.7. *The ith generalized order-3 Lucas lengths of the binary polyhedral group $\langle 2,n,2\rangle$ for every i integer such that $1 \le i \le 3$ and the generating triple $\{x,y,z\}$ are as follows:*

(i) $\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}\langle 2,n,2\rangle = 8$ *for* $1 \le \alpha_1, \alpha_2 \le 2$,

(ii) $\mathrm{LEN}_{\{x,y,z\}} l^{2,\{\alpha_1,\alpha_2\}}\langle 2,n,2\rangle = 8$ *for* $1 \le \alpha_1, \alpha_2 \le 2$,

(iii) $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{\alpha_1,\alpha_2\}}\langle 2,n,2\rangle = h_3(2n)$ *for* $1 \le \alpha_1, \alpha_2 \le 2$.

Proof. The proof is similar to the proof of Theorem 2.5 and is omitted. □

Theorem 2.8. *The ith generalized order-2 Lucas lengths of the binary polyhedral group $\langle 2,n,2\rangle$ for every i such that $1 \le i \le 2$ and the generating pair $\{x,y\}$ are 6.*

Proof. The proof is similar to the proof of Theorem 2.6 and is omitted. □

Theorem 2.9. *The ith generalized order-3 Lucas lengths of the binary polyhedral group $\langle 2,2,n\rangle$ for every i integer such that $1 \le i \le 3$ and the generating triple $\{x,y,z\}$ are as follows:*

(i)

$$\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}(\langle 2,2,n\rangle) = \begin{cases} 4n & \text{if } n \text{ is even,} \\ 8n & \text{if } n \text{ is odd} \end{cases} \quad \text{for } 1 \le \alpha_1, \alpha_2 \le 2, \tag{2.30}$$

(ii)

$$\mathrm{LEN}_{\{x,y,z\}} l^{2,\{\alpha_1,\alpha_2\}}(\langle 2,2,n\rangle) = \begin{cases} 2n & \text{if } n \equiv 0 \mod 4, \\ 4n & \text{if } n \equiv 2 \mod 4, \\ 8n & \text{otherwise} \end{cases} \quad \text{for } 1 \le \alpha_1, \alpha_2 \le 2, \tag{2.31}$$

(iii)

$$\mathrm{LEN}_{\{x,y,z\}} l^{3,\{\alpha_1,\alpha_2\}}(\langle 2,2,n\rangle) = 8 \quad \text{for } 1 \le \alpha_1, \alpha_2 \le 2 \tag{2.32}$$

Proof. We prove the result by direct calculation. We first note that in the group defined by $\langle x, y, z \mid x^2 = y^2 = z^n = xyz\rangle$, $|x| = 4$, $|y| = 4$, $|z| = 2n$, $x = yz$, $y = xz^{-1}$ and $z = yx^{-1}$.

(i) the 1st generalized order-3 Lucas orbits of the group $\langle 2, 2, n\rangle$ for generating triple $\{x, y, z\}$ and every constant α_1, α_2 such that $1 \leq \alpha_1, \alpha_2 \leq 2$ are the same and are as follows:

$$\begin{gathered} x_0 = x,\ x_1 = y, x_2 = z^{-1}, x_3 = yz^2y, x_4 = y^2z^3y, x_5 = y, x_6 = y^2z, \\ x_7 = y^2z^4, x_8 = xz^4, x_9 = y, x_{10} = z^{-1}, x_{11} = yz^6y, x_{12} = y^2z^7y, \\ x_{13} = y, x_{14} = y^2z, x_{15} = y^2z^8, x_{16} = xz^8, x_{17} = y, x_{18} = z^{-1}, \ldots. \end{gathered} \tag{2.33}$$

The sequence can be said to form layers of length eight. Using the above, the sequence becomes

$$\begin{gathered} x_0 = x, x_1 = y, x_2 = z^{-1}, \ldots, \\ x_8 = xz^4, x_9 = y, x_{10} = z^{-1}, \ldots, \\ x_{16} = xz^8, x_{17} = y, x_{18} = z^{-1}, \ldots, \\ x_{8i} = xz^{4i}, x_{8i+1} = y, x_{8i+2} = z^{-1}, \ldots. \end{gathered} \tag{2.34}$$

So, we need the smallest $i \in \mathbb{N}$ such that $4i = 2nk$ for $k \in \mathbb{N}$.

If n is even, then $i = n/2$. Thus, $8i = 4n$ and $\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}(\langle 2, 2, n\rangle) = 4n$ for $1 \leq \alpha_1, \alpha_2 \leq 2$.

If n is odd, then $i = n$. Thus, $8i = 8n$ and $\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}(\langle 2, 2, n\rangle) = 8n$ for $1 \leq \alpha_1, \alpha_2 \leq 2$.

(ii) The orbits $l^{2,\{1,1\}}_{\{x,y,z\}}(\langle 2, 2, n\rangle)$ and $l^{2,\{2,1\}}_{\{x,y,z\}}(\langle 2, 2, n\rangle)$ are the same and are as follows:

$$\begin{gathered} x_0 = x, x_1 = yz^{-1}, x_2 = z^2, x_3 = z^n, x_4 = xz^n, x_5 = z^2x,\ x_6 = xz^2x, \\ x_7 = xz^4x, x_8 = z^8x, x_9 = yz^{-1}, x_{10} = z^2, x_{11} = z^{n+8},\ x_{12} = xz^{n+8}, \\ x_{13} = z^2x, x_{14} = xz^2x, x_{15} = xz^{12}x, x_{16} = z^{16}x,\ x_{17} = yz^{-1}, x_{18} = z^2, \ldots. \end{gathered} \tag{2.35}$$

The sequence can be said to form layers of length eight. Using the above, the sequence becomes

$$\begin{gathered} x_0 = x, x_1 = yz^{-1}, x_2 = z^2, \ldots, \\ x_8 = z^8x, x_9 = yz^{-1}, x_{10} = z^2, \ldots, \\ x_{16} = z^{16}x, x_{17} = yz^{-1}, x_{18} = z^2,\ \ldots, \\ x_{8i} = z^{8i}x, x_{8i+1} = yz^{-1}, x_{8i+2} = z^2,\ \ldots. \end{gathered} \tag{2.36}$$

So, we need the smallest $i \in \mathbb{N}$ such that $4i = 2nk$ for $k \in \mathbb{N}$.

If $n \equiv 0 \bmod 4$, then $i = n/4$. Thus, $8i = 2n$ and $\mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,1\}}(\langle 2, 2, n\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,1\}}(\langle 2, 2, n\rangle) = 2n$.

If $n \equiv 2 \bmod 4$, then $i = n/2$. Thus, $8i = 4n$ and $\mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,1\}}(\langle 2,2,n\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,1\}}(\langle 2,2,n\rangle) = 4n$.

If $n \equiv 1 \bmod 4$ or $n \equiv 3 \bmod 4$, then $i = n$. Thus, $8i = 8n$ and $\mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,1\}}(\langle 2,2,n\rangle) = \mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,1\}}(\langle 2,2,n\rangle) = 8n$.

The orbits $l^{2,\{1,1\}}_{\{x,y,z\}}(\langle 2,2,n\rangle)$ and $l^{2,\{2,1\}}_{\{x,y,z\}}(\langle 2,2,n\rangle)$ are the same. The proofs for these orbits are similar to the above and are omitted.

(iii) The orbits $l^{3,\{1,1\}}(\langle 2,2,n\rangle)$, $l^{3,\{1,2\}}(\langle 2,2,n\rangle)$, $l^{3,\{2,1\}}(\langle 2,2,n\rangle)$, and $l^{3,\{2,2\}}(\langle 2,2,n\rangle)$, respectively, are as follows:

$$\begin{gathered}
x_0 = y, x_1 = xz, x_2 = e, x_3 = z^{n+2}, x_4 = xz^{n+3}, x_5 = xz, \\
x_6 = z^n, x_7 = xz^2x, x_8 = y, x_9 = xz, x_{10} = e, \dots, \\
x_0 = y, x_1 = z^2y, x_2 = e, x_3 = xzy, x_4 = z^4y^3, x_5 = z^2y, \\
x_6 = z^n, x_7 = z^{n+2}, x_8 = y, x_9 = z^2y, x_{10} = e, \dots, \\
x_0 = xz, x_1 = xz, x_2 = e, x_3 = z^n, x_4 = xz^{n+1}, x_5 = xz, \\
x_6 = z^n, x_7 = z^n, x_8 = xz, x_9 = xz, x_{10} = e, \dots, \\
x_0 = xz, x_1 = z^2y, x_2 = e, x_3 = yz^4y, x_4 = z^{n+6}y, x_5 = z^2y, \\
x_6 = z^n, x_7 = z^{n+4}, x_8 = xz, x_9 = z^2y, x_{10} = e, \dots,
\end{gathered} \tag{2.37}$$

which have period 8. □

Theorem 2.10. *The ith generalized order-2 Lucas lengths of the binary polyhedral group $\langle 2,2,n\rangle$ for every i integer such that $1 \le i \le 2$ and the generating triple $\{x,y\}$ are as follows:*

(i) $\mathrm{LEN}_{\{x,y\}} l^{1,\{1\}}(\langle 2,2,n\rangle) = \mathrm{LEN}_{\{x,y\}} l^{1\ \{2\}}(\langle 2,2,n\rangle) = 6$,

(ii) $\mathrm{LEN}_{\{x,y\}} l^{2,\{1\}}(\langle 2,2,n\rangle) = \mathrm{LEN}_{\{x,y\}} l^{2,\{2\}}(\langle 2,2,n\rangle) = h_2(2n)$.

Proof. We prove the result by direct calculation. We first note that in the group defined by $\langle x, y \mid x^2 = y^2 = (xy)^n\rangle$, $|x| = 4$, $|y| = 4$, and $|xy| = 2n$.

(i) The orbits $l^{1,\{1\}}_{\{x,y\}}(\langle 2,2,n\rangle)$ and $l^{1,\{2\}}_{\{x,y\}}(\langle 2,2,n\rangle)$ are the same and are as follows:

$$x_0 = x, x_1 = y^3, x_2 = xy^3, x_3 = yxy, x_4 = y^3, x_5 = yx, x_6 = x, x_7 = y^3, \dots, \tag{2.38}$$

which have period 6.

(ii) The orbits $l^{2,\{1\}}_{\{x,y\}}(\langle 2,2,n\rangle)$ and $l^{2,\{2\}}_{\{x,y\}}(\langle 2,2,n\rangle)$ are the same and are as follows:

$$x_0 = (xy)^{n-1}, x_1 = (xy)^n, \dots. \tag{2.39}$$

We consider the recurrence relation defined by the following:

$$u_m = u_{m-2} + u_{m-1}, \qquad u_0 = n - 1, \qquad u_1 = n. \tag{2.40}$$

Then a routine induction shows that $x_m = (xy)^{u_m}$. Using Lemma 1.4, we obtain $u_L = u_0 = n-1$ and $u_{L+1} = u_1 = n$. In this case the equalities $x_m = (xy)^{u_m}$ give

$$x_L = (xy)^{u_L} = (xy)^{n-1}, x_{L+1} = (xy)^{u_{L+1}} = (xy)^n. \tag{2.41}$$

The smallest nontrivial integer satisfying the above conditions occurs when the period is $h_2(2n)$. □

Theorem 2.11. *The ith generalized order-3 Lucas lengths of the polyhedral group $(n,2,2)$ for every i integer such that $1 \le i \le 3$ and the generating triple $\{x,y,z\}$ are as follows:*

(i) $\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}((n,2,2)) = 6$ *for* $1 \le \alpha_1, \alpha_2 \le 2$,

(ii) $\mathrm{LEN}_{\{x,y,z\}} l^{2,\{\alpha_1,\alpha_2\}}((n,2,2)) = h_3(n)$ *for* $1 \le \alpha_1, \alpha_2 \le 2$,

(iii) (1) $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{1,1\}}((n,2,2)) = \mathrm{LEN}_{\{x,y,z\}} l^{3,\{1,2\}}((n,2,2)) = 8$,
(2) $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,1\}}((n,2,2)) = \mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,2\}}((n,2,2)) = 4$.

Proof. (i) We follow the proof given in [13].

The proofs of (ii) and (iii) are similar to the proofs of Theorem 2.5(ii) and 2.5(iii) and are omitted. □

Theorem 2.12. *The ith generalized order-2 Lucas lengths of the polyhedral group $(n,2,2)$ for every i integer such that $1 \le i \le 2$ and the generating triple $\{x,y\}$ are as follows:*

(i) $\mathrm{LEN}_{\{x,y\}} l^{1,\{1\}}((n,2,2)) = \mathrm{LEN}_{\{x,y\}} l^{1,\{2\}}((n,2,2)) = 6$,

(ii) $\mathrm{LEN}_{\{x,y\}} l^{2,\{1\}}((n,2,2)) = \mathrm{LEN}_{\{x,y\}} l^{2,\{2\}}((n,2,2)) = 3$.

Proof. (i) The orbits $l^{1,\{1\}}((n,2,2))$ and $l^{1,\{2\}}((n,2,2))$ are the natural extension of the result of dihedral groups given in [16].

(ii) The orbits $l^{2,\{1\}}_{\{x,y\}}((n,2,2))$ and $l^{2,\{2\}}_{\{x,y\}}((n,2,2))$, respectively, are as follows:

$$\begin{gathered} x_0 = xy, x_1 = e, x_2 = xy, x_3 = xy, x_4 = e, \ldots, \\ x_0 = yx, x_1 = e, x_2 = yx, x_3 = yx, x_4 = e, \ldots, \end{gathered} \tag{2.42}$$

which have period 3. □

Theorem 2.13. *The ith generalized order-3 Lucas lengths of the polyhedral group $(2,n,2)$ for every i integer such that $1 \le i \le 3$ and the generating triple $\{x,y,z\}$ are as follows:*

(i) $\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}((2,n,2)) = 6$ *for* $1 \le \alpha_1, \alpha_2 \le 2$,

(ii) (1) $\mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,1\}}((2,n,2)) = LEN_{\{x,y,z\}} l^{2,\{2,1\}}((2,n,2)) = 4$,
(2) $\mathrm{LEN}_{\{x,y,z\}} l^{2,\{1,2\}}((2,n,2)) = \mathrm{LEN}_{\{x,y,z\}} l^{2,\{2,2\}}((2,n,2)) = 8$,

(iii) $\mathrm{LEN}_{\{x,y,z\}} l^{3,\{\alpha_1,\alpha_2\}}((2,n,2)) = h_3(n)$ *for* $1 \le \alpha_1, \alpha_2 \le 2$.

Proof. (i) We follow the proof given in [13].

The proofs of (ii) and (iii) are similar to the proofs of Theorem 2.5(ii) and 2.5(iii) and are omitted. □

Theorem 2.14. *The* i*th generalized order-2 Lucas lengths of the polyhedral group* $(2, n, 2)$ *for every* i *such that* $1 \le i \le 2$ *and the generating pair* $\{x, y\}$ *are 6.*

Proof. The proof is similar to the proof of Theorem 2.6 and is omitted. □

Theorem 2.15. *The* i*th generalized order-3 Lucas lengths of the polyhedral group* $(2, 2, n)$ *for every* i *integer such that* $1 \le i \le 3$ *and the generating triple* $\{x, y, z\}$ *are as follows:*

(i)

$$\mathrm{LEN}_{\{x,y,z\}} l^{1,\{\alpha_1,\alpha_2\}}(2,2,n) = \begin{cases} 2n \text{ if } n \equiv 0 \mod 4, \\ 4n \text{ if } n \equiv 2 \mod 4, \\ 8n \text{ otherwise}, \end{cases} \quad \text{for } 1 \le \alpha_1, \alpha_2 \le 2 \tag{2.43}$$

(ii)

$$\mathrm{LEN}_{\{x,y,z\}} l^{2,\{\alpha_1,\alpha_2\}}((2,2,n)) = \begin{cases} n \text{ if } n \equiv 0 \mod 8, \\ 2n \text{ if } n \equiv 4 \mod 8, \\ 4n \text{ if } n \equiv 2 \mod 8, \\ 8n \text{ otherwise}, \end{cases} \quad \text{for } 1 \le \alpha_1, \alpha_2 \le 2 \tag{2.44}$$

(iii) (1)

$$\begin{aligned} \mathrm{LEN}_{\{x,y,z\}} l^{3,\{1,1\}}((2,2,n)) &= \mathrm{LEN}_{\{x,y,z\}} l^{3,\{1,2\}}((2,2,n)) \\ &= \mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,2\}}((2,2,n)) = 8, \end{aligned} \tag{2.45}$$

(2)

$$\mathrm{LEN}_{\{x,y,z\}} l^{3,\{2,1\}}((2,2,n)) = 4. \tag{2.46}$$

Proof. The proof is similar to the proof of Theorem 2.9 and is omitted. □

Theorem 2.16. *The* i*th generalized order-2 Lucas lengths of the polyhedral group* $(2, 2, n)$ *for every* i *integer such that* $1 \le i \le 2$ *and the generating triple* $\{x, y\}$ *are as follows:*

(i) $\mathrm{LEN}_{\{x,y\}} l^{1,\{1\}}((2,2,n)) = \mathrm{LEN}_{\{x,y\}} l^{1,\{2\}}((2,2,n)) = 6$,

(ii) $\mathrm{LEN}_{\{x,y\}} l^{2,\{1\}}((2,2,n)) = \mathrm{LEN}_{\{x,y\}} l^{2,\{2\}}((2,2,n)) = h_2(n)$.

Proof. (i) The orbits $l^{1,\{1\}}((2,2,n))$ and $l^{1,\{2\}}((2,2,n))$ are the natural extension of the result of dihedral groups given in [16].

(ii) The proof is similar to the proof of Theorem 2.10(ii) and is omitted. □

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