Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 154670, 13 pages
doi:10.1155/2012/154670

Research Article

Note on the Choquet Integral as an Interval-Valued Aggregation Operators and Their Applications

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Received 2 November 2011; Revised 15 December 2011; Accepted 16 December 2011

Academic Editor: Hui-Shen Shen

The concept of an interval-valued capacity is motivated by the goal to generalize a capacity, and it can be used for representing an uncertain capacity. In this paper, we define the discrete interval-valued capacities, a measure of the entropy of a discrete interval-valued capacity, and, Choquet integral with respect to a discrete interval-valued capacity. In particular, we discuss the Choquet integral as an interval-valued aggregation operator and discuss an application of them.

1. Introduction

Let (X, Ω) be a measurable space. A capacity (or a fuzzy measure) on X is a nonnegative monotone set function $\mu : \Omega \to \overline{\mathbb{R}}^{+} = [0, \infty]$ with $\mu(\emptyset) = 0$. Many researchers have been studying a discrete capacity in many topics such as capacity functionals of random sets (see [1–5]) and entropy-like measures (see [6–9]).

The Choquet integral with respect to a capacity of a nonnegative measurable function f is given by

$$\mathcal{C}_{\mu}(f) = (\mathrm{C}) \int f \, d\mu = \int_{0}^{\infty} \mu_{f}(\alpha) d\alpha, \tag{1.1}$$

where $\mu_f(\alpha) = \mu(\{x \in X \mid f(x) > \alpha\})$ and the integral on the right-hand side is an ordinary one. If we take $N = \{1, 2, \ldots, n\}$ and $f : N \to \mathbb{R}^{+}$ by $f(i) = x_i$ for all $i \in N$, then we have

$$\mathcal{C}_{\mu}(f) = \sum_{i=1}^{n} x_{\pi(i)} \left[\mu\left(A_{\pi(i)}\right) - \mu\left(A_{\pi(i+1)}\right) \right], \tag{1.2}$$

where π is a permutation on $\{1,2,\ldots,n\}$ such that $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$ and $A_{\pi(i)} = \{\pi(i),\ldots,\pi(n)\}$ and $A_{\pi(n+1)} = \emptyset$ (see [1–4, 6, 9, 10]). Note that if we put $x_{\pi(0)} = 0$ and $A_{\pi(n+1)} = \emptyset$, then we obtain the following formula:

$$\mathcal{C}_\mu(f) = \sum_{i=0}^{n-1} \left(x_{\pi(i+1)} - x_{\pi(i)}\right)\mu\left(A_{\pi(i+1)}\right). \tag{1.3}$$

By using interval-valued functions to express uncertain functions, we have studied the Choquet integral with respect to a capacity of an interval-valued function which is able to better handle the representation of decision making and information theory (see [10–16]). During the last decade, it has been suggested to use intervals in order to represent uncertainty in the area of decision theory and information theory, for example, calculation of economic uncertainty [8], theory of interval probability as a unifying concept for uncertainty [17], and the Choquet integral of uncertain functions [3, 12–16, 18]. Recently, Xu et al. [19–24] have been studying the application of the Choquet integral with uncertain and fuzzy information.

The main idea of this paper is to use the concept of an interval-valued capacity in the entropy-like measure which is an aggregation defined by the discrete interval-valued capacities. In Section 2, we introduce the Choquet integral with respect to an interval-valued capacity and discuss some of its properties. In Section 3, we investigate the interval-valued weighted arithmetic mean, the interval-valued Shannon entropy, the interval-valued weighted averaging operator, and an interval-valued measure of the entropy of an interval-valued capacity. In Section 4, we give the problem of evaluating students as an example where interval-valued weights and some suitable interval-valued capacity are used in practical situation. In Section 5, we give a brief summary results and some conclusions.

2. The Choquet Integral with Respect to a Discrete Interval-Valued Capacity

Throughout this paper, $I(\mathbb{R}^+)$ is the set of all closed intervals in $\mathbb{R}^+ = [0,\infty)$, that is,

$$I(\mathbb{R}^+) = \{[a_1,a_2] \mid a_1,a_2 \in \mathbb{R}^+,\ a_1 \leq a_2\}. \tag{2.1}$$

For any $a \in \mathbb{R}^+$, we define $a = [a,a]$. Obviously, $a \in I(\mathbb{R}^+)$ (see [13–15]).

Definition 2.1. If $\overline{a} = [a_1,a_2]$, $\overline{b} = [b_1,b_2] \in I(\mathbb{R}^+)$, and $k \in \mathbb{R}^+$, then one defines arithmetic, minimum, order, and inclusion operations as follows:

(1) $\overline{a} + \overline{b} = [a_1 + b_1, a_2 + b_2]$,

(2) $k\overline{a} = [ka_1, ka_2]$,

(3) $\overline{a}\overline{b} = [a_1b_1, a_2b_2]$,

(4) $\overline{a} \wedge \overline{b} = [a_1 \wedge b_1, a_2 \wedge b_2]$,

(5) $\overline{a} \vee \overline{b} = [a_1 \vee b_1, a_2 \vee b_2]$,

(6) $\overline{a} \leq \overline{b}$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$,

(7) $\overline{a} < \overline{b}$ if and only if $\overline{a} \leq \overline{b}$ and $\overline{a} \neq \overline{b}$,

(8) $\overline{a} \subset \overline{b}$ if and only if $b_1 \leq a_1$ and $a_2 \leq b_2$.

Let U be a countably infinite set as the universe of discourse and $\mathcal{P}(U)$ the power set of U. We propose an interval-valued capacity and discuss some of its properties.

Definition 2.2. (1) An interval-valued set function $\overline{\mu} = [\mu_l, \mu_r] : \mathcal{P}(U) \rightarrow I(\overline{\mathbb{R}}^+)$ is said to be a discrete interval-valued capacity on U if it satisfies the following conditions:

(i) $\overline{\mu}(\emptyset) = \overline{0}$,

(ii) $\overline{\mu}(S) \leq \overline{\mu}(T)$, whenever $S, T \in \mathcal{P}(U)$ and $S \subset T$.

(2) A set $D(\in \mathcal{P}(U))$ is said to be a carrier (or support) of an interval-valued capacity $\overline{\mu}$ if $\overline{\mu}(S) = \overline{\mu}(D \cap S)$ for all $S \in \mathcal{P}(U)$.

(3) An interval-valued capacity $\overline{\mu}$ with nonempty finite carrier $D(\in \mathcal{P}(U))$ is said to be normalized if $\overline{\mu}(D) = \overline{1}$.

For any integer $k \geq 1$, the set $\{1, \ldots, k\}$ will simply be denoted by $[k]$ and $I([0,1]) = \{[a_1, a_2] \mid a_1, a_2 \in [0,1] \text{ and } a_1 \leq a_2\}$. For the sake of convenience, we will henceforth assume that $D = N$ is the n-element set $[n]$. We denote by IVC the set of interval-valued capacities with a nonempty finite carrier on U and by IVC_N the set of normalized interval-valued capacities having $N \subset U$ as a nonempty finite carrier.

Definition 2.3. (1) An interval-valued capacity $\overline{\mu} \in \text{IVC}_N$ is said to be additive if $\overline{\mu}(S \cup T) = \overline{\mu}(S) + \overline{\mu}(T)$ for all disjoint subsets $S, T \subset N$.

(2) $\overline{\mu} \in \text{IVC}_N$ is said to be cardinality based if for all $T \subset N$, $\overline{\mu}(T)$ depends only on the cardinality of T; that is, there exists $\overline{\mu}_0, \overline{\mu}_1, \ldots, \overline{\mu}_n \in I([0,1])$ such that $\overline{\mu}(T) = \overline{\mu}_t = [\mu_{lt}, \mu_{rt}]$ for all $T \subset N$ such that $|T| = t$, where $|T|$ is the cardinality of T.

In [6, p. 135], there is only one normalized capacity μ_N^* with a nonempty finite carrier N which is both additive and cardinality based, and in this case, μ_N^* is given by $\mu_N^*(T) = t/n$ for all $T \subset N$ such that $|T| = t$. Thus we can obtain the following theorem.

Theorem 2.4. *If $\overline{\mu} \in \text{IVC}_N$ is both additive and cardinality based, then $\overline{\mu}(T) = [t/n, t/n]$, for all $T \subset N$ with $|T| = t$.*

Theorem 2.4 implies that if a discrete interval-valued normalized capacity $\overline{\mu}$ is both additive and cardinality based, then it is a discrete real-valued capacity (or a real-valued monotone set function). By Definition 2.3, we can easily obtain the following theorem.

Theorem 2.5. *(1) An interval-valued set function $\overline{\mu} = [\mu_l, \mu_r]$ is a discrete interval-valued capacity if and only if μ_l and μ_r are discrete capacities.*

(2) A set $N(\subset U)$ is a carrier of $\overline{\mu} = [\mu_l, \mu_r]$ if and only if N is a carrier of both μ_l and μ_r.

(3) $\overline{\mu} = [\mu_l, \mu_r]$ is normalized if and only if μ_l and μ_r are normalized.

(4) $\overline{\mu} = [\mu_l, \mu_r]$ is additive if and only if μ_l and μ_r are additive.

(5) $\overline{\mu} = [\mu_l, \mu_r]$ is cardinality based if and only if μ_l and μ_r are cardinality based.

By using formula (1.3) of the Choquet integral and a discrete interval-valued capacity with a nonempty finite carrier N, we will define the Choquet integral with respect to a discrete interval-valued capacity.

Definition 2.6. Let $x : N \to \mathbb{R}^+$ be a function such that $x(i) = x_i$ for all $i \in N$ and $\overline{\mu}$ a discrete interval-valued capacity with a nonempty finite carrier N. The Choquet integral with respect to $\overline{\mu}$ of x is defined by

$$\mathcal{C}_{\overline{\mu}}(f) = \sum_{i=0}^{n-1}\left(x_{\pi(i+1)} - x_{\pi(i)}\right)\overline{\mu}\left(A_{\pi(i+1)}\right), \tag{2.2}$$

where π is a permutation on N such that $x_{\pi(0)} = 0 \le x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(n)}$ and $A_{\pi(i)} = \{\pi(i), \ldots, \pi(n)\}$ and $A_{\pi(n+1)} = \emptyset$.

By (2.2), we can easily obtain the following basic property of $\mathcal{C}_{\overline{\mu}}$.

Theorem 2.7. *If $\overline{\mu} \in \mathrm{IVC}_N$, then one has*

$$\mathcal{C}_{\overline{\mu}}(x) = \left[\mathcal{C}_{\mu_l}(x), \mathcal{C}_{\mu_r}(x)\right], \tag{2.3}$$

where a function $x : N \to \mathbb{R}^+$ by $x(i) = x_i$ for all $i \in N$.

From the right-hand side of (2.2), we note that

$$\sum_{i=0}^{n}\left(x_{\pi(i+1)} - x_{\pi(i)}\right)\overline{\mu}\left(A_{\pi(i+1)}\right) \ne \sum_{i=1}^{n} x_{\pi(i)}\left(\overline{\mu}\left(A_{\pi(i+1)}\right) - \overline{\mu}\left(A_{\pi(i)}\right)\right), \tag{2.4}$$

in general. Because of this note, we consider a new difference operation $\ominus$ defined by

$$[a,b] \ominus [c,d] = \begin{cases} [a-c, b-d] & \text{if } 0 \le a-c \le b-d, \\ \overline{0} & \text{if otherwise,} \end{cases} \tag{2.5}$$

where $a, b, c, d \in \mathbb{R}^+$. From this difference operation, we can easily see that $[ka, kb] \ominus [kc, kd] = k([a,b] \ominus [c,d])$ for all $k, a, b, c, d \in \mathbb{R}^+$. We denote by $\mathrm{IVC}_N^1(x)$ the set of normalized interval-valued capacities $\overline{\mu} = [\mu_l, \mu_r]$ with a nonempty finite carrier N satisfying the following condition:

$$\mu_l\left(A_{\pi(i)}\right) - \mu_l\left(A_{\pi(i+1)}\right) \le \mu_r\left(A_{\pi(i)}\right) - \mu_r\left(A_{\pi(i+1)}\right), \tag{2.6}$$

for all $i \in N$ and where π is a permutation on $\{1, 2, \ldots, n\}$ such that $x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(n)}$ and $A_{\pi(i)} = \{\pi(i), \ldots, \pi(n)\}$ and $A_{\pi(n+1)} = \emptyset$. We remark that $\mathrm{IVC}_N^1(x) \subset \mathrm{IVC}_N$ and that if $\overline{\mu} \in \mathrm{IVC}_N^1(x)$, then we have

$$\overline{\mu}\left(A_{\pi(i)}\right) \ominus \overline{\mu}\left(A_{\pi(i+1)}\right) = \left[\mu_l\left(A_{\pi(i)}\right) - \mu_l\left(A_{\pi(i+1)}\right), \mu_r\left(A_{\pi(i)}\right) - \mu_r\left(A_{\pi(i+1)}\right)\right]. \tag{2.7}$$

By Theorem 2.7 and (1.2), (2.5), and (2.7), we derive the following theorem.

Theorem 2.8. *If there exists function* $x : N \to \mathbb{R}^+$ *by* $x(i) = x_i$ *for all* $i \in N$ *and* $\overline{\mu} \in \mathrm{IVC}_N^1(x)$, *then one has*

$$\mathcal{C}_{\overline{\mu}}(x) = \sum_{i=1}^{n} x_{\pi(i)} \left(\overline{\mu}\left(A_{\pi(i)}\right) \ominus \overline{\mu}\left(A_{\pi(i+1)}\right)\right), \tag{2.8}$$

where π *is a permutation on* N *such that* $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$ *and* $A_{\pi(i)} = \{\pi(i), \ldots, \pi(n)\}$ *and* $A_{\pi(n+1)} = \emptyset$.

Proof. By Theorem 2.7 and the definition (1.2) and the difference (2.5) operation, we have

$$\begin{aligned}
\mathcal{C}_{\overline{\mu}}(x) &= \left[\mathcal{C}_{\mu_l}(x), \mathcal{C}_{\mu_r}(x)\right] \\
&= \left[\sum_{i=1}^{n} x_{\pi(i)} \left(\mu_l\left(A_{\pi(i)}\right) - \mu_l\left(A_{\pi(i+1)}\right)\right), \sum_{i=1}^{n} x_{\pi(i)} \left(\mu_r\left(A_{\pi(i)}\right) - \mu_r\left(A_{\pi(i+1)}\right)\right)\right] \\
&= \sum_{i=1}^{n} x_{\pi(i)} \left[\mu_l\left(A_{\pi(i)}\right) - \mu_l\left(A_{\pi(i+1)}\right), \mu_r\left(A_{\pi(i)}\right) - \mu_r\left(A_{\pi(i+1)}\right)\right] \\
&= \sum_{i=1}^{n} x_{\pi(i)} \left[\mu_l\left(A_{\pi(i)}\right), \mu_r\left(A_{\pi(i)}\right)\right] \ominus \left[\mu_l\left(A_{\pi(i+1)}\right), \mu_r\left(A_{\pi(i+1)}\right)\right] \\
&= \sum_{i=1}^{n} x_{\pi(i)} \left(\overline{\mu}\left(A_{\pi(i)}\right) \ominus \overline{\mu}\left(A_{\pi(i+1)}\right)\right).
\end{aligned} \tag{2.9}$$

□

3. The Choquet Integral as an Interval-Valued Aggregation Operator

In this section, we define the interval-valued weighted arithmetic mean (IWAM), which is the concept of a generalized aggregation (or an uncertain aggregation), as follows:

$$\mathrm{IWAM}_{\overline{w}}(x) = \sum_{i=1}^{n} \overline{w}_i x_i, \tag{3.1}$$

where $\overline{w}_i = [w_{li}, w_{ri}]$, $0 \leq w_{li} \leq w_{ri} \leq 1$ for all $i \in N$, $\sum_{i=1}^{n} w_{li} \leq \sum_{i=1}^{n} w_{ri} = 1$, and $x : N \to \mathbb{R}^+$ by $x(i) = x_i$ for all $i \in N$ is a function such that $x_1, \ldots, x_n$ represent the arguments. We denote by $x = (x_1, \ldots, x_n)$ an n-dimensional vector in $(\mathbb{R}^+)^n$.

Note that the arguments $x_1, \ldots, x_n$ that are used in such an interval-valued aggregation process strongly depend upon the interval-valued (or uncertain) weight vector $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_n) \in (I([0,1]))^n$. Then the interval-valued Shannon-entropy $\overline{H}_S$ of $\overline{w}$ defined on N is given by

$$\overline{H}_S(\overline{w}) = \sum_{i=1}^{n} \overline{h}(\overline{w}_i), \tag{3.2}$$

where $\overline{h}(\overline{w}_i) = [h(w_{li}), h(w_{ri})]$ for $0 \le w_{li} \le w_{ri} \le 1$ for all $i \in N$, and

$$h(w) = \begin{cases} -w \ln w & \text{if } 0 < w \le 1, \\ \overline{0} & \text{if } w = 0, \end{cases} \tag{3.3}$$

for all $w \in [0,1]$. Then it means a measure of dispersion associated to the interval-valued weight vector of the interval-valued weighted arithmetic mean $\text{IWAM}_{\overline{w}}$. We also easily see that

$$\overline{H}_S(\overline{w}) = [H_S(w_l), H_S(w_r)], \tag{3.4}$$

where $\overline{w} = [w_l, w_r]$.

Now, we define the following interval-valued (or uncertain) ordered weight averaging (IOWA) operator.

Definition 3.1. Let $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_n) \in (I([0,1]))^n$ be an interval-valued weight vector such that $\overline{w}_i = [w_{li}, w_{ri}]$, $0 \le w_{li} \le w_{ri} \le 1$ for all $i \in N$, and $\sum_{i=1}^n w_{li} \le \sum_{i=1}^n w_{ri} = 1$. The interval-valued ordered weighted averaging (IOWA) operator on $(\mathbb{R}^+)^n$ is defined by

$$\text{IOWA}_{\overline{w}}(x) = \sum_{i=1}^{n} \overline{w}_i x_{\pi(i)}, \tag{3.5}$$

where $x = (x_1, \ldots, x_n) \in (\mathbb{R}^+)^n$ and π is a permutation on N such that $0 \le x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(n)}$.

From Definition 3.1, we have

$$\text{IOWA}_{\overline{w}}(x) = \left[\sum_{i=1}^{n} w_{li} x_{\pi(i)}, \sum_{i=1}^{n} w_{ri} x_{\pi(i)}\right]. \tag{3.6}$$

By (3.5) and (3.6), we obtain that if $s \in \{l, r\}$ and we write $w_s = (w_{s1}, \ldots, w_{sn})$, then OWA_{w_s} is an ordered weighted averaging operator associated to a weight vector w_s, proposed by Yager [18]. Remark that if $w_{li} = w_{ri} = w_i$ for all $i \in N$ and we write $w = (w_1, \ldots, w_n)$, then $\text{IWAM}_{\overline{w}}(x) = \sum_{i=1}^n w_i x_i$ is the weight arithmetic mean (WAM), $\overline{H}_S(\overline{w}) = \sum_{i=1}^n h(w_i)$ is the Shannon entropy of w, and $\text{IOWA}_{\overline{w}}(x) = \sum_{i=1}^n w_i x_{\pi(i)}$ is the ordered weighted averaging (OWA) operator (see [3, 5]).

Theorem 3.2. *(1) If one takes $w_l = w_r = (1/n, \ldots, 1/n)$, then $\overline{H}_S(\overline{w}) = \overline{\ln} n$ is maximum.*

(2) If one takes $w_{li} = 1$ and $w_{rj} = 1$ for some $i, j \in N$, then $\overline{H}(\overline{w}) = \overline{0}$ is minimum.

Proof. (1) $\overline{H}_S(\overline{w}) = [\sum_{i=1}^n (-w_{l1} \ln w_{li}), \sum_{i=1}^n (-w_{ri} \ln w_{ri})] = [\ln n, \ln n] = \overline{\ln} n$.

(2) $\overline{H}_S(\overline{w}) = [\sum_{k=1}^n h(w_{lk}), \sum_{k=1}^n h(w_{rk})] = [h(w_{li}), h(w_{rj})] = [0, 0] = \overline{0}$. □

From Theorem 3.2, the interval-valued measure of dispersion can be normalized into

$$\frac{1}{\ln n}\overline{H}_S(\overline{w}) = \frac{1}{\ln n}\sum_{i=1}^{n}\overline{h}(\overline{w}_i), \tag{3.7}$$

so that it ranges in $I([0,1])$. Finally, we will define the interval-valued entropy of an interval-valued capacity which is the generalization of the entropy proposed by Marichal [3] as follows:

$$H_M(\mu) = \sum_{i\in N}\sum_{T\subset N\setminus\{i\}} p_t h(\mu(T\cup\{i\}) - \mu(T)), \tag{3.8}$$

where μ is a capacity on N, $|T| = t$ is the cardinality of T, the coefficients $p_t(|T| = t)$ are nonnegative, and $\sum_{T\subset N\setminus\{i\}} p_t = 1$.

Definition 3.3. The interval-valued entropy of an interval-valued capacity $\overline{\mu}$ is defined by

$$\overline{H}_M(\overline{\mu}) = \sum_{i\in N}\sum_{T\subset N\setminus\{i\}} p_t \overline{h}(\overline{\mu}(T\cup\{i\}) \ominus \overline{\mu}(T)), \tag{3.9}$$

where $\ominus$ is the same operation in (2.5), the coefficients $p_t(|T| = t)$ are nonnegative, $\sum_{T\subset N\setminus\{i\}} p_t = 1$, and $\overline{h}$ is the same function in (3.3).

From Definition 3.3, if we take $\mu_l = \mu_r = \mu$, then $\overline{H}_M(\overline{\mu}) = H_M(\mu)$. We consider the following assumption of $\overline{\mu} = [\mu_l, \mu_r]$:

$$\sum_{T\subset N\setminus\{i\}} p_t h(\mu_l(T\cup\{i\}) - \mu_l(T)) \leq \sum_{T\subset N\setminus\{i\}} p_t h(\mu_r(T\cup\{i\}) - \mu_r(T)), \quad \forall i \in N, \tag{3.10}$$

and let $\mathrm{IVC}_N^2 = \{\overline{\mu} \in \mathrm{IVC}_N \mid \overline{\mu} \text{ satisfy assumption (3.10)}\}$.

Theorem 3.4. *If $\overline{\mu} = [\mu_l, \mu_r] \in \mathrm{IVC}_N^2$, then one has*

$$\overline{H}_M(\overline{\mu}) = [H_M(\mu_l), H_M(\mu_r)]. \tag{3.11}$$

Proof. By Definition 3.3, we can directly calculate $\overline{H}_M(\overline{\mu})$ as follows:

$$\begin{aligned}\overline{H}_M(\overline{\mu}) &= \sum_{i\in N}\sum_{T\subset N\setminus\{i\}} p_t \overline{h}(\overline{\mu}(T\cup\{i\}) \ominus \overline{\mu}(T)) \\ &= \sum_{i\in N}\sum_{T\subset N\setminus\{i\}} p_t \overline{h}([\mu_l(T\cup\{i\}) - \mu_l(T), \mu_r(T\cup\{i\}) - \mu_r(T)])\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in N}\sum_{T\subset N\setminus\{i\}} p_t\big[h\big(\mu_l(T\cup\{i\})-\mu_l(T)\big), h\big(\mu_r(T\cup\{i\})-\mu_r(T)\big)\big] \\
&= \left[\sum_{i\in N}\sum_{T\subset N\setminus\{i\}} p_t h\big(\mu_l(T\cup\{i\})-\mu_l(T)\big), \sum_{i\in N}\sum_{T\subset N\setminus\{i\}} p_t h\big(\mu_r(T\cup\{i\})-\mu_r(T)\big)\right] \\
&= \big[H_M(\mu_l), H_M(\mu_r)\big].
\end{aligned}
\tag{3.12}
$$

□

From Theorem 3.4, we can see that if we take $\overline{\mu} = [\mu_l, \mu_r] \in \mathrm{IVC}_N$ such that $H_M(\mu_r) < H_M(\mu_l)$, then $\overline{H}_M(\overline{\mu})$ is not defined. Thus, the Assumption (3.10) of $\overline{\mu}$ is a sufficient condition for defining the interval-valued entropy $\overline{H}_M(\overline{\mu})$ of an interval-valued capacity $\overline{\mu}$. We also suggest that that $\overline{H}_M(\overline{\mu})$ is interpreted as an interval-valued measure of dispersion for $\mathcal{C}_{\overline{\mu}}$ a sum over $i \in N$ of an average value of $\overline{h}(\overline{\mu}(T\cup\{i\}) \ominus \overline{\mu}(T))(T \subset N \setminus \{i\})$ as follows: for all $i \in N$,

$$
\big(\overline{\mu}\big(A_{\pi(i)}\big) \ominus \overline{\mu}\big(A_{\pi(i+1)}\big)\big) = \sum_{T\subset N\setminus\{i\}} p_t \overline{h}\big(\overline{\mu}(T\cup\{i\}) \ominus \overline{\mu}(T)\big). \tag{3.13}
$$

Theorem 3.5. *If* $x = (x_1, \ldots, x_n)$ *and* $\overline{\mu} \in \mathrm{IVC}_N^1(x)$, *then* $\overline{\mu} \in \mathrm{IVC}_N^2$.

Proof. Let $x = (x_1, \ldots, x_n) \in (\mathbb{R}^+)^n$ and π be a permutation on N such that $0 \le x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(n)}$. Since $\overline{\mu} \in \mathrm{IVC}_N^1(x)$, we get $\mu_l(A_{\pi(i)}) - \mu_l(A_{\pi(i+1)}) \le \mu_r(A_{\pi(i)}) - \mu_r(A_{\pi(i+1)})$ for all $i \in N$. Thus, by (3.13),

$$
\begin{aligned}
\sum_{T\subset N\setminus\{i\}} p_t h\big(\mu_l(T\cup\{i\})-\mu_l(T)\big) &= \mu_l\big(A_{\pi(i)}\big) - \mu_l\big(A_{\pi(i+1)}\big) \\
&\le \mu_r\big(A_{\pi(i)}\big) - \mu_r\big(A_{\pi(i+1)}\big) \\
&= \sum_{T\subset N\setminus\{i\}} p_t h\big(\mu_r(T\cup\{i\})-\mu_r(T)\big).
\end{aligned}
\tag{3.14}
$$

This implies $H_M(\mu_l) \le H_M(\mu_r)$. Therefore $\overline{\mu} \in \mathrm{IVC}_N^2$. □

Theorem 3.6. *If* $x = (x_1, \ldots, x_n)$, $\overline{\mu} \in \mathrm{IVC}_N^1(x)$, *and* $\mathcal{C}_{\overline{\mu}} = \mathrm{IOWA}_{\overline{w}}$, *then one has* $\overline{H}_M(\overline{\mu}) = \overline{H}_S(\overline{w})$, *that is,*

$$
\overline{H}_M(\overline{\mu}) = \sum_{i=1}^{n} \overline{h}\big(\overline{\mu}\big(A_{\pi(i)}\big) \ominus \overline{\mu}\big(A_{\pi(i+1)}\big)\big), \tag{3.15}
$$

where $\ominus$ *is the same operation in* (2.3), $\overline{h}$ *is the same function in* (2.8), *and* $x = (x_1, \ldots, x_n) \in (\mathbb{R}^+)^n$, *and* π *is a permutation on* N *such that* $0 \le x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(n)}$

Table 1

Student	Mathematics (M)	Physics (P)	Literature (L)	$\text{IWAM}_{\overline{w}}$
a	18	16	10	[9.75, 16.25]
b	10	12	18	[7.75, 11.75]
c	14	15	15	[9.13, 14.50]

Proof. Since $\mathcal{C}_{\overline{\mu}} = \text{IOWA}_{\overline{w}}$, $\overline{w}_i = \overline{\mu}(A_{\pi(i)}) \ominus \overline{\mu}(A_{\pi(i+1)})$. By (3.13), we get

$$\begin{aligned} \overline{H}_M(\overline{\mu}) &= \sum_{i\in N} \sum_{T\subset N\setminus\{i\}} p_t \overline{h}(\overline{\mu}(T\cup\{i\}) \ominus \overline{\mu}(T)) \\ &= \sum_{i=1}^{n} \overline{h}(\overline{\mu}(A_{\pi(i)}) \ominus \overline{\mu}(A_{\pi(i+1)})) \\ &= \sum_{i=1}^{n} \overline{h}(\overline{w}) = \overline{H}_S(\overline{w}). \end{aligned} \tag{3.16}$$

□

4. Applications

In this section, we consider the problem of evaluating students in a high school with respect to three subjects: mathematics (M), physics (P), and literature (L), proposed by Marichal [3]. Suppose that the school is more scientifically than literary from somewhat oriented to extremely oriented, so that interval-valued weights could be, for example, $\overline{w}_{\text{M}} = [1/4, 1/2]$, $\overline{w}_{\text{P}} = [1/4, 3/8]$, and $\overline{w}_{\text{L}} = [1/8, 1/8]$, respectively. We note that $\overline{w} = (\overline{w}_{\text{M}}, \overline{w}_{\text{P}}, \overline{w}_{\text{L}})$ and $\text{IWAM}_{\overline{w}}(x) = [\text{WAM}_{w_l}(x), \text{WAM}_{w_r}(x)]$ for all $x \in \{a, b, c\}$.

If we take $\overline{w} = [w_l, w_r]$, then $w_l = (1/4, 1/4, 1/8)$ and $w_r = (1/2, 3/8, 1/8)$. Then the interval-valued weighted arithmetic mean will give the results for three students a, b, and c (marks are given on a scale from 0 to 20) (see Table 1).

We note that $\text{IWAM}_{\overline{w}}(a) > \text{IWAM}_{\overline{w}}(c) > \text{IWAM}_{\overline{w}}(b)$. The total interval-valued weight is from rather well distributed to quite well distributed over three subjects since we have

$$\frac{1}{\ln n} \overline{H}_S(\overline{w}) = [0.868, 0.887]. \tag{4.1}$$

We consider the α-mean evaluation E_α of $\text{IWAM}_{\overline{w}_l}$ as follows:

$$E_\alpha(x) = \alpha \text{WAM}_{w_l}(x) + (1-\alpha)\text{WAM}_{w_r}(x) \tag{4.2}$$

for all $\alpha \in [0, 1]$ and $x \in \{a, b, c\}$. The α-mean evaluation E_α implies that we can interpret the difference of the degree of favor for students. Indeed, if $\alpha = 0$, that is, if the school is more scientifically than literary extremely oriented, then the school wants to favor more student c as $E_0(a) - E_0(c) = 1.75$; if $\alpha = 1$, that is, if the school is more scientifically than literary somewhat oriented, then the school wants to favor more student c as $E_1(a) - E_1(c) = 0.575$.

Table 2

Student	Mathematics (M)	Physics (P)	Literature (L)	$\mathcal{C}_{\overline{\mu}}$ (student)
a	18	16	10	[7.80, 14.60]
b	10	12	18	[6.85, 12.25]
c	14	15	15	[8.15, 14.75]

Now, if the school wants to favor somewhat well-equilibrated over extremely well equilibrated students without weak points, then student c should be considered better than student a, who has a severe weakness in literature. Unfortunately, no interval-valued vector $(\overline{w}_{\mathrm{M}}, \overline{w}_{\mathrm{P}}, \overline{w}_{\mathrm{L}})$ satisfying $\overline{w}_{\mathrm{M}} > \overline{w}_{\mathrm{P}} > \overline{w}_{\mathrm{L}}$ is able to favor student c. Indeed, it is possible that

$$\mathrm{IWAM}_{\overline{w}}(14, 15, 15) > \mathrm{IWAM}_{\overline{w}}(18, 16, 10) \Longleftrightarrow \overline{w}_{\mathrm{L}} > \overline{w}_{\mathrm{M}}. \tag{4.3}$$

The reason of this problem is that much importance is given to mathematics and physics, which present some overlap effect since, usually, students from little good to rather good at mathematics are also from little good to rather good at physics (and vice versa), so that the interval-valued evaluation is overestimated (resp., underestimated) for students from little good to rather good (resp., from little bad to rather bad) at mathematics and/or physics.

This problem can be overcome by using a suitable interval-valued capacity $\overline{\mu}$ and the Choquet integral $\mathcal{C}_{\overline{\mu}}$ as follows.

(i) Since scientific subjects are more important than literature, the following interval-valued weights can be put on subjects taken individually: $\overline{\mu}(\{\mathrm{M}\}) = [0.25, 0.5]$, $\overline{\mu}(\{\mathrm{P}\}) = [0.25, 0.375]$ and $\overline{\mu}(\{\mathrm{L}\}) = [0.125, 0.125]$. Note that the initial interval-valued ratio of interval-valued weight $([2, 4], [2, 3], [1, 1])$ is kept unchanged.

(ii) Since mathematics and physics overlap, the interval-valued weight attributed to the pair $\{\mathrm{M}, \mathrm{P}\}$ should be less than the sum of the interval-valued weight of mathematics and physics: $\overline{\mu}(\{\mathrm{M}, \mathrm{P}\}) = [0.3, 0.6]$.

(iii) Since students equally good at scientific subjects and literature must be favored, the interval-valued weight attributed to the pair $\{\mathrm{L}, \mathrm{M}\}$ should be greater than the sum of individual interval-valued weights (the same for physics and literature): $\overline{\mu}(\{\mathrm{M}, \mathrm{L}\}) = [0.45, 0.75] = \overline{\mu}(\{\mathrm{P}, \mathrm{L}\})$.

(iv) $\overline{\mu}(\emptyset) = \overline{0}$ and $\overline{\mu}(\{\mathrm{M}, \mathrm{P}, \mathrm{L}\}) = [0.55, 1]$.

If we take $\overline{\mu} = [\mu_l, \mu_r]$, and $\mu_l(\{\mathrm{L}, \mathrm{P}, \mathrm{M}\}) = 0.55$, $\mu_r(\{\mathrm{L}, \mathrm{P}, \mathrm{M}\}) = 1$, $\mu_l(\{\mathrm{P}, \mathrm{M}\}) = 0.3$, $\mu_r(\{\mathrm{P}, \mathrm{M}\}) = 0.6$, $\mu_l(\{\mathrm{M}, \mathrm{L}\}) = 0.45$, $\mu_r(\{\mathrm{M}, \mathrm{L}\}) = 0.75$, $\mu_l(\{\mathrm{P}, \mathrm{L}\}) = 0.45$, $\mu_r(\{\mathrm{P}, \mathrm{L}\}) = 0.75$, $\mu_l(\{\mathrm{P}\}) = 0.25$, $\mu_r(\{\mathrm{P}\}) = 0.375$, $\mu_l(\{\mathrm{M}\}) = 0.25$, $\mu_r(\{\mathrm{M}\}) = 0.5$, $\mu_l(\{\mathrm{L}\}) = 0.125$, and $\mu_r(\{\mathrm{L}\}) = 0.125$, then $H_{\mathrm{M}}(\mu_l) = 0.8802 < 0.9974 = H_{\mathrm{M}}(\mu_r)$ and hence $\overline{\mu} \in \mathrm{IVC}_N^2$.

Applying the Choquet integral with respect to the above interval-valued capacity leads to the Choquet integrals see Table 2.

Since $\mathrm{IWAM}_{\overline{w}}(a) > \mathrm{IWAM}_{\overline{w}}(c) > \mathrm{IWAM}_{\overline{w}}(b)$ and $\mathcal{C}_{\overline{\mu}}(c) > \mathcal{C}_{\overline{\mu}}(a) > \mathcal{C}_{\overline{\mu}}(b)$, we can see that if we use $\mathrm{IWAM}_{\overline{w}}$, then student a has the best rank, but if we use $\mathcal{C}_{\overline{\mu}}$, then student c has the best rank. We also consider the α-mean Choquet evaluation E_α^c of $\mathcal{C}_{\overline{\mu}}$ as follows:

$$E_\alpha^c(x) = \alpha \mathcal{C}_{\mu_l}(x) + (1 - \alpha)\mathcal{C}_{\mu_r}(x) \tag{4.4}$$

for all $\alpha \in [0, 1]$ and $x \in \{a, b, c\}$.

The α-mean Choquet evaluation E^c_α implies that we can interpret the difference of the degree of favor for student a and student c. Indeed, if $\alpha = 1$, that is, if the school wants to favor extremely well-equilibrated students, then the school wants to favor student c than student a as $E^c_1(c) - E^c_1(a) = 0.35$; if $\alpha = 0$, that is, if the school wants to favor somewhat more well-equilibrated students, then the school wants to favor student c more than student a as $E^c_0(c) - E^c_0(a) = 0.15$.

Finally, we have the normalized entropy of interval-valued capacity in IVC^2_N as follows:

$$\begin{aligned}
\frac{1}{\ln n}\overline{H}_{\mathrm{M}}(\overline{\mu}) &= \left[\frac{1}{\ln n}H_{\mathrm{M}}(\mu_l), \frac{1}{\ln n}H_{\mathrm{M}}(\mu_r)\right], \\
H_{\mathrm{M}}(\mu_l) &= \sum_{T\subset N\setminus\{\mathrm{P}\}} \frac{(3-t-1)!t!}{3!} h\left(\mu_l(T\cup\{\mathrm{P}\}) - \mu_l(T)\right) \\
&\quad + \sum_{T\subset N\setminus\{\mathrm{M}\}} \frac{(3-t-1)!t!}{3!} h\left(\mu_l(T\cup\{\mathrm{M}\}) - \mu_l(T)\right) \\
&\quad + \sum_{T\subset N\setminus\{\mathrm{L}\}} \frac{(3-t-1)!t!}{3!} h\left(\mu_l(T\cup\{\mathrm{L}\}) - \mu_l(T)\right), \\
H_{\mathrm{M}}(\mu_l) &= \sum_{T\subset N\setminus\{\mathrm{P}\}} \frac{(3-t-1)!t!}{3!} h\left(\mu_r(T\cup\{\mathrm{P}\}) - \mu_r(T)\right) \\
&\quad + \sum_{T\subset N\setminus\{\mathrm{M}\}} \frac{(3-t-1)!t!}{3!} h\left(\mu_r(T\cup\{\mathrm{M}\}) - \mu_r(T)\right) \\
&\quad + \sum_{T\subset N\setminus\{\mathrm{L}\}} \frac{(3-t-1)!t!}{3!} h\left(\mu_r(T\cup\{\mathrm{L}\}) - \mu_r(T)\right),
\end{aligned} \tag{4.5}$$

where $N = \{\mathrm{P}, \mathrm{M}, \mathrm{L}\}$. Thus, we have

$$\frac{1}{\ln n}\overline{H}_{\mathrm{M}}(\overline{\mu}) = [0.8012, 0.9079], \tag{4.6}$$

which shows that the total interval-valued weight is from still rather well distributed to very quite well distributed.

5. Conclusions

In this paper we consider the new interval-valued measure of the entropy of an interval-valued capacity which generalizes a measure of the entropy proposed by Marichal's [3]. From (3.1), (3.5), and (3.9) and Theorems 3.4, 3.5, 3.6, we investigate the interval-valued weighted arithmetic mean and interval-valued ordered weighted averaging operator for representing uncertain weight vectors which are used in the concept of an uncertain aggregations.

From an example in Section 4, it is possible that we use from somewhat oriented to extremely oriented instead of oriented, from rather well distributed to quite well distributed instead of well distributed, and from somewhat well equilibrated to extremely well equilibrated instead of well equilibrated in the problem of evaluating students.

In the future, by using these results of this paper, we can develop various problems and models for representing uncertain weights related to interacting criteria.

Acknowledgment

This paper was supported by the Konkuk University in 2012.

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