Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 587890, 10 pages
doi:10.1155/2011/587890

Research Article

Approximate Boundary Controllability for Semilinear Delay Differential Equations

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Received 22 March 2011; Accepted 3 July 2011

Academic Editor: Alexandre Carvalho

This paper considers the approximate controllability for a class of semilinear delay control systems described by a semigroup formulation with boundary control. Sufficient conditions for approximate controllability are established provided the approximate controllability of corresponding linear systems.

1. Introduction

In this paper, we consider the boundary control system described by the following delay differential equation:

$$y'(t) = \sigma y(t) + f(t, y_t), \quad \tau y(t) = B_1 u(t) \quad \text{for } t \in I = [0, T],\ y_0 = \xi, \tag{1.1}$$

where system state $y(t)$ takes values in a Banach space E; control function $u(t)$ takes values in another Banach space U and $u(\cdot) \in L^p(I; U)$ for $p \geq 1$; $\sigma : D(\sigma) \to E$ is a closed, densely defined linear operator; $\tau : E \to X$ is a linear operator from E to a Banach space X; $B_1 : U \to X$ is a linear bounded operator; $f : I \times C \to E$ is a nonlinear perturbation function, where $C := C([-\Delta, 0]; E)$ is the Banach space of all continuous functions from $[-\Delta, 0]$ to E endowed with the supremum norm. For any $y \in C([-\Delta, b]; E)$ and $t \in I$, $y_t \in C$ is defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in [-\Delta, 0]$.

In most applications, the state space E is a space of functions on some domain Ω of the Euclidean space $\mathbb{R}^n$, σ is a partial differential operator on Ω, and τ is a partial differential operator acting on the boundary Γ of Ω.

Several abstract settings have been developed to describe control systems with boundary control; see Barbu [1], Fattorini [2], Lasiecka [3], and Washburn [4]. In this paper, we use the setting developed in [2] to discuss the approximate controllability of system (1.1).

The norms in spaces E and C are denoted by $\|\cdot\|$ and $|\cdot|$, respectively. In other spaces, we use the norm notation with a space name in the subindex such as $\|\cdot\|_U$, $\|\cdot\|_X$, and $\|\cdot\|_{L^p}$.

Let $A : E \to E$ be the linear operator defined by

$$D(A) = \{y \in D(\sigma) : \tau y = 0\}, \quad Ay = \sigma y, \quad \forall y \in D(A). \tag{1.2}$$

We impose the following assumptions throughout the paper.

(H1) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to the graph norm of $D(\sigma)$.

(H2) The operator A is the infinitesimal generator of an analytic semigroup $S(t)$ for $t \geq 0$ on E.

(H3) There exists a linear continuous operator $B : U \to E$ and a positive constant K such that

$$\begin{gathered} \sigma B \in L(U, E), \quad \tau(Bu) = B_1 u, \quad \forall u \in U, \\ \|Bu\|_E \leq K \|B_1 u\|_X, \quad \forall u \in U. \end{gathered} \tag{1.3}$$

(H4) For each $t \in (0, T]$ and $u \in U$, one has $S(t)Bu \in D(A)$. Also, there exists a positive function $\gamma(\cdot) \in L^q(I)$ with $1/p + 1/q = 1$ such that

$$\|AS(t)B\|_{L(U,E)} \leq \gamma(t) \quad \text{a.e. } t \in (0, T]. \tag{1.4}$$

(H5) There exists a positive number L such that

$$\|f(t, \eta_1) - f(t, \eta_2)\| \leq L|\eta_1 - \eta_2| \tag{1.5}$$

for all $\eta_1, \eta_2 \in C$ and $t \in I$.

Based on the discussions in [2], system (1.1) can be reformulated as

$$\begin{gathered} y(t) = S(t)\xi(0) + \int_0^t S(t-s)[\sigma Bu(s) + f(s, y_s)]ds + \int_0^t AS(t-s)Bu(s)ds, \quad t \in I, \\ y_0 = \xi. \end{gathered} \tag{1.6}$$

The following system is called the corresponding linear system of (1.6)

$$y(t) = S(t)\xi(0) + \int_0^t S(t-s)\sigma Bu(s)ds + \int_0^t AS(t-s)Bu(s)ds. \tag{1.7}$$

Approximate controllability for semilinear control systems with distributed controls has been extensively studied in the literature under different conditions; see Fabre et al. [5], Fernandez and Zuazua [6], Li and Yong [7], Mahmudov [8], Naito [9], Seidman [10], Wang [11, 12], and many other papers. However, only a few papers dealt with approximate boundary controllability for semilinear control systems, in particular, semilinear delay control systems; the main difficulty is encountered in the construction of suitable integral equation to apply for different versions of fixed-point theorem. Balachandran and Anandhi [13] considered the controllability of boundary control integrodifferential system, Han and Park [14] studied the boundary controllability of nonlinear systems with nonlocal initial condition. MacCamy et al. [15] discussed the approximate controllability for the heat equations. The purpose of this paper is to study the approximate controllability for a class of semilinear delay systems with boundary control.

2. Mild Solutions

By solutions of system (1.6) we mean mild solutions, that is, solutions in the space $C([-\Delta, b]; E)$. In the following, we provide an existence and uniqueness theorem for (1.6).

Theorem 2.1. *If (H1)–(H5) are satisfied, then system* (1.6) *has a unique solution for each control* $u(\cdot) \in L^p(I; U)$.

Proof. Define

$$\widehat{\xi}(t) = \begin{cases} S(t)\xi(0), & t \in I, \\ \xi(t), & t \in [-\Delta, 0], \end{cases} \tag{2.1}$$

and define $y(t) = x(t) + \widehat{\xi}(t)$. It is easy to know that x satisfies

$$\begin{gathered} x(t) = \int_0^t S(t-s)\left[\sigma Bu(s) + f\left(s, x_s + \widehat{\xi}_s\right)\right]ds + \int_0^t AS(t-s)Bu(s)ds, \quad t \in I, \\ x_0 = 0. \end{gathered} \tag{2.2}$$

Let $Y = \{x \in C([-\Delta, b]; E) : x(t) = 0,\ \forall t \in [-\Delta, 0]\}$. Then, Y is a Banach space with supremum norm. For any $u(\cdot) \in L^p(I; U)$, define an operator $J : Y \to Y$ as follows:

$$(Jx)(t) = \begin{cases} \displaystyle\int_0^t S(t-s)\left[\sigma Bu(s) + f\left(s, x_s + \widehat{\xi}_s\right)\right]ds + \int_0^t AS(t-s)Bu(s)ds, & t \in I, \\ 0, & t \in [-\Delta, 0]. \end{cases} \tag{2.3}$$

We need to show that J is well defined. First, we show that $(Jx)(t) \in E$ for any $x \in Y$ and $t \in I$. Indeed, we have from (H5) that $\|f(t, \eta)\| \leq L|\eta| + M_1$, where $M_1 = \sup_{t\in I}\|f(t, 0)\|$. For any $s \in I$ and $\theta \in [-\Delta, 0]$, we have

$$\|x(s+\theta)\| \leq \sup_{t\in[-\Delta, T]} \|x(t)\| \leq \|x\|_Y, \tag{2.4}$$

and $\|\widehat{\xi}(s+\theta)\| \leq \max(|\xi|, M\|\xi(0)\|)$, where $M = \max_{t\in I}\|S(t)\|$.

Note that

$$\left\|\int_0^t AS(t-s)Bu(s)ds\right\| \leq \left(\int_0^t \|AS(t-s)B\|^q\right)^{1/q}\left(\int_0^t \|u(s)\|^p\right)^{1/p} \leq \|\gamma\|_{L^q}\|u\|_{L^p}, \tag{2.5}$$

and that

$$\begin{aligned} &\left\|\int_0^t S(t-s)\left[\sigma Bu(s)+f\left(s,x_s+\widehat{\xi}_s\right)\right]ds\right\| \\ &\quad\leq M\int_0^t \left[\|\sigma Bu(s)\| + L\left|x_s+\widehat{\xi}_s\right| + M_1\right]ds \\ &\quad\leq M\|\sigma B\|_{L(U,E)}\sqrt[q]{T}\|u\|_{L^p} + MM_1T + MLT(\max(|\xi|, M\|\xi(0)\|) + \|x\|_Y). \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we prove that $(Jx)(t) \in E$ for any $x \in Y$ and $t \in I$.

Next, we show that J maps Y into Y, in other words, $Jx \in Y$ for any $x \in Y$. Taking t, $t+\delta \in I$ with $\delta > 0$, then

$$\begin{aligned} &\|(Jx)(t+\delta)-(Jx)(t)\| \\ &\quad= \left\|\int_0^{t+\delta} S(t+\delta-s)\left[\sigma Bu(s)+f\left(s,x_s+\widehat{\xi}_s\right)\right]ds + \int_t^{t+\delta} AS(t+\delta-s)Bu(s)ds \right.\\ &\qquad\left. -\int_0^t S(t-s)\left[\sigma Bu(s)+f\left(s,x_s+\widehat{\xi}_s\right)\right]ds - \int_0^t AS(t-s)Bu(s)ds\right\| \\ &\quad\leq \left\|\int_0^t [S(t+\delta-s)-S(t-s)]\left[\sigma Bu(s)+f\left(s,x_s+\widehat{\xi}_s\right)\right]ds\right\| \\ &\qquad+ \left\|\int_0^t [AS(t+\delta-s)Bu(s)-AS(t-s)Bu(s)]ds\right\| \\ &\qquad+ \left\|\int_t^{t+\delta} S(t+\delta-s)\left[\sigma Bu(s)+f\left(s,x_s+\widehat{\xi}_s\right)\right]ds\right\| \\ &\qquad+ \left\|\int_t^{t+\delta} AS(t+\delta-s)Bu(s)ds\right\| \\ &\quad:= I_1+I_2+I_3+I_4. \end{aligned} \tag{2.7}$$

Since $S(t)$ is an analytic semigroup, (2.6) implies that as $\delta \to 0$

$$I_1 = (S(\delta)-I)\int_0^t S(t-s)\left[\sigma Bu(s)+f\left(s,x_s+\widehat{\xi}_s\right)\right]ds \longrightarrow 0. \tag{2.8}$$

Also, from (2.5), we have

$$I_2 = (S(\delta) - I)\int_0^t AS(t-s)Bu(s)ds \longrightarrow 0. \tag{2.9}$$

Notice that $I_3 \to 0$ and $I_4 \to 0$ as $\delta \to 0$ follow from estimates

$$\begin{aligned} I_3 &\le M\int_t^{t+\delta}\left(\|\sigma B\|\|u(s)\| + L\left|x_s + \hat{\xi}_s\right| + M_1\right)ds \\ &\le M\|\sigma B\|_{L(U,E)}\sqrt[q]{\delta}\ \|u\|_{L^p} + ML(\max(|\xi|, M\|\xi(0)\|) + \|x\|_Y)\delta + MM_1\delta, \\ I_4 &\le \left(\int_t^{t+\delta}\gamma^q(s)ds\right)^{1/q}\|u\|_{L^p}. \end{aligned} \tag{2.10}$$

We have $\|(Jx)(t+\delta) - (Jx)(t)\| \to 0$ as $\delta \to 0$ and, hence, $Jx \in Y$.

Now, we prove that J^n is a contraction mapping for sufficiently large n. In fact, for any $x_1, x_2 \in Y$,

$$|x_{1s} - x_{2s}| = \sup_{\theta\in[-\Delta,0]}\|x_1(s+\theta) - x_2(s+\theta)\| \le \|x_1 - x_2\|_Y. \tag{2.11}$$

Therefore,

$$\begin{aligned} \|(Jx_1)(t) - (Jx_2)(t)\| &= \left\|\int_0^t S(t-s)\left[f\left(s, x_{1s} + \hat{\xi}_s\right) - f\left(s, x_{1s} + \hat{\xi}_s\right)\right]ds\right\| \\ &\le ML\int_0^t |x_{1s} - x_{2s}|ds \le MLt\|x_1 - x_2\|_Y, \\ |(Jx_1)_s - (Jx_2)_s| &= \sup_{\theta\in[-\Delta,0]}\|(Jx_1)(s+\theta) - (Jx_2)(s+\theta)\| \le MLs\|x_1 - x_2\|_Y. \end{aligned} \tag{2.12}$$

Similarly,

$$\left\|\left(J^2x_1\right)(t) - \left(J^2x_2\right)(t)\right\| \le ML\int_0^t|(Jx_1)_s - (Jx_2)_s|ds \le \frac{M^2L^2t^2}{2}\|x_1 - x_2\|_Y. \tag{2.13}$$

By mathematical induction, we have

$$\|(J^nx_1)(t) - (J^nx_2)(t)\| \le \frac{(MLT)^n}{n!}\|x_1 - x_2\|_Y. \tag{2.14}$$

Hence,

$$\|J^nx_1 - J^nx_2\|_Y \le \frac{(MLT)^n}{n!}\|x_1 - x_2\|_Y, \tag{2.15}$$

and J^n is a contraction mapping for sufficiently large n. The contraction mapping principle implies that J has a unique fixed-point in Y, which is the unique solution of (1.6). The proof of the theorem is complete. □

3. Approximate Controllability

The solution of (1.6) is denoted by $y(t; t_0, \xi, u)$ to emphasize the initial time t_0, initial state $\xi \in C$, and control function $u(\cdot)$. $y(t_1; t_0, \xi, u)$ is called the system state at time t_1 corresponding to initial pair (t_0, ξ) and the control function u. The set

$$R(t_1; t_0, \xi)(N) = \{y(t_1; t_0, \xi, u) : u(\cdot) \in L^p(t_0, T; U)\} \tag{3.1}$$

is called the reachable set of system (1.6) at time t_1 corresponding to initial pair (t_0, ξ). $\overline{R(t_1; t_0, \xi)(N)}$ is the closure of $R(t_1; t_0, \xi)(N)$ in E.

Definition 3.1. System (1.6) is said to be approximately controllable on $[t_0, t_1]$ *if* $\overline{R(t_1; t_0, \xi)(N)} = E$ for any $\xi \in C$.

Definition 3.2. System (1.6) is said to be approximately null controllable on $[t_0, t_1]$ if for any $\xi \in C$ and $\epsilon > 0$, there is a control function $u(\cdot) \in L^p(t_0, t_1; U)$ such that $\|y(t_1; t_0, \xi, u)\| < \epsilon$.

Similar to nonlinear system (1.6), we define the reachable set of system (1.7) at time t_1 corresponding to the initial pair (t_0, y_0) as $R(t_1; t_0, y_0)(L)$. The approximate controllability and approximate null controllability for system (1.7) can also be defined similarly.

To consider the approximate controllability of system (1.6), we need two new operators. For any $t_1, t_2 \in I$ with $t_2 > t_1$, $E(t_1, t_2) : L^p(t_1, t_2; U) \to E$, and $N(t_1, t_2) : L^p(t_1, t_2; U) \to E$ are defined as:

$$\begin{gathered} E(t_1, t_2)u = \int_{t_1}^{t_2} S(t_2 - s)\sigma Bu(s)ds + \int_{t_1}^{t_2} AS(t_2 - s)Bu(s)ds, \\ N(t_1, t_2)u = \int_{t_1}^{t_2} S(t_2 - s)f(s, y_s)ds, \end{gathered} \tag{3.2}$$

where $y(t; u)$ is the solution of (1.6) with the initial pair (t_1, ξ) and control function $u(\cdot) \in L^p(t_1, t_2; U)$ in the definition of $N(t_1, t_2)$.

The following result provides sufficient conditions for the approximate controllability of system (1.6).

Theorem 3.3. *Assume that system* (1.7) *is approximately controllable on the interval* $[b, T]$ *for any* $b \geq 0$. *If there exists a function* $Q(\cdot) \in L^1(I)$ *such that*

$$\|f(t, z)\| \leq Q(t), \quad \forall (t, z) \in I \times C, \tag{3.3}$$

then system (1.6) *is approximately controllable on* I.

Proof. We need to show that the reachable set of system (1.6) at time T is dense in Banach space E, in other words,

$$\overline{R(T;0,\xi)(N)} = E \tag{3.4}$$

for any $\xi \in C$. To this end, given any $\epsilon > 0$ and $\overline{x} \in E$. Since (1.7) is approximately controllable on $[0,T]$, there exists a control function $v_0(\cdot) \in L^p(0,T;U)$ such that

$$\|S(T)\xi(0) + E(0,T)v_0 - \overline{x}\| < \frac{\epsilon}{2}. \tag{3.5}$$

Note that $Q(\cdot) \in L^1(I)$, we can select a sequence $t_n \in I$ such that $t_n > t_{n-1}$ and

$$\int_{t_n}^{T} Q(t)dt \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.6}$$

Let $y_1 := y(t_1;0,\xi,v_0)$. Again, the approximate controllability of (1.7) on $[t_1,T]$ implies that a control $v_1(\cdot) \in L^p(t_1,T;U)$ exists such that

$$\left\|S(T-t_1)y_1 + E(t_1,T)v_1 - \overline{x}\right\| < \frac{\epsilon}{2}. \tag{3.7}$$

Define

$$u_1(t) = \begin{cases} v_0(t), & 0 \le t \le t_1, \\ v_1(t), & t_1 < t \le T. \end{cases} \tag{3.8}$$

Then $u_1(\cdot) \in L^p(0,T;U)$. Repeating the procedure, we have three sequences y_n, v_n, and u_n such that $v_n(\cdot) \in L^p(t_n,T;U)$, $u_n(\cdot) \in L^p(0,T;U)$,

$$\begin{gathered} u_n(t) = \begin{cases} u_{n-1}(t), & 0 \le t \le t_n, \\ v_n(t), & t_n < t \le T, \end{cases} \\ y_n = y(t_n;0,\xi,u_{n-1}), \qquad \left\|S(T-t_n)y_n + E(t_n,T)v_n - \overline{x}\right\| < \frac{\epsilon}{2}. \end{gathered} \tag{3.9}$$

The solution of (1.6) under the control function $u_n(\cdot)$ is

$$\begin{aligned} y(t;0,\xi,u_n) &= S(t-t_n)[S(t_n)\xi(0) + E(0,t_n)u_n + N(0,t_n)u_n] + E(t_n,t)u_n + N(t_n,t)u_n \\ &= S(t-t_n)[S(t_n)\xi(0) + E(0,t_n)u_{n-1} + N(0,t_n)u_{n-1}] + E(t_n,t)u_n + N(t_n,t)u_n \\ &= S(t-t_n)y_n + E(t_n,t)v_n + N(t_n,t)u_n. \end{aligned} \tag{3.10}$$

Therefore,

$$
\begin{aligned}
\left\|y(T;0,\xi,u_n)-\overline{x}\right\| &\le \left\|S(T-t_n)y_n+E(t_n,T)v_n-\overline{x}\right\|+\|N(t_n,T)u_n\| \\
&< \frac{\epsilon}{2}+\int_{t_n}^{T}\left\|S(T-s)f(s,y_s)\right\|ds \\
&\le \frac{\epsilon}{2}+M\int_{t_n}^{T}Q(s)ds<\epsilon
\end{aligned}
\tag{3.11}
$$

for a sufficient large n such that $M\int_{t_n}^{T}Q(s)ds<\epsilon/2$. Hence, (3.4) follows, and the proof is complete. □

The next theorem is about the approximate null controllability of system (1.6).

Theorem 3.4. *Assume that system* (1.7) *is approximately null controllable on the interval* $[b,T]$ *for any* $b\ge 0$*, and* (3.3) *is satisfied. Then system* (1.6) *is approximately null controllable on* I*.*

Proof. For any $\epsilon>0$ and $\xi\in C$, we need to show that there exists a control function $u(\cdot)\in L^p(I;U)$ such that $\|S(T)\xi(0)+E(0,T)u+N(0,T)u\|<\epsilon$. Since system (1.7) is approximately null controllable on $[0,T]$, there is a control function $v_0(\cdot)\in L^p(0,T;U)$ such that $\|S(T)\xi(0)+E(0,T)v_0\|<\epsilon/2$. Select a sequence t_n as in the proof of Theorem 3.3. Let $y_1:=y(t_1;0,\xi,v_0)$. There exists a control function $v_1(\cdot)\in L^p(t_1,T;U)$ such that

$$
\left\|S(T-t_1)y_1+E(t_1,T)v_1\right\|<\frac{\epsilon}{2}
\tag{3.12}
$$

due to the assumption that (1.7) is approximately null controllable on $[t_1,T]$.

Similar to the proof of Theorem 3.3, we obtain three sequences y_n, v_n, and u_n such that $v_n(\cdot)\in L^p(t_n,T;U)$, $u_n(\cdot)\in L^p(I;U)$,

$$
\begin{gathered}
u_n(t)=\begin{cases} u_{n-1}(t), & 0\le t\le t_n,\\ v_n(t), & t_n<t\le T,\end{cases}\\
y_n=y(t_n;0,\xi,u_{n-1}),\qquad \left\|S(T-t_n)y_n+E(t_n,T)v_n\right\|<\frac{\epsilon}{2}.
\end{gathered}
\tag{3.13}
$$

Note that

$$
\begin{aligned}
y(t;0,\xi,u_n)&=S(t)\xi(0)+E(0,t)u_n+N(0,t)u_n\\
&=S(t-t_n)y_n+E(t_n,t)v_n+N(t_n,t)u_n,
\end{aligned}
\tag{3.14}
$$

we have

$$
\begin{aligned}
\left\|y(T;0,\xi,u_n)\right\| &\le \left\|S(T-t_n)y_n+E(t_n,T)v_n\right\|+\|N(t_n,T)u_n\|\\
&<\frac{\epsilon}{2}+M\int_{t_n}^{T}Q(s)ds<\epsilon.
\end{aligned}
\tag{3.15}
$$

The proof of the theorem is complete. □

4. Example

In this section, we provide an example to illustrate the application of the results established in Section 3.

Example 4.1. Consider the following heat control system:

$$
\begin{aligned}
y_t(t,x) &= \Delta y(t,x) + f\left(t, y(t,x), y(t-\Delta,x)\right), \quad t \in I,\ x \in \Omega, \\
y(t,x) &= u(t), \quad t \in I,\ x \in \Gamma, \\
y(t,x) &= \xi(t,x), \quad t \in [-\Delta, 0],\ x \in \Omega,
\end{aligned}
\tag{4.1}
$$

where Ω is a bounded and open subset of the Euclidean space $\mathbb{R}^n$ with a sufficiently smooth boundary Γ.

To formulated this system as a boundary control system (1.1), we let $E = L^2(\Omega)$, $X = H^{-1/2}(\Gamma)$, $U = L^2(\Gamma)$, $B_1 = I$, $D(\sigma) = \{y \in L^2(\Omega) : \Delta y \in L^2(\Omega)\}$, and $\sigma = \Delta$. The operator A is given by $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, $A = \Delta$. Then A generates an analytic semigroup $S(t)$ in E. The operator τ is the trace operator $\gamma_0 y$ which is well defined and belongs to $H^{-1/2}(\Gamma)$ for each $y \in D(\sigma)$. Clearly, assumptions (H1) and (H2) are satisfied. Define the linear operator $B : L^2(\Gamma) \to L^2(\Omega)$ by $Bu = w_u$, where $w_u \in L^2(\Omega)$ is the unique solution to the Dirichlet boundary-value problem

$$
\begin{aligned}
\Delta w_u &= 0 \quad \text{in } \Omega, \\
w_u &= u \quad \text{on } \Gamma.
\end{aligned}
\tag{4.2}
$$

It is proved in [1] that for every $u \in H^{-1/2}(\Gamma)$, (4.2) has a unique solution $w_u \in L^2(\Omega)$ satisfying $\|Bu\|_{L^2(\Omega)} = \|w_u\|_{L^2(\Omega)} \leq C_1 \|u\|_{H^{-1/2}(\Gamma)}$. This shows that (H3) is satisfied. It is proved in [4] that there exists a positive constant K_1 independent of u and t such that

$$
\|AS(t)Bu\|_{L^2(\Omega)} \leq K_1 t^{-3/4} \|u\|_{L^2(\Gamma)}
\tag{4.3}
$$

for all $u \in L^2(\Gamma)$ and $t > 0$. In other words, (H4) holds with $\gamma(t) = K_1 t^{-3/4}$. Therefore, system (4.1) can be formulated to the form (1.6). Since the corresponding linear system of (4.1)

$$
\begin{aligned}
y_t(t,x) &= \Delta y(t,x), \quad t \in I,\ x \in \Omega, \\
y(t,x) &= u(t), \quad t \in I,\ x \in \Gamma, \\
y(0,x) &= \xi(0,x), \quad x \in \Omega
\end{aligned}
\tag{4.4}
$$

is approximately controllable on any interval $[b, T]$ with $b \geq 0$; see [15]. It follows from Theorem 3.3 that system (4.1) is approximately controllable on I if the nonlinear perturbation function f satisfies (H5).

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